

# VECTOR FIELDS ON SPHERES AND ALLIED PROBLEMS

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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **7 (1961)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **17.04.2024**

Persistenter Link: <https://doi.org/10.5169/seals-37127>

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# VECTOR FIELDS ON SPHERES AND ALLIED PROBLEMS <sup>1)</sup>

by Raoul BOTT <sup>2)</sup>

The problem on whose development I would like to report is very easily stated.

A set of  $k$  maps  $f_i: E_n \rightarrow E_n, i = 1, \dots, k$ ; of Euclidean  $n$ -space into itself will be called an orthogonal  $k$ -system if:

$$f_1(x) = x \quad \text{for all } x \in E_n, \quad \text{and}$$

$f_1(x), \dots, f_k(x)$  form an orthonormal system whenever  $x$  is a unit vector.

With this terminology our question is the following one:

*Find the greatest integer  $k$ , so that  $E_n$  admits an orthogonal  $k$ -system.*

Geometrically an orthogonal  $k$ -system on  $E_n$  is precisely a continuous  $(k - 1)$  frame on the unit sphere  $S_{n-1} \subset E_n$ , as is seen immediately once the tangent space of  $S_{n-1}$  at  $x \in S_{n-1}$  is identified with the orthogonal complement to the subspace generated by  $x$ . We are therefore dealing with a very special case of the general question of how many independent vector fields exist on a manifold, and this central position of our question has made it the favorite testing ground of progress in Algebraic Topology. It is not in the spirit of this talk to recount the precise evolution of the problem, or pay tribute to the many people who have contributed to it, be it through the general theory of vector fields, or through a specific attack—Poincaré, Hopf, Stiefel, Whitney, Steenrod, Whitehead, Wu, Addem, are just a few names which come to mind—rather, I would first like to recall an old algebraic result in this direction and then go on to some of the most recent work which topologically confirms the algebraic findings.

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<sup>1)</sup> Talk delivered at the Zurich Colloquium on Differential Geometry and Topology, June 1960.

<sup>2)</sup> The Author holds an A. P. Sloan Fellowship.

First of all it is convenient to recast the question in this manner:

PROBLEM I. *Given  $k$ , for what  $n$  does  $E_n$  admit an orthogonal  $k$ -system?*

Before discussing this question, let us formulate its linear version. If we call an orthogonal  $k$ -system linear whenever each of the functions  $f_i$ ,  $i = 1, \dots, k$ ; comprising it is linear, then this query is:

*For which  $n$ , does  $E_n$  admit a linear orthogonal  $k$ -system?*

This purely algebraic question can also be expressed as follows. A linear map

$$\mu : E_k \otimes E_n \rightarrow E_n \quad k \leq n$$

will be said to define  $E_n$  as an  $E_k$ -module if for all  $x$  in  $E_k$  and  $y$  in  $E_n$

$$|\mu(x \otimes y)| = |x| \cdot |y|,$$

The vertical bars denoting the Euclidean norm. It is then quite easy to check that  $E_n$  admits the structure of an  $E_k$ -module if and only if  $E_n$  admits a linear orthogonal  $k$ -system. In this guise then, the associated algebraic problem is stated as follows:

PROBLEM I<sup>L</sup>. *What are the dimensions of the possible  $E_k$ -modules.*

The complete solution of the linear problem is given by the theorem of Hurwitz-Radon [10, 11, 16].

THEOREM I. *If  $A_k^L$  denotes the set of dimensions of possible  $E_k$ -modules then there exist integers  $a_k^L$  so that:*

$$1. \quad A_k^L = \{n a_k^L\} \quad n = 1, 2, \dots$$

*One has:*

$$2. \quad a_{k+8}^L = 16 a_k^L \quad k > 1.$$

$$3. \quad \text{The first eight values of } a_k^L \text{ are: } 1, 2, 4, 4, 8, 8, 8, 8.$$

Immediate corollaries are:

a) *The integer  $a_k^L$  is always a power of 2.*

b)  *$E_k$  occurs as an  $E_k$ -module (i.e.,  $a_k^L = k$ ) if and only if  $k = 1, 2, 4, 8$ .*

A lovely proof of this theorem is given by Eckmann in [8]. Very briefly, his argument takes this form.

