SPECIAL CASE OF KUMMER'S CONGRUENCE

A SPECIAL CASE OF KUMMER'S CONGRUENCE (mod 2^e)

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Put

$$\frac{x}{\sinh x} = \sum_{1}^{\infty} D_{2n} \frac{x^{2n}}{(2n)!}, \quad D_{2n+1} = 0, \tag{1}$$

where [3, p. 27-28]

$$(D+1)^n - (D-1)^n = \begin{cases} 2 & (n=1) \\ 0 & (n<1); \end{cases}$$

also

$$D^n = (2B+1)^n = (2-2^n) B_n \tag{2}$$

and B_n is defined by

$$\frac{x}{e^x - 1} = \sum_{0}^{\infty} B_n \frac{x^n}{n!}.$$

The first few values of D_{2n} are

$$D_0 = 1,\ D_2 = -\frac{1}{3}, \quad D_4 = \frac{7}{15}, \quad D_6 = \frac{31}{21}, \quad D_8 = \frac{127}{15}.$$

If we define

$$\Delta_n = (D^2 - 1)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} D_{2r} \tag{3}$$

we find that

$$\Delta_0 = 1, \quad \Delta_1 = \frac{2^2}{3}, \quad \Delta_2 = \frac{2^5}{15}, \quad \Delta_3 = \frac{2^9}{105}, \quad \Delta_4 = \frac{2^{11}}{105}.$$

(Compare [1, p. 821].)

By the second part of (2), both numerator and denominator of D_{2n} (in reduced form) are odd. Moreover by the Staudt-Clausen theorem for the Bernoulli numbers B_n, we have

$$pD_{2n} \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid 2n) \\ 0 \pmod{p} & (p-1 \nmid 2n) \end{cases},$$

where p is any odd prime. Thus the denominator of Δ_n can be determined (see below). The numerators of Δ_n for $0 \geqslant n \geqslant 4$ are all powers of 2. However this is really too good to be true; indeed we find that

$$\Delta_5 = -\frac{2^{15}}{3.7.11}, \quad \Delta_6 = \frac{2^{17}.191}{3.5.7.11.13}.$$

It may be of interest to determine the highest power of 2 dividing Δ_n. A result of this sort is analogous to Kummer's congruences for the Bernoulli or Euler numbers [2, Chapter 14]; however the standard results on Kummer's congruences are restricted to odd moduli.

It is convenient to first transform Δ_n. We have, using the symbolic notation,

$$\Delta_n = (D^2 - 1)^n = ((2B+1)^2 - 1)^n = (4B^2 + 4B)^n = 4^n (B^2 + B)^n$$

and therefore

$$\Delta_n = 2^{2n} \sum_{r=0}^{n} \binom{n}{r} B_{n+r} \cdot \tag{4}$$

Since $2B_{2r} \equiv 1 \pmod 2$ it follows readily that the denominator of

$$\sum_{r+0}^{n} \binom{n}{r} B_{n+r}$$

(in reduced form) is odd. Consequently

$$\Delta_n \equiv 0 \pmod{2^{2n}}. \tag{5}$$

Incidentally (4) is a bit more convenient for computation than (2). For example

$$\Delta_5 = 2^{10} (5B_6 + 10B_8 + B_{10}) = -\frac{2^{15}}{3.7.11},$$

$$\Delta_6 = 2^{12} (B_6 + 15B_8 + 15B_{10} + B_{12})$$

$$= 2^{12}\left(\frac{1}{42} - \frac{15}{30} + \frac{75}{66} - \frac{691}{2730}\right) = \frac{2^{17}.191}{3.5.7.11.13}.$$

The congruence (5) can be improved. We make use of the explicit formula

$$B_n = \sum_{r=0}^{n} \frac{1}{r+1} \sum_{s=0}^{r} (-1)^s \binom{r}{s} s^n,$$

which yields

$$(\mathrm{B}^2 + \mathrm{B})^n = 2^n \sum_{r=0}^{2n} \frac{1}{r+1} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{1}{2} s (s+1)\right)^n . \quad (6)$$

However allowance must be made for the denominator $r + 1$. Clearly the highest power of 2 contained in $r + 1$, where $0 \leqslant r \leqslant 2n$, is at most 2^e, where e is determined by

$$2^e \leqslant 2n < 2^{e+1} . \quad (7)$$

Hence by (4) and (6),

$$\Delta_n \equiv 0 \pmod{2^{3n-e}} . \quad (8)$$

This result also can be improved for $n \geqslant 2$. Suppose first that n is even and let

$$2^k \mid n, 2^{k+1} \nmid n \qquad (k \geqslant 1) . \quad (9)$$

Then for arbitrary odd u we have

$$u^n \equiv 1 \pmod{2^{k+2}} . \quad (10)$$

Put

$$\mathrm{S}_j = \mathrm{S}_j^{(r)} = \sum_{s \equiv j \pmod 4} \binom{r}{s} \qquad (0 \leqslant j \leqslant 3) .$$

It is easily verified that

$$2(\mathrm{S}_1 - \mathrm{S}_2) = (1+i)^{r-1} + (1-i)^{r-1} .$$

Hence for r odd

$$\mathrm{S}_1 - \mathrm{S}_2 = \begin{cases} 0 & (r \equiv 3 \pmod 4)) \\ (-1)^{1/4(r-1)}\, 2^{1/2(r-1)} & (r \equiv 1 \pmod 4)) \end{cases} .$$

