5. NON-REALIZABILITY AS COHOMOLOGY ALGEBRAS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **7 (1961)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.09.2024**

$= 2i(p-1)$; and the degree of a monomial in the generators is the sum of the degrees of the factors. After these definitions, it follows readily that, for each p, the cohomology $H^*(X; Z_p)$ of a space X is a graded $\mathscr{A}_p$-module.

As an abstract algebra, $\mathscr{A}_p$ has a complicated structure. It is, of course, non-commutative. The Adem-Cartan relations give a kind of commutation law. A monomial in the generators

$$\beta^{\varepsilon_0} \mathscr{P}^{r_1} \beta^{\varepsilon_1} \mathscr{P}^{r_2} \ldots \mathscr{P}^{r_k} \beta^{\varepsilon_k} \qquad (\varepsilon_j = 0 \quad \text{or} \quad 1)$$

is called *admissible* if $r_j \geqq pr_{j+1} + \varepsilon_j$ for $j = 1, 2, \ldots, k-1$ and $r_k \geqq 1$. The Adem-Cartan relations are rules for expressing inadmissible monomials in terms of admissible ones. Cartan has shown [9] that the admissible monomials form a vector space basis for $\mathscr{A}_p$. Thus there is a *normal form* for an element of $\mathscr{A}_p$.

Another consequence of the relations is the following result of Adem [3]:

4.12. *The algebra $\mathscr{A}_p$ is generated by β and the $\mathscr{P}^{p^i}$ for* $i = 0, 1, 2, \ldots$; *and $\mathscr{A}_2$ is generated by the* Sq^{2^i} *for* $i = 0, 1, 2, \ldots$.

Let us see how this is proved for $\mathscr{A}_2$. Assume, inductively, that, for $j < n$, each Sq^j is in the subalgebra generated by the Sq^{2^i}. If n is not a power of 2, then $n = a + 2^k$ where $0 < a < 2^k$. Set $b = 2^k$ and apply 4.5. The coefficient in 4.5 of $\mathrm{Sq}^{a+b} = \mathrm{Sq}^n$ is congruent to 1 mod 2. It follows that Sq^n is decomposable as a sum of products of Sq^j with $j < n$. The inductive hypothesis now implies that Sq^n is in the subalgebra of the Sq^{2^i}.

5. Non-realizability as cohomology algebras.

The preceding results will now be used to show that many of the graded algebras $F(R, n)^h$ on one generator of dimension n and height h are not realizable. Recall that $F(R, n)^2$ is realized by the n-sphere for each n and any ring R. So we shall restrict attention to the cases $2 < h \leqq \infty$.

First let $R = Z_2$, and assume that $F(Z_2, n)^h$ is realized by a space X. Let $x \in H^n(X; Z_2)$ be the generator of $H^*(X; Z_2)$. Since $h > 2$, x^2 is not zero. By 4.3, $\mathrm{Sq}^n x = x^2$ is not zero.

By 4.12, Sq^n is a sum of monomials in the Sq^{2^i} $(i = 0, 1, 2, \ldots)$. This implies that $\mathrm{Sq}^{2^i} x$ is not zero for some $2^i \leqq n$. Its dimension $n + 2^i$ is $\leqq 2n$. Since the groups $H^q(X; Z_2) = 0$ for $n < q < 2n$, it follows that $2^i = n$. This proves

5.1. *If* n *is not a power of* 2, *and* $2 < \mathrm{h} \leqq \infty$, *then* $\mathrm{F}(\mathrm{Z}_2, \mathrm{n})^h$ *cannot be realized.*

Now let p be a prime > 2, and consider $F(Z_p, 2n)^h$. Suppose it is realized by a space X for a certain n and $h > p$. Then the generator $x \in H^{2n}(X; Z_p)$ is such that x^p is non-zero in $H^{2np}(X; Z_p)$. By 4.8, $\mathscr{P}^n x = x^p$ is not zero. By 4.12, $\mathscr{P}^n$ is a sum of monomials in the $\mathscr{P}^{p^i}$ $(i = 0, 1, 2, \ldots)$. It follows that some $\mathscr{P}^{p^i} x \neq 0$ where $p^i \leqq n$. Therefore the dimension $2n + 2p^i(p - 1)$ of $\mathscr{P}^{p^i} x$ must coincide with one of the non-zero dimensions $2ns$ of $H^*(X; Z_p)$. Then

$$n(s-1) = p^i(p-1) .$$

Since $p^i \leqq n$, and n divides $p^i(p - 1)$, it follows that $n = p^i m$ where m divides $p - 1$. This proves