Let  $G_k$  be the abstract group generated by the symbols  $1, \varepsilon, \sigma_1, \dots, \sigma_k$ , subject to the relations:

$$1 \text{ — identity; } \varepsilon^2 = 1; \varepsilon \sigma_i = \sigma_i \varepsilon \quad (i = 1, \dots, k)$$

$$\sigma_i \sigma_j = \varepsilon \sigma_j \sigma_i, \quad i \neq j; \quad \sigma_i^2 = \varepsilon, \quad i = 1, \dots, k.$$

Next let a  $G_k$ -module,  $W$ , be called special if  $\varepsilon$  acts as  $-1$  on  $W$ . It is then easy to verify that;

$E_n$  admits the structure of an  $E_k$ -module, if and only if there exists a special  $G_k$ -module of dimension  $n$  over the real numbers.

(In one direction this correspondence is obtained by sending  $\sigma_i$  into  $f_i$  and  $\varepsilon$  into  $-1$ , whenever  $f_1, \dots, f_k$  is an orthogonal  $k$ -system. This function is then seen to define a representation of  $G_k$ .)

We are thus led to seek the special  $G_k$ -modules and it will clearly suffice to find the irreducible ones among them. Eckmann determines these with the aid of the representation theory of finite groups. He first finds the irreducible special complex  $G_k$ -modules—it turns out that there is only one isomorphism class of these if  $k$  is odd, and that there are two such classes, however of the same dimension, when  $k$  is even—and then determines the real irreducible special  $G_k$ -modules by the Schur criterion: A complex  $G_k$ -module,  $W$ , is the complexification of a real one if and only if the character,  $\chi_w$ , of  $W$ , satisfies the condition:

$$\sum \chi_w(g^2) > 0, \quad g \in G_k.$$

It is at this point that the mod 8 dependance of the answer emerges.

So much, then, for the linear case. The theorem of Radon-Hurwitz of course also gives us information about problem I. Indeed if we denote by  $A_k$  the set of dimensions  $n$ , for which  $E_n$  admits an orthogonal  $k$ -system, then  $A_k$  contains  $A_k^L$ , so that  $A_k^L$  furnishes a lower bound for the set  $A_k$ .

Actually, at the present time there is no counter example to the conjecture that  $A_k$  equals  $A_k^L$ , however we are still far from a proof of such a fact. (Added in Proof: F. Adams has just estab-

lished the validity of this conjecture.) The following theorem, due to I. M. James [12, 13, 14] possibly best describes the presently known information in the direction of this conjecture.

**THEOREM II.** *Let  $A_k$  denote the set of integers  $n$ , for which  $E_n$  admits an orthogonal  $k$ -system. Then there exist integers  $a_k$  with the property:*

$$\text{Either} \quad A_k = \{n a_k\} \quad k = 1, 2, \dots$$

$$\text{or} \quad A_k = \{n a_k\} \quad n = 2, 3, 4, \dots$$

*further in the latter (exceptional) case,  $k \leq a_k \leq 2k - 1$ . Finally, for all  $k$ ,  $a_{k+1}/a_k = 1$  or  $2$ .*

The James theorem is clearly a great step towards the conjecture that  $A_k = A_k^L$ . The next step, one hopes, will be the elimination of the exceptional cases. (In this direction Adams has quite recently shown that in the exceptional case,  $a_k$  must actually equal  $(2k - 1)$ .)