Since $s(s+1)/2$ is odd if and only if $s \equiv 1$ or $2 \pmod 4$, it follows from (10) that $(n > 2)$

$$\mathrm{S} = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{1}{2} s (s+1)\right)^n \equiv \mathrm{S}_2 - \mathrm{S}_1 \pmod{2^{k+2}} ,$$

and therefore

$$\mathrm{S} \equiv 0 \pmod{2^{k+2}} \quad (r \equiv 3 \pmod 4)) , \quad (11)$$

$$\mathrm{S} \equiv 0 \pmod{(2^{k+2}, 2^{\frac{1}{2}(r-1)})} \quad (r \equiv 1 \pmod 4)) . \quad (12)$$

Now when $r \equiv 1 \pmod 4$, $r + 1$ is divisible by 2 but not by 2^2; hence the term $\mathrm{S}/(r+1)$ is integral (mod 2) provided

$r > 1$. When $r \equiv 3 \pmod 4$, let 2^{e_r} denote the highest power of 2 dividing $r+1$. Then it follows from (11) that $S/(r+1)$ is of the form $2^{k+2-e_r}A$, where A is integral (mod 2). The least favorable case arises when e_r has its maximum value e as determined by (7). We have therefore

$$\Delta_n \equiv 0 \pmod{2^{3n-e+k+2}} \tag{13}$$

provided $k+2 > e$; when $k+2 \leqslant e$ we have

$$\Delta_n \equiv 0 \pmod{2^{3n-1}} \tag{14}$$

since for example the term $r = 1$ has denominator 2.

In the next place for n odd, let

$$2^k \mid n-1, \quad 2^{k+1} \nmid n-1 \qquad (k \geqslant 1);$$

Then (10) is replaced by

$$u^{n-1} \equiv 1 \pmod{2^{k+2}},$$

where u is odd. Put

$$\begin{aligned} T_j = T_j^{(r)} &= \frac{1}{2} \sum_{s \equiv j \pmod 4} \binom{r}{s} s(s+1) \\ &= \binom{r}{2} S_{j-2}^{(r-2)} + r S_{j-1}^{(r-1)}. \end{aligned}$$

Thus

$$\begin{aligned} T_1 - T_2 &= \binom{r}{2}\left(S_3^{(r-2)} - S_0^{(r-2)}\right) + r\left(S_0^{(r-1)} - S_0^{(r-1)}\right) \\ &= \frac{i}{4}\binom{r}{2}\left\{(1+i)^{r-1} - (1-i)^{r-1}\right\} + \frac{r}{4}\left\{(1+i)^r + (1-i)^r\right\}. \end{aligned}$$

Simplifying we get (for $r \geqslant 3$)

$$T_1 - T_2 = \begin{cases} \frac{1}{4} r(r+1)\, 2^{\frac{1}{2}(r-1)}\, i^{\frac{1}{2}(r+1)} & (r \equiv 3 \pmod 4)) \\ \frac{1}{2} r\, (2i)^{\frac{1}{2}(r-1)} & (r \equiv 1 \pmod 4)). \end{cases}$$

Also $(n > 3)$

$$T = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(\frac{1}{2} s(s+1)\right)^n \equiv T_2 - T_1 \pmod{2^{k+2}}$$

and therefore in particular

$$T \equiv 0 \pmod{(2^{\frac{1}{2}(r-1)}, 2^{k+2})} \quad (r \equiv 3 \pmod 4)). \tag{15}$$

As in the case n even, it will suffice to take $r \equiv 3 \pmod 4$. Let 2^{e_r} denote the highest power of 2 dividing $r + 1$, and consider $2^{-e_r}T$. For $e_r \leqslant 2$, we get the exponent -1; for $e_r \geqslant 3$ it follows that $\frac{1}{2}(r - 1) \geqslant e_r$. Hence if $k + 2 < e_r$, we get the exponent $k + 2 - e_r$, while if $k + 2 \geqslant e_r$ we again get -1 (at most). Consequently

$$\Delta_n \equiv 0 \pmod{2^{3n-e+k+2}} \quad (k + 2 < e_r) , \tag{16}$$

$$\Delta_n \equiv 0 \pmod{2^{3n-1}} \quad (k + 2 \geqslant e_r) . \tag{17}$$

Comparing (16) and (17) with (13) and (14) we may accordingly state the following

THEOREM. *Let* $2^e \leqslant 2n < 2^{e+1}$ *and let* 2^k *denote the highest power of 2 dividing* n *or* $n - 1$ *according as* n *is even or odd. Then* Δ_n *as defined by (1) and (3) satisfies (16) and (17).*

Applying the Staudt-Clausen theorem, it follows from (3) that

$$p\Delta_n \equiv -\sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \pmod p ,$$

where $p = 2m + 1$ is an odd prime. It can be shown that

$$\sum_{r>0} (-1)^{n-rm} \binom{n}{rm} \equiv \begin{cases} 2 & (p - 1 \mid n) \\ \binom{m}{k} & (p - 1 \mid n + k,\ 0 < k \leqslant m) \\ 0 & \text{otherwise.} \end{cases}$$

In particular the prime factors of the denominator of Δ_n are simple and cannot exceed $2n + 1$.

In connection with formula (4) above it may be of interest to cite the formula [2, p. 189]

$$(-1)^n \sum_{r=0}^{n} \binom{n}{r} \frac{B_{n+r+1}}{n + r + 1} + \frac{n!\,n!}{2\,(2n + 1)!} = 0 .$$

REFERENCES

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