5.2. *If* n *is not of the form* $\mathrm{p}^i \mathrm{m}$ *where* m *divides* $\mathrm{p} - 1$, *and* $\mathrm{p} < \mathrm{h} \leqq \infty$, then $\mathrm{F}(\mathrm{Z}_p, 2\mathrm{n})^h$ *cannot be realized.*

Passing to integer coefficients, we shall derive the following complete result:

5.3. *If* $3 < \mathrm{h} \leqq \infty$, *then* $\mathrm{F}(\mathrm{Z}, 2\mathrm{n})^h$ *is realizable if and only if* $\mathrm{n} = 1$ *or* 2.

We have seen in § 2 that $F(Z, 2)^h$ ($F(Z, 4)^h$) is realized by the complex (quaternionic) projective $(h - 1)$-space. Conversely, suppose X realizes $F(Z, 2n)^h$. As $H^*(X; Z)$ has no torsion, the universal coefficient theorem states that

$$H^*(X; Z) \otimes Z_p \approx H^*(X; Z_p) .$$

Since the reduction mod p: $H^*(X; Z) \to H^*(X; Z_p)$ is a ring homomorphism, it follows that X realizes $F(Z_p, 2n)^h$. Taking $p = 2$, 5.1 asserts that $2n = 2^s$ for some s. Taking $p = 3$, 5.2 asserts that $n = 3^t$ or 2.3^t for some t. Since both hold, we have $2^{s-1} = 3^t$ or 2.3^t. This implies $t = 0$, and therefore $n = 1$ or 2.

If we knew only that $x^2 \neq 0$, the above argument with $p = 2$ shows that n is a power of 2. Therefore

5.4. *If* n *is not a power of* 2, *then* $F(Z, 2n)^3$ *is not realizable.*

Recall, by § 2, that $F(Z, 8)^3$ and $F(Z_p, 8)^3$ are realized by the Cayley projective plane. However, by 5.3, $F(Z, 8)^4$ is not realizable. This is in accord with the fact that there is no projective 3-space over the Cayley numbers (due to non-associativity).

We turn next to the case of odd dimensional generators. Recall that $F(Z, 2n + 1)^h$ is zero except for a Z in dimensions 0 and $2n + 1$, and a Z_2 in dimensions $(2n + 1) k$ for $1 < k < h$.

5.5. *If* $2 < h \leqq \infty$, *then* $F(Z, 1)^h$ *is not realizable.*

Assume X realizes $F(Z, 1)^h$. Let η: $H^*(X; Z) \rightarrow H^*(X; Z_2)$ be reduction mod 2, and let $x \in H^1(X; Z)$ be the generator. Then x^2 is not zero and $2x^2 = 0$. It follows that ηx and $\eta(x^2) = (\eta x)^2$ are not zero. By 4.3 and 4.2,

$$(\eta x)^2 = \mathrm{Sq}^1 \eta x = \beta \eta x .$$

But $\beta\eta$ is identically zero by the definition of β. This contradiction proves 5.5.

A second proof of 5.5 is based on the Hopf theorem that there exists a mapping f: $X \rightarrow S^1$ (assuming X is a complex) such that $x = f^* y$ where y generates $H^1(S^1, Z)$. Since $y^2 = 0$, it follows that $x^2 = 0$.

5.6. $F(Z, 3)^3$ *is realizable.*

To see this, let Y be the suspension of the complex projective plane CP^2. If the latter is represented in the form $S^2 \cup e_4$ (a 2-sphere with a 4-cell attached by the Hopf mapping $S^3 \rightarrow S^2$), then $Y = S^3 \cup e_5$ where e_5 is attached by the suspension of the Hopf mapping. As this has order 2 in $\pi_4(S^3)$, the 5-cycle $2e_5$ is spherical. Hence we may adjoin a 6-cell to Y obtaining a space $X = S^3 \cup e_5 \cup e_6$ such that $\partial e_6 = 2e_5$. It is easily checked that $H^*(X; Z)$ has Z in dimensions 0 and 3, Z_2 in dimension 6, and is otherwise 0. We must show that the square of the generator $x \in H^3(X; Z)$ is non-zero in $H^6(X; Z)$. It is easily checked that the diagram

$$\begin{array}{ccccccc} H^3(X;Z) & \xrightarrow{\eta} & H^3(X;Z_2) & & \xrightarrow{g} & & H^3(Y;Z_2) \\ \downarrow f & & Sq^3 \swarrow & & \searrow Sq^2 & & \downarrow Sq^2 \\ H^6(X;Z) & \xrightarrow{\eta'} & H^6(X;Z_2) & \xleftarrow{\beta} & H^5(X;Z_2) & \xrightarrow{g'} & H^5(Y;Z_2) \end{array}$$

is commutative where f is the squaring operation, η and η' are reduction mod 2, and g, g' are induced by the inclusion $Y \subset X$. The relation $\beta Sq^2 = Sq^1 Sq^2 = Sq^3$ follows from 4.2, 4.5. All of the indicated groups except $H^3(X; Z)$ are isomorphic to Z_2.