We sketch the main lines of the proof briefly. Let  $O_{n,k}$  be the Stiefel manifold of  $k$ -frames in  $E_n$ , and let  $\pi: O_{n,k} \rightarrow O_{n,1}$  be the fiber-projection on the first element of this frame. Then an orthogonal  $k$ -system on  $E_n$  is equivalent to a section  $s: O_{n,1} \rightarrow O_{n,k}$  of this fibering. (If  $f_1, \dots, f_k$  is an orthogonal  $k$ -system,  $s$  is defined by  $s(x) = \{x, f_2(x), \dots, f_k(x)\}$ .) Now by the covering homotopy theorem the problem can be formulated entirely in terms of homotopy groups. Indeed,  $\pi: O_{n,k} \rightarrow O_{n,1}$  admits a section if and only if  $\pi_{n-1}(O_{n,k})$  maps onto  $\pi_{n-1}(O_{n,1}) = \mathbb{Z}$  under  $\pi_*$ . (Any element  $\alpha$ , projecting onto the generator can be deformed into a section.) To recapitulate—from this point of view  $A_k$  consists of those integers  $n$  for which

$$\pi_*: \pi_{n-1}(O_{n,k}) \rightarrow \pi_{n-1}(O_{n,1})$$

is surjective.

The first step is now to show that if  $n$  and  $m$  are in  $A_k$ , then  $n + m$  is again in  $A_k$ . In the linear case this is trivial enough—if  $E_n$  and  $E_m$  are  $E_k$  modules, then  $E_n \oplus E_m$  is again an  $E_k$  module. The topological counter part to this argument is given by the join map  $\lambda$  of James, which takes  $O_{n,k} * O_{m,k}$  into  $O_{n+m,k}$ . Here  $*$  denotes the join, and  $\lambda$  is defined by:

If  $x = \{x_i\}$  and  $y = \{y_i\}$ ,  $i = 1, \dots, k$ , are  $k$ -frames in  $E_n$  and  $E_m$  respectively, then  $\lambda(x, t, y)$ ;  $0 \leq t \leq 1$ ; is the frame  $\{x_i \cos \pi t/2 \oplus y_i \sin \pi t/2\}$  in  $E_n \oplus E_m$ .

In the usual manner the map

$$\lambda: O_{n,k} * O_{m,k} \rightarrow O_{n+m,k}$$

defines a pairing

$$\lambda_*: \pi_r(O_{n,k}) \oplus \pi_s(O_{m,k}) \rightarrow \pi_{r+s+1}(O_{n+m,k})$$

and the naturality conditions of  $\lambda_*$  relative to  $\pi_*$  easily yield the fact that  $A_k$  is closed under addition. To get further, one needs at least a partial subtraction law. The basic result in this direction is James's extension of the Freudenthal theorem:

**GENERALIZED FREUDENTHAL THEOREM.** *Suppose that  $n \in A_k$  and let  $s \in \pi_{n-1}(O_{n,k})$  project on a generator of  $\pi_{n-1}(O_{n,1})$ . Then  $s_*: \pi_i(O_{m,k}) \rightarrow \pi_{i+n}(O_{n+m,k})$  defined by:  $s_*(y) = \lambda_*(s \otimes y)$ , is a bijection for  $i \leq 2(m - k + 1)$ .*

Roughly this theorem enables James to conclude that if  $n + m \in A_k$  and  $n$  is small relative to  $n + m$  then  $m$  is also in  $A_k$ . By subtracting the lowest integer in  $A_k$  successively as far as possible he then obtains theorem II.

James prove the generalized Freudenthal theorem by induction on  $k$ . For  $k = 1$ , we have precisely the Freudenthal theorem. The crucial fact here is a "boundary" formula of the type

$$\nabla \lambda_*(a \otimes b) = \lambda_*(\nabla a \otimes b) \pm a \otimes \lambda_* \nabla b$$

where  $\nabla$  is the boundary in the homotopy sequence of fiberings of the type  $O_{n,k} \rightarrow O_{m,1}$ . I will not describe it more precisely. However, this formula is the hardest and its proof the most geometric part of the whole theory.

Theorem II does not include all the presently known information about our vector-field problem. By means of cohomology operations one can, for instance find restrictions on the set  $A_k$ . I will not attempt to do justice to these but, rather, say a word about the parallelizability question which was settled two years ago.

