3. The implicit function theorem.

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **9 (1963)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **07.05.2024**

Condition (2.4) is satisfied if (2.3) holds in the sphere

$$S\colon \| u - u_0 \| \leqq (1-l)^{-1} \| Vu_0 - u_0 \| . \tag{2.5}$$

Moreover, u is the limit of the sequence $\{ u_n \}$ where

$$u_{u+1} = Vu_n , \qquad n = 0, 1, 2, \ldots ,$$

and there results the estimate

$$\| u - u_{n+1} \| \leqq l(1-l)^{-1} \| u_{n+1} - u_n \| \leqq l^{n+1}(1-l)^{-1} \| u_1 - u_0 \| . \tag{2.6}$$

3. The implicit function theorem.

Theorem 3.1. Let T^* be an operator with domain $D \subset B_1$ and range in B_2, let $S^* = S(u^*, r^*) \subset D$ and

$$T^* u^* = \theta . \tag{3.1}$$

We assume furthermore that there exists a linear operator K on S^* into B_2 with the following properties:

α) K has a bounded inverse, K^{-1}, defined on B_2 and
β) There exists a constant $m < \| K^{-1} \|^{-1}$ such that

$$\| T^* v - T^* u - K(v-u) \| \leqq m \| v - u \| \quad \text{for} \quad u, v \in S^* . \tag{3.2}$$

Then there exists an $\Omega = (u^*, r, a, b)$-neighborhood of T^*, such that for all $T \in \Omega$ the equation

$$Tu = \theta , \tag{3.1a}$$

has a unique solution $u = u(T)$ in $S(u^*, r)$. This solution is continuous in T at $T = T^*$ in the sense

$$\| u(T) - u^* \| \to 0 \quad \text{as} \quad \| Tu^* \| \to 0 . \tag{3.3}$$

In this theorem the operators T and K need not be continuous.

Proof. Let T lie in a (u^*, r, a, b)-neighborhood of T^* with $r \leqq r^*$. Then by (3.2), with $\Delta T = T - T^*$, we have

$$\begin{aligned} \| Tv - Tu - K(v-u) \| &\leqq \| \Delta Tv - \Delta Tu \| \\ + \| T^* v - T^* u - K(v-u) \| &\leqq (b+m) \cdot \| v-u \| \\ \text{for} \quad u, v \in S(u^*, r) &\subset S^*, \end{aligned} \tag{3.4}$$

and the equation

$$u = Vu \equiv K^{-1}(K-T)u\,, \quad u \in S(u^*, r) = S\,, \tag{3.5}$$

is equivalent to (3.1 a), $u \in S$.

For every $b \geqq 0$ with $l = (b+m) \, \| K^{-1} \| < 1$, (3.4) yields

$$\| Vu - Vv \| = \| K^{-1}[K(u-v) - Tu + Tv] \| \leqq l \, \| u-v \|\,,$$
$$l < 1, \qquad \text{for} \qquad u,\ v \in S(u^*, r)\,.$$

If

$$\| Vu^* - u^* \| = \| K^{-1} \, Tu^* \| < (1-l)\, r\,, \tag{3.6}$$

then the assumptions of the contraction mapping theorem [Section 2 f] are satisfied. Thus, under these conditions, there exists a unique solution $u = u(T)$ in S satisfying the condition

$$\| u - u^* \| \leqq (1-l)^{-1} \, \| K^{-1} \, Tu^* \| \leqq (1-l)^{-1} \, \| K^{-1} \| \cdot \| Tu^* \|\,. \tag{3.7}$$

This implies the continuity (3.3).

The inequality (3.6) is satisfied if $T \in \Omega$ with

$$a = [\, \| K^{-1} \|^{-1} - (b+m)]\, r\,.$$

This completes the proof.

This proof also gives quantitative conditions for r, a, b which are sufficient for the existence of a unique and continuous solution u of (3.1 a) in $S(u^*, r)$.

Supplement. The assertion of Theorem 3.1 is true for each Ω-neighborhood of T^* with $0 < r \leqq r^*$ and a, b satisfying

$$a = [\, \| K^{-1} \|^{-1} - (b+m)]\, r > 0\,. \tag{3.8}$$

Then, for the solution $u = u(T)$ in S, the estimate (3.7) holds.

A unique solution of (3.1 a) in $S(u^*, r)$ also exists for such r and b if (3.6) holds, but in (3.8) the sign " $>$ " cannot be replaced by " $\geqq$ ", nor can the constant a in (3.8) be replaced by any larger one.

The last statement can be proved by simple examples in the one-dimensional case and with an operator T which is linear in $S(u^*, r)$.

4. Inverse function theorems.

Under the conditions of the implicit function Theorem 3.1, the operator T has a local inverse defined in a neighborhood of a point w_0 for which

$$Tu = w. \tag{4.1}$$

has a solution u_0. This inverse has its range in a neighborhood of u_0. For the proof set $T^*u = Tu - w_0$ in Theorem 3.1. However, the conditions of this theorem are still not sufficient for the existence of a solution u of equation (4.1) for all w in B_2 even if T is defined on the whole Banach space B_1 and the conditions are satisfied at each point u of B_1.[1])

However, this actually is not necessary for the existence of at least one solution u of (4.1) for all $w \in B_2$ as is indicated by the following theorem.

Theorem 4.1. Let the operator T, mapping a non-empty domain $D \subset B_1$ into B_2, satisfy the following conditions:

For each $u \in D$ there exist a sphere $S(u, r) \subset D$, a linear operator K, and a constant m such that the following conditions hold:

α) K has a bounded inverse K^{-1} on $TS(u, r)$

β) $\| Tv - T\tilde{v} - K(v - \tilde{v}) \| \leqq m \| v - \tilde{v} \|$ for v, $\tilde{v} \in S(u, r)$

γ) $(\| K^{-1} \|^{-1} - m)\, r \geqq c > 0$ where the constant c is independent of $u \in D$.

1) Example: Tu $\equiv$ arctan u $=$ w, with $B_1 = B_2 = \{$real numbers$\}$, is not solvable for all w$\in B_2$, although the conditions of Theorem 3.1 are satisfied at each point (u, w $=$ arctan u) for $T^*u = Tu - w$.