1. Introduction.

ON THE CONSTRUCTION OF RELATED EQUATIONS FOR THE ASYMPTOTIC THEORY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS ABOUT A TURNING POINT

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(Reçu le 17 mars 1962)

1. INTRODUCTION.

A common type of linear ordinary differential equation—indeed the one which includes the great majority of such equations that figure prominently in applications, and for which literatures therefore exist—is

$$\frac{d^n u}{dz^n} + \lambda p_1(z, \lambda) \frac{d^{n-1} u}{dz^{n-1}} + \ldots + \lambda^n p_n(z, \lambda) u = 0 , \qquad (1.1)$$

in which λ is a parameter, and the coefficients $p_j(z, \lambda)$ are either:

(a) free from λ;
(b) polynomials in $1/\lambda$;
(c) analytic functions of $1/\lambda$ in a domain $|\lambda| \geqq M > 0$;
(d) functions that can be asymptotically represented by power series in $1/\lambda$ in a sector of the complex λ plane.

This paper is concerned with equations of this type for large values of $|\lambda|$.

The consideration of any differential equation— unless its coefficients be constants—must be relevant to a domain of the variable. The objective of this paper can be made clear only after some description of this domain has been given.

An equation (1.1) has associated with it the so-called *auxiliary equation*

$$\chi^n + p_1(z, \infty) \chi^{n-1} + \ldots + p_n(z, \infty) = 0 . \qquad (1.2)$$

Sponsored by the Mathematics Research Center, U. S. Army, Madison, Wisconsin, under Contract No. DA-11-022-ORD-2059.

As a polynomial equation in χ, this has roots which are functions of z. In a given z-region, if z is complex, or on a given interval, if z is real, these *auxiliary roots* may all be simple or various multiplicities may occur among them. A multiplicity may be permanent in z over the region, or over a sub-region, or, on the other hand, it may be isolated, in which case it maintains at a point without doing so at other points of the neighborhood. An isolated point of multiplicity is called a *turning-point*.

Fully developed theory is extant, and can justifiably be referred to as classical, for the determination of the asymptotic forms of the solutions of a differential equation (1. 1) over any closed z-region which completely excludes turning-points. This theory applies, of course, irrespective of the region, to all equations (1. 1) with constant coefficients. The state of the theory is very different, namely quite fragmentary, when a turning-point is lodged within the region. For this reason, and also because modern physical theories require it, the study of the solution forms of an equation (1. 1) in a region about a turning-point is of emminent contemporary interest. The classical algorithms fail irretrievably in such a region, a fact which has been shown to be inevitable by results otherwise obtained, because the forms yielded by those algorithims lack adequacy to reflect the intricate functional metamorphoses which characterize the solutions of the differential equation in a turning-point neighborhood. The profundity of these changes is suggested by even so simple an example as the differential equation

$$\frac{d^2 u}{dx^2} + \lambda^2 xu = 0,$$

with x and λ^2 real. The origin is in this case a turning point, and about this point the solutions undergo transitions between oscillatory and exponential function types.

A turning-point is to be characterized in the first instance by the configuration of the auxiliary roots in its neighborhood. In the case of a differential equation of the second order, for which there are just two auxiliary roots, $\chi_1(z)$, $\chi_2(z)$, the grounds for distinction are limited, being in fact restricted to the degree to which the root difference $(\chi_1 - \chi_2)$ vanishes. When equations of higher and higher orders are taken into consideration, the grounds for distinction rapidly multiply. Between the possible extreme configurations in which the roots all coincide with each other, and in which only two roots come into a coincidence, there are all the intermediate ones in which certain roots may come into one coincidence, while other roots come into other coincidences quite apart. A simple example of this kind is afforded by the differential equation

$$\frac{d^4 u}{dz^4} - 2\lambda \frac{d^3 u}{dz^3} + \lambda^2 \left(1 - \frac{1}{\lambda}\right) \frac{d^2 u}{dz^2} - 2z\lambda^3 \frac{du}{dz}$$

$$+ \lambda^4 \left[z - z^2 + \frac{z-1}{\lambda}\right] u = 0 . \tag{1.3}$$

The auxiliary roots of this are $\chi_1 = iz^{\frac{1}{2}}$, $\chi_2 = -iz^{\frac{1}{2}}$, $\chi_3 = 1 + z^{\frac{1}{2}}$, $\chi_4 = 1 - z^{\frac{1}{2}}$.

At the turning-point $z = 0$, $\chi_1 = \chi_2$ and $\chi_3 = \chi_4$, but $\chi_1 \neq \chi_3$.

Algebraically the characteristics of an auxiliary root configuration are, of course, expressible in terms of the reducibility of the auxiliary equation (1. 2). In these terms the objective of the present paper can now be explained. A methodical key to the deduction of the asymptotic solution forms of a given differential equation has been brought to hand when a so-called *related equation* has been found. The term *related equation* in this context signifies another differential equation which

(a) has coefficients that are the same as those of the given equation out to terms of an arbitrarily pre-specified degree in $1/\lambda$, and

(b) whose solution forms are known.

The analysis through which such an equation may be applied is not included in this paper, mainly because it is essentially systematic and has been set forth in a number of instances in the literature [1], [2], [3], [4], [5], [6]. The finding, or the construction, of a related equation is not, and could not be expected to be, systematic. It is in this that ingenuity is an essential requisite. As might well be expected, its difficulty mounts rapidly with the order of the differential equation, and therefore any means for referring the problem from a given equation to ones of lower order are treasurable. This brings us to the point. In an earlier proper [5] I have shown that if, in a given region, p of the auxiliary roots of a differential equation (1. 1) are simple, the construction of a related equation is referrable to such a construction for an equation of the lower order $(n-p)$. The present paper goes materially further. It shows that if the auxiliary polynomial in χ factors into relatively prime factors of the degrees p and q, the coefficients of which are analytic, the construction of a related equation is referrable to such constructions for differential equations of the lower orders p and q. It will be seen at once, by iteration, that if the auxiliary polynomial has analytic factors of the degrees $p_1, p_2, \dots p_k$, the construction of a related equation is referrable to such constructions for differential equations of the orders $p_1, p_2, \cdots p_k$. The equation (1. 3) serves again as an example. Its auxiliary equation may be written

$$(\chi^2+z)(\chi^2-2\chi+1-z) = 0 .$$

A related equation for it can be constructed, because that can be done for the two equations of lower order

$$\frac{d^2 v}{dz^2} + \lambda^2 \left(z - \frac{1}{\lambda}\right) v = 0 ,$$

$$\frac{d^2 y}{dz^2} - 2\lambda \frac{dy}{dz} + \lambda^2 (1-z) y = 0 .$$

2. The hypotheses.

The given differential equation (1. 1) may be conveniently denoted by

$$L(u) = 0 , \qquad (2.1)$$

with

$$L(u) \equiv \sum_{j=0}^{n} \lambda^j p_j(z, \lambda) D^{n-j} u , \quad p_0 \equiv 1 , \qquad (2.2)$$