The problem is: *For what  $n$ , does  $S_{n-1}$  admit an  $(n-1)$  field, or put more geometrically, for what  $n$  can a global parallelism be defined on  $S_{n-1}$ ?* In our notation the question is simply: *When does  $A_k$  contain  $k$ ?*

The answer, due independently to Milnor [7] and Kervaire [15] asserts that, just as in the linear case, this phenomenon occurs only if  $k = 1, 2, 4$  and  $8$ .

At present several proofs of this result are known. The most topological proof is obtained by applying the work of Adams [1] on the decomposability of certain primary operations in terms of secondary ones. (This result becomes pertinent in view of the following construction. Corresponding to  $\alpha \in \pi_{n-1}(O_{n,n})$  let  $\xi_\alpha$  be the bundle determined over  $S_n$ , and let  $X_\alpha = S_n \cup_\alpha e_{2n}$  be the complex obtained by forming the 1-point compactification of  $\xi_\alpha$ . Then if  $\alpha$  represents a section of  $\pi: O_{n,n} \rightarrow O_{n,1}$  it is well known that  $Sq^n: H^n(X_\alpha) \rightarrow H^{2n}(X_\alpha)$  is nontrivial. Now by Adams, this can occur only if  $n = 1, 2, 4$  or  $8$ .)

The original solutions of the parallelizability question were based on divisibility properties of the characteristic classes of vector bundles. Quite recently Atiyah and Hirzebruch brought another proof based on this principle, which is possibly the most satisfactory one. The main steps are:

If  $\xi$  is a (real) vector bundle over a complex  $X$  then  $w(\xi)$ —the Stiefel Whitney class—is a well determined element of  $H^*(X; \mathbb{Z}_2)$  which has component 1 in dimension 0. This class is not affected by adding a trivial bundle to  $\xi$ . Now it is not hard to see that  $k \in A_k$  is equivalent to the assertion:  *$S_k$  admits a vector bundle with  $w(\xi) \neq 1$ .* (If  $\alpha \in \pi_{n-1}(O_{n,n})$  is a section, then  $\pi_* \alpha \pmod{2}$  can be identified with the component in dimension  $n$  of  $w(\xi_\alpha)$ — $\xi_\alpha$  being the bundle over  $S_n$  determined by  $\alpha$ .) With this as a starting point, the hard part of the parallelizability question is to show that  $w(\xi) = 1$  whenever  $\xi$  is a vector bundle over a sphere of dimension  $> 8$ . Suppose then that  $S_n = S_{8+m}$ ,  $m \geq 1$ , so that  $S_n = S_m \# S_8$  where  $\#$  denotes the identification space obtained from  $S_m \times S_8$  by collapsing the wedge  $S_m \vee S_8$  in  $S_m \times S_8$  to a point. Now the periodicity theorem for the stable orthogonal group asserts [5/6]:

Let  $X$  be a finite CW complex, and let  $\xi$  be a (real) vector bundle over  $X \# S_8$ . Then there is a bundle,  $\xi/\lambda$ , over  $X$ , and a canonical 8 dimensional bundle  $\lambda$  over  $S_8$ ; so that  $\xi \equiv \xi/\lambda \otimes \lambda$ , where  $\otimes$  denotes the reduced tensor product and the congruence is taken modulo trivial bundles.

Concerning the reduced tensor product of two bundles  $\xi$  and  $\eta$  on  $X$  and  $Y$  one has to recall that  $\xi \otimes \eta$  determines a bundle on  $X \# Y$ , and that  $w(\xi \otimes \eta)$  is determined by  $\xi$  and  $\eta$  according to the law:

Let

$$w(\xi) = \prod_1^n (1+x_i), \quad n = \dim \xi; \quad w(\eta) = \prod_1^m (1+y_j), \quad m = \dim \eta.$$

Then

$$w(\xi \oplus \eta) = \prod_{i,j} (1+x_i+y_j) / \{w(\xi)\}^m \cdot \{w(\eta)\}^n$$

Now, if one takes a bundle  $\xi$  over  $S_{m+8}$ ,  $m > 1$ ; it follows from the periodicity formula that  $w(\xi) = w(\xi/\lambda \otimes \lambda)$ , and a purely algebraic estimate, using the fact that  $w(\lambda) = 1 + u$ , where  $u$  is the generator of  $H^8(S_8; \mathbb{Z}_2)$ , and that  $\lambda$  has dimension 8, shows that  $w(\xi) = 1$  under these conditions.