It follows that η is an epimorphism, and η' is an isomorphism. Since Y has the same 5-skeleton as X, g is an isomorphism and g' is a monomorphism. But both groups being Z_2, g' is an isomorphism. Since $\partial e_6 = 2e_5$, it follows that β is an isomorphism. Because Sq^2 commutes with suspension and is an isomorphism in CP^2, it gives an isomorphism in Y. Thus all the mappings of the diagram excepting f and η are isomorphisms. Since η is an epimorphism, commutativity implies that $fx = x^2$ is not zero.

The preceding results are about as far as one can go using only the *primary* cohomology operations. There are secondary cohomology operations corresponding to the relations among the primary operations, and they are defined on a cohomology class on which certain primary operations are zero. The secondary operations have been exploited by J. F. Adams [1] to show that there are no mappings $S^{2n-1} \to S^n$ of Hopf invariant 1 in cases other than $n = 1, 2, 4$ and 8. He proves this by showing that Sq^{2^i}, which is not decomposable in $\mathscr{A}_2$, is decomposable in terms of secondary operations for each $i \geqq 4$. Using an argument similar to the proof of 5.1, Adams obtains the result

5.7. *If* $i \geqq 4$ *and* $2 < h \leqq \infty$, *then* $F(Z_2, 2^i)^h$ *is not realizable.*

This and preceding results settle all cases for $F(Z_2, n)^h$. It is realizable precisely in the cases $n = 1, 2$, and 4 with $3 \leqq h \leqq \infty$, and $n = 8$ with $h = 3$.

The result of Adams has been extended to primes $p > 2$ by Liulevicius [13] and Shimada [17]. They have shown that $\mathscr{P}^{p^i}$

is decomposable in terms of secondary operations for each $i \geqq 1$. Using this result, 5.2 can be improved as follows:

5.8. *If* n *is not a divisor of* $\mathrm{p} - 1$, *and* $\mathrm{p} < \mathrm{h} \leqq \infty$, *then* $\mathrm{F}(\mathrm{Z}_p, 2\mathrm{n})^h$ *cannot be realized.*

This leaves a good many unsettled cases. For example can $F(Z_p, 2(p-1))^3$ be realized for some $p > 5$? Can $F(Z_5, 8)^4$ be realized ? The cohomology of such a space would necessarily have torsion involving the prime 3. Likewise unsettled are the cases of $F(Z, 2n+1)^h$ where $n > 1$, $h > 2$ and $n = 1$, $h > 3$. In view of the preceding results, it seems unlikely that any of these can be realized.

For a rough summary, let us exclude the trivial cases $h = 1, 2$. Then the only n's for which $F(R, n)^h$ is known to be realizable are included among the integers 1, 2, 4 and 8. If $R = Z, Z_2$, or Z_3 it is not realizable for any other n. If $R = Z_p$, it is not realizable for $h > p$ and $n > 2(p-1)$. In short, $F(R, n)^h$ is not realizable except in rare cases involving small values of n or h.

These negative conclusions have interesting implications in algebra. The successful realizations were obtained by using projective spaces over the real numbers, complex numbers, quaternions, and Cayley numbers. If there is a real division algebra on n units, we can use it to realize $F(Z_2, n)^3$; hence our non-existence results imply that $n = 1, 2, 4$ or 8. Again, since $F(Z_3, 8)^4$ is not realizable, it follows that there is no real, associative division algebra on 8 units.

6. Hopf algebras.

Historically, we started with the preconception that the cohomology of a space is nothing more than a graded algebra, and we asked if certain simple graded algebras could be realized. On the whole we found that the answer was negative; and this was shown by using the fact that the algebra $\mathscr{A}_p$ of reduced powers operates in $H^*(X; Z_p)$. Our preconception was misleading, the cohomology algebra of a space is something more than a graded algebra. Just how much more is not yet clear.