

3. The implicit function theorem.

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Condition (2.4) is satisfied if (2.3) holds in the sphere

$$S: \|u - u_0\| \leq (1-l)^{-1} \|Vu_0 - u_0\|. \quad (2.5)$$

Moreover, u is the limit of the sequence $\{u_n\}$ where

$$u_{n+1} = Vu_n, \quad n = 0, 1, 2, \dots,$$

and there results the estimate

$$\|u - u_{n+1}\| \leq l(1-l)^{-1} \|u_{n+1} - u_n\| \leq l^{n+1}(1-l)^{-1} \|u_1 - u_0\|. \quad (2.6)$$

3. THE IMPLICIT FUNCTION THEOREM.

THEOREM 3.1. Let T^* be an operator with domain $D \subset B_1$ and range in B_2 , let $S^* = S(u^*, r^*) \subset D$ and

$$T^* u^* = \theta. \quad (3.1)$$

We assume furthermore that there exists a linear operator K on S^* into B_2 with the following properties:

- $\alpha)$ K has a bounded inverse, K^{-1} , defined on B_2 and
- $\beta)$ There exists a constant $m < \|K^{-1}\|^{-1}$ such that

$$\|T^*v - T^*u - K(v-u)\| \leq m \|v-u\| \quad \text{for } u, v \in S^*. \quad (3.2)$$

Then there exists an $\Omega = (u^*, r, a, b)$ -neighborhood of T^* , such that for all $T \in \Omega$ the equation

$$Tu = \theta, \quad (3.1a)$$

has a unique solution $u = u(T)$ in $S(u^*, r)$. This solution is continuous in T at $T = T^*$ in the sense

$$\|u(T) - u^*\| \rightarrow 0 \quad \text{as} \quad \|Tu^*\| \rightarrow 0. \quad (3.3)$$

In this theorem the operators T and K need not be continuous.

Proof. Let T lie in a (u^*, r, a, b) -neighborhood of T^* with $r \leq r^*$. Then by (3.2), with $\Delta T = T - T^*$, we have

$$\begin{aligned} \|Tv - Tu - K(v - u)\| &\leq \|\Delta T v - \Delta T u\| \\ + \|T^*v - T^*u - K(v - u)\| &\leq (b + m) \cdot \|v - u\| \quad (3.4) \\ \text{for } u, v \in S(u^*, r) &\subset S^*, \end{aligned}$$

and the equation

$$u = Vu \equiv K^{-1}(K - T)u, \quad u \in S(u^*, r) = S, \quad (3.5)$$

is equivalent to (3.1 a), $u \in S$.

For every $b \geq 0$ with $l = (b + m) \|K^{-1}\| < 1$, (3.4) yields

$$\begin{aligned} \|Vu - Vv\| &= \|K^{-1}[K(u - v) - Tu + Tv]\| \leq l \|u - v\|, \\ l &< 1, \quad \text{for } u, v \in S(u^*, r). \end{aligned}$$

If

$$\|Vu^* - u^*\| = \|K^{-1}Tu^*\| < (1 - l)r, \quad (3.6)$$

then the assumptions of the contraction mapping theorem [Section 2 f] are satisfied. Thus, under these conditions, there exists a unique solution $u = u(T)$ in S satisfying the condition

$$\|u - u^*\| \leq (1 - l)^{-1} \|K^{-1}Tu^*\| \leq (1 - l)^{-1} \|K^{-1}\| \cdot \|Tu^*\|. \quad (3.7)$$

This implies the continuity (3.3).

The inequality (3.6) is satisfied if $T \in \Omega$ with

$$a = [\|K^{-1}\|^{-1} - (b + m)]r.$$

This completes the proof.

This proof also gives quantitative conditions for r, a, b which are sufficient for the existence of a unique and continuous solution u of (3.1 a) in $S(u^*, r)$.

Supplement. The assertion of Theorem 3.1 is true for each Ω -neighborhood of T^* with $0 < r \leq r^*$ and a, b satisfying

$$a = [\|K^{-1}\|^{-1} - (b + m)]r > 0. \quad (3.8)$$

Then, for the solution $u = u(T)$ in S , the estimate (3.7) holds.

A unique solution of (3.4 a) in $S(u^*, r)$ also exists for such r and b if (3.6) holds, but in (3.8) the sign " $>$ " cannot be replaced by " $\geq$ ", nor can the constant a in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator T which is linear in $S(u^*, r)$.

4. INVERSE FUNCTION THEOREMS.

Under the conditions of the implicit function Theorem 3.1, the operator T has a local inverse defined in a neighborhood of a point w_0 for which

$$Tu = w. \quad (4.1)$$

has a solution u_0 . This inverse has its range in a neighborhood of u_0 . For the proof set $T^*u = Tu - w_0$ in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution u of equation (4.1) for all w in B_2 even if T is defined on the whole Banach space B_1 and the conditions are satisfied at each point u of B_1 .¹⁾

However, this actually is not necessary for the existence of at least one solution u of (4.1) for all $w \in B_2$ as is indicated by the following theorem.

THEOREM 4.1. Let the operator T , mapping a non-empty domain $D \subset B_1$ into B_2 , satisfy the following conditions:

For each $u \in D$ there exist a sphere $S(u, r) \subset D$, a linear operator K , and a constant m such that the following conditions hold:

- $\alpha)$ K has a bounded inverse K^{-1} on $TS(u, r)$
- $\beta)$ $\|T\varphi - T\tilde{\varphi} - K(\varphi - \tilde{\varphi})\| \leq m \|\varphi - \tilde{\varphi}\|$ for $\varphi, \tilde{\varphi} \in S(u, r)$
- $\gamma)$ $(\|K^{-1}\|^{-1} - m)r \geq c > 0$ where the constant c is independent of $u \in D$.

1) Example: $Tu \equiv \arctan u = w$, with $B_1 = B_2 = \{\text{real numbers}\}$, is not solvable for all $w \in B_2$, although the conditions of Theorem 3.1 are satisfied at each point $(u, w = \arctan u)$ for $T^*u = Tu - w$.