I would like to take up the "allied" problems next. We have been concerned with the question whether  $\pi: O_{n,k} \rightarrow O_{n,1}$  has a section. Now this problem has an obvious analogue for the other two fields over the real numbers. The spaces  $O_{p,q}$  are perfectly well defined over the complex numbers (unitary  $q$ -frames in Hermitian  $p$ -space) and also over the Quaternions (Symplectic  $q$ -frames in Symplectic  $p$ -space). Hence the question of whether  $O_{n,k} \rightarrow O_{n,1}$  has a section is also meaningful over the complex numbers and the quaternions.

The method of James turns out to be applicable to these cases also. In fact, it yields the following stronger result:

**THEOREM III.** Let  $B_k, [C_k]$  denote the set of integers for which  $O_{n,k} \rightarrow O_{n,1}$  has a section in the complex [quaternion] case. Then there exist positive integers  $b_k [c_k]$  so that

$$B_k = \{n b_k\} \\ C_k = \{n c_k\}.$$

In these two instances, then, there are no exceptional cases. The proof of James follows the earlier pattern. The result is stronger because the extended Freudenthal theorem is applicable in a greater range of dimensions. However, there is this great difference: In these two cases there is no known information about the linear case—in its strongest formulation there are no linear orthogonal  $k$ -systems for  $k > 1$ —in particular it is not apriori clear that  $B_k$  is nonempty for  $k > 1$ . To show that indeed  $O_{n,k} \rightarrow O_{n,1}$  has a section for some  $n$ , James again uses his boundary formula essentially to derive this fact from the finiteness of the stable homotopy groups of the spheres.

There is another path to this theorem. Motivated by certain other results of James, this approach has recently been perfected by M. Atiyah [2]. It proves that  $B_k$  consists of multiples of a certain integers  $b_k$  by essentially identifying  $B_k$  with the kernel of a homomorphism of a cyclic group. I will discuss only the complex case, as the quaternion case is entirely similar.

Several steps are involved. First we reformulate our problem once again. As our concern is now with the behavior of  $\pi_{2n-1}$  under the projection  $O_{n,k} \rightarrow O_{n,1}$  we may replace  $O_{n,k}$  by its  $2n$ -skeleton, which can be constructed in a very simple manner when  $n/k$  is large. Let  $P_n$  denote the projective space of the one-dimensional subspaces of complex  $n$ -space  $C_n$ . (Thus  $P_n$  is the projective space of complex dimension  $n - 1$ .) Next, define  $P_{n,k}$  to be the identification space  $P_n/P_{n-k}$ , the inclusion  $P_{n-k} \subset P_n$  being induced by the inclusion  $C_{n-k} \subset C_n$ . We clearly have a natural projection  $\pi': P_{n,k} \rightarrow P_{n,1}$ , and it can be shown that the suspension of  $\pi'$ , that is  $E \circ \pi': EP_{n,k} \rightarrow EP_{n,1} = S_{2n-1}$ , represents the projection  $O_{n,k} \rightarrow O_{n,1}$  in the pertinent dimension—at least if  $n/k$  is large. Using cohomology operations to eliminate low ratios of this integer, one concludes, that  $O_{n,k} \rightarrow O_{n,1}$  has a section, if and only if  $(E' \circ \pi)_*$  is surjective in dimension  $2n - 1$ . Finally, again using the apriori estimate on  $n/k$  one can redefine  $B_k$  in terms of purely “stable” notions.

*The integer  $n \in B_k$  if and only if some suspension of the map  $\pi': P_{n,k} \rightarrow S_{2n-1}$ , admits a right homotopy inverse.*

Precisely then, the condition is that there should exist an  $m$ , and a map  $f: E^m S_{2n-1} \rightarrow E^m P_{n,k}$  so that  $E^m \pi' \circ f$  be homotopic

to 1. Colloquially one may also put it this way: "The top cycle of  $P_{n,k}$  should become spherical after a suitable number of suspensions." James calls this condition  $S$ -reducibility.

