

3. Recurrences.

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powers of Ψ are lowered into subscripts; thus $\Psi^{(1)}$ would then be written $\Psi_1(t)$.

If (18) and (19) are substituted in (13), there results the following umbral identity:

$$(20) \quad \Psi (1 - \cos \Psi_s) + 2 \frac{\partial}{\partial t} \left(\frac{1 - \cos \Psi_s}{\Psi} \right) = 2 \sin \Psi_s * \sin \Psi_s ,$$

where the asterisk (*) indicates umbral multiplication.

For the cases $r = 2, 3$, rather extensive calculations show that umbral identities of the same form exist. We may therefore state the following result which is implied by the non-linear equation (13).

Theorem 1: "Let Ψ , X , Φ be respectively the umbrae of the sequences $\{\Psi_{2k-1}(t)\}$, $\{X_{2k-1}(t)\}$, and $\{\Phi_{2k-1}(t)\}$. If γ is one of these umbrae, then the following umbral identity holds:

$$(21) \quad \gamma (1 - \cos \gamma_s) + 2 \frac{\partial}{\partial t} \left(\frac{1 - \cos \gamma_s}{\gamma} \right) = 2 \sin \gamma_s * \sin \gamma_s .$$

3. RECURRENCES.

It is clear that (20) implies a recurrence relation for the functions $\Psi_j(t)$, and indeed, Theorem 1 yields the following.

Theorem 2: "Let $\gamma_j(t)$ be $\Psi_j(t)$, $X_j(t)$ or $\Phi_j(t)$; then the following recurrence holds:

$$(22) \quad \frac{d}{dt} \gamma_{2n-1}(t) + \frac{1}{2} \gamma_{2n+1}(t) = \sum_{k=0}^{n-1} \binom{2n}{2k+1} \gamma_{2k+1}(t) \gamma_{2n-2k-1}(t) ,$$

and hence $\gamma_{2n+1}(t)$ is a polynomial in $\gamma_1(t)$ and its derivatives up to order n ."

This result, in turn, implies the following

Theorem 3: "Let $\rho_{2k-1}(n)$ be either of the arithmetical functions $\beta_{2k-1}(n)$ or $\xi_{2k-1}(n)$ defined by (8) and (10) respectively; then $\rho_{2k-1}(n)$ satisfies a recurrence relation of the form:

$$(23) \quad \rho_{2k+1}(n) - n \rho_{2k-1}(n) = 2 \sum_{s=0}^{k-1} \sum_{j=1}^{n-1} \binom{2k}{2s+1} \rho_{2s+1}(j) \rho_{2k-2s-1}(n-j) ,$$

for all n and $k \geq 1$. Moreover, the arithmetical function $\zeta_{2k-1}(n)$ defined by (9) satisfies the recurrence

$$(24) \quad \zeta_{2k+1}(n) - 2n \zeta_{2k-1}(n) = 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left\{ U_{k-s} \zeta_{2s+1}(n) + \right. \\ \left. + U_{s+1} \zeta_{2k-2s-1}(n) + \sum_{j=1}^{n-1} \zeta_{2s+1}(j) \zeta_{2k-2s-1}(n-j) \right\}$$

where U_k is defined by (11) and $n, k \geq 1$.

Incidentally, the comparison of coefficients which yields (24) also gives:

$$(25) \quad U_{n+1} = 2 \sum_{k=0}^{n-1} \binom{2n}{2k+1} U_{k+1} U_{n-k}, \quad n \geq 1,$$

which is equivalent to a result given by NIELSEN [7].

Finally, the case $r = 1$, has been discussed by VAN DER POL [1] who finds an expression analogous to (22) as follows:

$$(26) \quad \frac{d}{dt} h_{2n-1}(t) + \frac{2n+3}{4n+2} h_{2n+1}(t) = \sum_{k=0}^{n-1} \binom{2n}{2k+1} h_{2k+1}(t) h_{2n-2k-1}(t), \quad n \geq 1,$$

where,

$$(27) \quad h_{2n-1}(t) = \frac{(-1)^n B_n}{4n} \alpha_{2n-1}(t),$$

$\alpha_{2n-1}(t)$ being defined by (2).

We find the analogue of (24) for this case to be:

$$(28) \quad \frac{2k+3}{2k+1} \sigma_{2k+1}(n) - 2n \sigma_{2k-1}(n) = 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left\{ V_{k-s} \sigma_{2s+1}(n) + \right. \\ \left. + V_{s+1} \sigma_{2k-2s-1}(n) + \sum_{j=1}^{n-1} \sigma_{2s+1}(j) \sigma_{2k-2s-1}(n-j) \right\}, \quad n \geq 1.$$

Corresponding to (25), we find

$$(29) \quad V_{n+1} = \frac{4n+2}{2n+3} \sum_{k=0}^{n-1} \binom{2n}{2k+1} V_{k+1} V_{n-k},$$

which is equivalent to a known recurrence for the BERNOULLI numbers [8].

4. THE FUNCTIONS $\Psi_{2k-1}(t)$, $X_{2k-1}(t)$, $\Phi_{2k-1}(t)$ AS DOUBLE SUMS.

The results which are stated as (4), (5), (6) follow readily from (1) and (2) which are known to be equivalent (see [1], [2]). It is to be observed first that a comparison of (4) and (5) with (1) taking into account (27) gives the relations:

$$(30) \quad \Psi_{2k-1}(t) = h_{2k-1}(t/2) - h_{2k-1}(t) = V_k(\alpha_{2k-1}(t/2) - \alpha_{2k-1}(t)),$$

$$(31) \quad X_{2k-1}(t) = 2^{2k} h_{2k-1}(2t) - h_{2k-1}(t) = V_k(2^{2k} \alpha_{2k-1}(2t) - \alpha_{2k-1}(t)).$$

From (4) and (6) we also have,

$$(32) \quad \Phi_{2k-1}(t) = 2^{2k} \Psi_{2k-1}(2t) - \Psi_{2k-1}(t).$$

By (30), we may write

$$(33) \quad \Phi_{2k-1}(t) = -V_k(\alpha_{2k-1}(t/2) - (2^{2k} + 1)\alpha_{2k-1}(t) + 2^{2k}\alpha_{2k-1}(2t)).$$

Thus, our functions (4), (5), (6) are expressed in terms of $\alpha_{2k-1}(u)$. These relations in conjunction with (1) and (2) identify them with $(4)_1$, $(5)_1$, and $(6)_1$ respectively.

It is of interest to note that (31) with $k = 2$ permits, with the aid of a result of VAN DER POL [1], the deduction of Jacobi's famous theorem on the number of representations $r_8(n)$ of the integer n as the sum of eight squares. Thus,

$$(34) \quad 240 X_3(t) = 16 \alpha_3(2t) - \alpha_3(t) = 15 \theta_0^8(0, q)$$

where $q = \exp(-t)$. Hence,

$$\theta_0^8(0, q) = 16 X_3(t) = 1 + 16 \sum_{n=1}^{\infty} q^n \zeta_3(n),$$

and

$$\theta_3^8(0, q) = 1 + 16 \sum_{n=1}^{\infty} (-1)^n q^n \zeta_3(n).$$