To proceed further we need the notion of the generalized  $J$ -homomorphism, and of the twisted suspension.

Let  $X$  be a finite  $CW$  complex. We write  $\widetilde{KO}(X)$  for the suspension classes of real vector bundles over  $X$  (see [3, 9]). Thus two bundles  $\xi$  and  $\eta$  determine the same element,  $[\xi] = [\eta]$ , in  $\widetilde{KO}(X)$  if after suitable trivial bundles are added to both they become isomorphic. Next define  $J(X)$  as the set of equivalence classes of vector-bundles over  $X$ , in which two bundles are considered equal if after suitable trivial bundles are added to them, their unit sphere-bundles are of the same fiber-homotopy type. Finally  $J$  shall denote the projection  $\widetilde{KO}(X) \rightarrow J(X)$ . The Whitney sum now defines a group structure in both these sets and makes  $J$  into a homomorphism. So interpreted  $J$  is the generalized  $J$ -homomorphism. A first observation is now,

PROPOSITION 1.  $J(X)$  is a finite group.

The proof follows more or less directly from finiteness of the stable homotopy of the spheres and the definition of fiber homotopy type.

Finally the twisted suspension of  $X$ , by a vector bundle  $\xi$  (over  $X$ ) is defined as the one-point compactification of  $\xi$ , and will be denoted by  $X^\xi$ . The terminology is justified by this formula:

$$X^{(\xi+1)} = E \cdot X^\xi$$

where 1 stands for the trivial bundle and  $E$  denotes suspension as before. One also needs the convention that when  $\xi$  has dimension 0, then  $X^\xi$  is to be the disjoint union of  $X$  and a point.

This construction is pertinent to our discussion for the following reason: Let  $P_k$  be the orthogonal projective space to  $P_{n-k}$  in  $P_n$ , and let  $\tau$  denote its normal bundle. Then it is geometrically clear that  $\tau$  can be identified with  $P_n - P_{n-k}$ . Hence  $P_{n,k} = P_k^\tau$ . The bundle  $\tau$  splits into the direct sum of (complex) line-bundles as is also evident because  $P_k$  is a complete intersection in  $P_n$ .

Thus, if  $\xi$  is the normal bundle of in  $P_k P_{k+1}$  then  $P_{n,k}$  is given by:

$$P_{n,k} = P_k^{(n-k)\xi}$$

To return to the general situation—First there is the following rather easy relation between  $J$  and the twisted suspension:

PROPOSITION 2. *Let  $\xi$  be a vector bundle over  $X$ . Then  $X^\xi$  is of the same  $S$ -type as  $X$  if and only if  $J[\xi] = 0$ .*

(Here, as usual, two spaces are of the same  $S$ -type if suitable suspensions of them are of the same homotopy type.)

Suppose now that  $M$  is a compact differentiable manifold. If  $\xi$  is a vector-bundle over  $M$ , the  $S$ -reducibility of  $M^\xi$  is defined exactly as it was for  $P_{n,k}$ —the top cycle of  $M^\xi$  has to be stably spherical. Let  $\nu$  be the normal bundle of  $M$  imbedded in some high dimensional sphere  $S_N$ . By collapsing the exterior of a tubular neighborhood of  $M$  in  $S_N$  we obtain a map  $S_N \rightarrow M^\nu$  which clearly establishes  $M^\nu$  as  $S$ -reducible. This argument makes the following proposition plausible:

*$M^\xi$  is  $S$ -reducible if and only if  $J([\xi] - [\nu]) = 0$ .*

By replacing  $M$  with  $P_k$  this last formula now immediately yields the theorem of James. Indeed, in this case  $[\nu] = -k[\xi]$  as is well known. Thus  $P_{n,k} = P_k^{(n-k)\xi}$  is  $S$ -reducible if and only if  $J(n[\xi]) = 0$ . Because  $J(P_k)$  is finite, and  $J$  is a homomorphism the theorem follows.

The last formula is really a special case of the following *duality theorem* of Atiyah, which was also independently proved by A. Shapiro and the Author.

DUALITY THEOREM. *Let  $X$  be a differentiable manifold, and let  $\xi$  and  $\xi'$  be two vector bundles over  $X$  so that  $[\xi'] = -([\xi] + [\tau])$ , where  $\tau$  is the tangent bundle of  $X$ . Then the  $S$ -types of  $X^\xi$  and  $X^{\xi'}$  are dual to each other in the sense of Spanier Whitehead [17]:*

$$D[X^\xi] = [X^{-\xi-\tau}].$$

A remark is now in order as to why the real case cannot be treated in this manner. Actually one can proceed quite similarly at first. In the real case  $O_{n,k}$  is approximated by the real

analogue of  $P_{n,k}$ , rather than by  $EP_{n,k}$  as it was in the present situation, however this is no serious drawback. The exceptional case can occur precisely because one is not always able to eliminate low ratios of  $n/k$  a priori, so that the  $S$ -reducibility of  $P_{n,k}$  is not necessarily an equivalent problem to the existence of a section to the fibering  $O_{n,k} \rightarrow O_{n,1}$ . The  $S$ -reducibility of the real  $P_{n,k}$  of course has the same sort of answer as in the complex case.

In conclusion let me report on estimates which Atiyah and Todd obtained for the  $b_k$  of theorem II [3]. Let  $\lambda_p(N)$  denote the power to which the prime  $p$  occurs in the integer  $N$ . Now let integers  $M_k$  be defined by the formula:

$$\lambda_p(M_k) = \begin{cases} \sup (r + \lambda_p(r)), & 1 \leq r \leq \left[ \frac{k-1}{p-1} \right], \quad \text{if } p \leq k \\ 0 & \text{if } p > k. \end{cases}$$

**THEOREM III.** *The integers  $b_k$  of theorem II (for the complex case) are divisible by  $M_k$ .*

The principle on which this estimate is based is the following one. As we have seen, our question is really: For what values of  $n$  is the top cycle of  $P_{n,k}$  "stably spherical". That is, when does this homology class become spherical after a suitable number of suspensions.

In short, we need criteria for stably spherical homology classes. The following simple procedure clearly yields such criteria. Suppose  $B$  is a space in which the stably spherical classes are already known. Then if  $u \in H_i(X)$  is a homology class in  $X$ , it will be stably spherical only if for every map  $f: X \rightarrow B$ ,  $f_* u$  is stably spherical in  $H_*(B)$ . Such a criterion is of course only effectively applicable if we know how to compute (1) the set of homotopy classes of maps of  $X$  into  $B$  and (2) the homomorphisms these classes induce in cohomology.

The best known application occurs when  $B$  is an Eilenberg-MacLane space. Here there are no stable spherical classes except the lowest dimensional ones—(in the stable range). In this way one obtains the criterion that  $u$  is stably spherical only if the value of any stable primary cohomology operation on a lower

dimensional class, vanishes on  $u$ . (This criterion can be applied to our problem, however it yields considerably weaker results than those given by theorem III.)

The results of Atiyah Todd, are in fact obtained by using for their testing space  $B$ , the universal base space,  $B_U$ , of the infinite unitary group. As a consequence of the periodicity [5]:  $\Omega^2(Z \times B_U) = Z \times B_U$  one can determine the spherical classes in  $B_U$  rather easily, and for stable spherical classes one can derive this criterion: *There is a rational cohomology class  $ch$  (with components in all dimensions) in  $H^*(B_U; Q)$ , such that if  $u$  is a stably spherical class in  $H_k(B_U)$  then  $ch(u)$  must be an integer.*

Thus, if we write  $\widetilde{KU}(X)$  for the homotopy classes of maps of  $X$  into  $B_U$ , and for  $\xi \in \widetilde{KU}(X)$  write  $ch(\xi) = \xi_* \cdot ch$ , then we have the criterion:

$u \in H_k(X)$  is stably spherical only if for each  $\xi \in \widetilde{KU}(X)$ ,  $ch(\xi) \cdot u$  is an integer.

It is this criterion which yields the Atiyah Todd theorem modulo some rather delicate number theory.

How does one carry out the steps (1) and (2) of our program in this case? Here it is only fair to admit, that the space  $B_U$  is not an ad-hoc testing space, but rather that it is more or less God given. Indeed, by virtue of the classifying theorems for bundles,  $\widetilde{KU}(X)$  can be interpreted geometrically as the suspension classes of complex vector bundles over  $X$ . Further, if  $\xi \in \widetilde{KU}(X)$  then the element  $ch(\xi)$  in  $H^*(X; Q)$  is a particular characteristic class of  $\xi$  about which a lot is known. For instance if  $\xi$  is the normal bundle of  $P_n$  in  $P_{n+1}$ , then  $ch(\xi) = e^x - 1$  where  $x \in H^2(P_n; Z)$  is a generator. Now, using the known functorial properties of  $ch$ , it follows that  $ch(\xi^k) = (e^x - 1)^k$ , where  $\xi^k$  is  $\xi \otimes \dots \otimes \xi$  ( $k$  times) in the sense of the reduced tensorproduct. Thus, if we restrict  $\xi^m$  with  $m \geq n - k$  to  $P_{n-k}$  we get an element of  $\widetilde{KU}(P_{n-k})$  with vanishing character.

It is a theorem that if  $X$  is a torsion free finite  $CW$  complex then  $ch: \widetilde{KU}(X) \rightarrow H^*(X; Q)$  is injective. This was first proved by F. Peterson—directly by obstruction theory from the evaluation of  $\pi_i(Z \times B_U)$  as  $Z$  if  $i$  is even and 0 when  $i$  is odd. A

proof which possibly goes more to the heart of the matter emerges from the point of view of Atiyah and Hirzebruch. They define the groups:

$$KU^i(X) = \pi[E^{-i}X; Z \times B_U] \quad i \leq 0$$

where  $\pi[A, B]$  denotes homotopy classes of maps. In this terminology the periodicity formula:  $\Omega^2(Z \times B_U) = Z \times B_U$  is expressed by:

$$KU^i(X) = KU^{i+2}(X) \quad i \leq -2.$$

Now Atiyah and Hirzebruch use this recurrence to define  $KU^i(X)$  for all integers  $i$ , and then show that the resulting functor  $X \rightarrow \{KU^i(X)\}$  satisfies all the axioms of a cohomology theory—except for the dimension axiom. Further they observe that the uniqueness theorem of Eilenberg-Steenrod can be generalized to yield a spectral sequence relating  $E_2 = H^*(X; KU^*(p))$  to  $KU^*(X)$ . (Here  $KU^i(p)$ —the  $KU$ —theory of a point—is  $Z$  if  $i$  is even and 0 otherwise.) This sequence immediately implies the proposition. (See [8].)

To return to our bundles  $\xi^m$  on  $P_n$ . By the proposition just discussed the restriction of  $\xi^m$  to  $P_{n-k}$  will be trivial if  $m \geq n - k$ . By trivializing this element on  $P_{n-k}$  we obtain bundles  $\xi^m$  on  $P_{n,k}$  which under the projection  $\pi: P_n \rightarrow P_{n,k}$  go over into  $\xi^m$ ,  $m \geq n - k$ . In particular,  $\pi^* ch(\xi^m) = (e^x - 1)^m$ . Thus in any case we obtain these criteria the  $S$ -reducibility of  $P_{n,k}$ :

$P_{n,k}$  is  $S$ -reducible only if the coefficient of  $x^{n-1}$  in  $(e^x - 1)^m$ ,  $m \geq n - k$ , is an integer.

This is the number theoretical condition from which Atiyah and Todd deduce theorem III. Their result is the best possible one obtainable from the test-space  $B_U$ , because one can show quite easily, with the spectral sequence alluded to earlier, that the elements  $1, \xi^m; n - 1 \geq m \geq n - k$ ; form a base of  $KU(P_{n,k})$ .

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