I. – THE THEORETIC SIDE

THE TRAINING OF TEACHERS OF MATHEMATICS IN THE UNITED STATES OF AMERICA

BY

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When an American writes on the topic standing above, it would seem natural that he give an account of what is currently done in such preparation in the United States. But at the present moment, it seems superfluous to present such an account to the readers of *L'Enseignement Mathématique*. The field has already been well covered in the various reports from the United States submitted to the International Commission on the Teaching of Mathematics, and presented to the Congress at Cambridge in 1912. The readers of *L'Enseignement Mathématique* have also very recently (nos 5-6, 1915), found in its pages an excellent account by Professor G. A. MILLER, of current facilities and tendencies in the United States along this line. Still further, a summary and restatement of the whole matter will be included in the reply which is to be made to the Questionnaire [1] on this subject sent out by the Central Committee some time since.

With the past, the present, and the future thus well provided for in the matter of accounts of what is being done, it would seem unnecessary to offer the readers of *L'Enseignement Mathématique* an additional account at the present juncture; an account which could at most restate in different words the facts included in the different accounts mentioned above.

If, on the other hand, a writer from America were to discuss the needs at the present day in the matter of the theoretic train-

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[1] *L'Ens. Mathém.*, no 2, 1915.

ing of teachers of mathematics in his country and were to consider plans for supposed improvement and progress, what he could say would be of limited international interest since the conditions and the problems in the United States differ so widely from those in the European nations.

Does there then remain anything of general interest on which I can write in response to the Editor's kind invitation?

There does remain in fact, one matter in which the conditions in the leading European nations (so far as I know them) seem to be sufficiently like those in the United States to warrant the belief that in its broad lines the problem is one that is common to all these countries; and that is, the question of the relation of the needs of teachers of mathematics, prospective or in service, to the work in mathematics done in Universities and other institutions of like grade. (In what follows, I shall allow myself to use the single term « Universities » as a convenient, compact designation for all institutions of University grade, whatever their technical appellation. The work in mathematics done in the last two years of the best American Colleges, and in fact some of the work of the earlier years, parallels work that in Europe is usually done in Universities; the term « University » will accordingly be used as including this work, irrespective of the technical designation of the institution or division of an institution giving it.)

It is held everywhere that the teacher's knowledge of mathematical theory should extend far beyond the confines of what he teaches. The range of knowledge to be attained as a minimum requirement is fixed by legislative enactment in some countries; in others, it is held up as a desirable standard by recommendations from more or less authoritative sources.

It is not necessary to enter here into a discussion of what specific subjects are indispensable in the adequate preparation of the future teacher of mathematics, what ones are highly desirable and what ones are less essential. The actual status of usage in this respect throughout the mathematical world is sufficiently described in the reports mentioned above and there appears to be less need for discussing whether or not this or that special subject shall be included among those that the prospective teacher *must* know, than there is for asking whether or not the preparation which he actually receives in those subjects that are currently required is as thorough and as well adapted to his future needs as teacher of mathematics as it is feasible to make it.

The great bulk of the students who take those University courses beyond the elements of the Calculus that cover the field of mathematical theory which either custom or definite prescription require for appointment as teacher, will later actually teach mathematics in the pre-University field. Whatever may be the

aims of the instructor giving the course, for the students the work of these courses constitutes *de facto* their professional training in mathematical theory for their future life work, and the majority of those attending would in all probability not do so if the possibility of teaching mathematics later on were excluded from their plans. So that, from the quantitative standpoint at least, there would seem to be a certain warrant for regarding preparation for teaching as standing in the very forefront of the reasons for the existence of such courses. It is accordingly quite in place to ask what is the way in which these courses can be conducted so as to furnish the best preparation for teaching. Is it by presenting an exposition that shall be as well rounded as possible from the point of view of the subject itself; is it by throwing the light of the higher mathematics as strongly as possible upon the elementary field; or is it by a mode of treatment that shall stimulate the hearer as strongly as possible to independent research in mathematics and be as helpful to him as possible in its prosecution?

Undoubtedly the University is the normal gathering place of men who are engaged in pure research and who communicate the results of this research to the world through the written and the spoken word without concern as to the uses to which the hearer may put the knowledge. It is certainly desirable that the future teacher of mathematics should have a feeling for the subject as a self-sufficient and growing entity and that he should have a sympathetic knowledge of the methods of independent research by which the confines of the subject have been and are being widened. He can obtain these things in no better way than by coming into touch with men who are devoting their lives to the investigation of unsolved problems of mathematics and by working for a time in an atmosphere in which mathematics is studied, taught and acquired solely for its own sake.

It is not so evident, however, that the future teacher of mathematics will secure the best preparation in mathematical theory if, throughout his period of preparation, his thoughts and those of his instructors are focussed entirely upon his own *acquisition* of knowledge, by learning or by discovery, to the exclusion of serious consideration of his future activities in *imparting* knowledge. On the contrary, it seems reasonable to believe that he will control his knowledge of advanced mathematics more effectively for the work of teaching if during its acquisition there is frequent explicit consideration of its bearing on the elementary mathematics which he is later to impart.

Granting that it is desirable that the University preparation in mathematical theory should pay some attention to the subject matter which the hearer is later to teach, the question next arises

as to how this may best be done. The problem may be attacked in various ways, and the ideal solution would no doubt take account of all of them.

First, it may lead the instructor to consider from time to time the relation of the topic in hand to the elementary field. This may sometimes take the form of a discussion of the mode of presenting the related portion of elementary mathematics; at others, of a discussion of the scope of that work and of what should be included and what omitted; at still others, of a warning that the advanced material is quite unsuited to be taken up at all in the elementary work.

Second, it may modify the fundamental plan or the structure of the course, inclusions and omissions being made with the avowed purpose of adapting the instruction more closely to the needs of the future teacher.

Third, it may take the form of entire courses specially designed for the future teacher. Such courses may be so-called « survey » or « encyclopedic » courses, giving a broad account of the scope and fundamental material of a subject without the obligation to give that measure of completeness and fulness of detail which the needs of the advanced student of mathematics would require. On the other hand, they may be courses taking up certain topics and phases in more detail than the non-teacher would need. Or still further, they may be courses in the history or the philosophy or the psychology of mathematics and the mathematical processes, specially destined to give the future teacher of mathematics a more comprehensive view of his subject from all these angles.

Fourth, it may take the form of looking to the improvement of the mathematical equipment of the teachers already in service. Obviously such courses must be given when teachers in service can attend, but there is no need apparent for differentiating them otherwise from courses intended for prospective teachers.

In a communication like the present, all that can be done is to point out how desirable it is that in University instruction some measure of attention be given to the needs of students as prospective teachers of mathematics. It is difficult to specify the nature and extent of such attention and certainly quite impossible to lay down any hard and fast rules concerning its details. After those who believe in such attention have, in whatever ways may be open to them, brought their views before University instructors and those who are influential in shaping the policy of the Universities as to the character of the courses offered, they have, in the United States at least, done all that is in their power. What is actually done depends in the main upon the views and temperament of the individual University instructors. And I wish to repeat here that it seems to me just as important that the student should

come into touch with instructors whose attention is focussed exclusively upon advanced mathematics and the advancement of mathematical knowledge through research, as it is that somewhere in the University training the mathematical theories of the more elementary field in which he is to teach should be illuminated from the higher standpoint. The giving of « survey » courses and other courses intended especially for the prospective teacher will depend, of course, primarily upon the willingness of instructors to give them, but also upon the other needs of instruction and the feasibility of incorporating such courses in the general program.

Finally, a few words as to conditions along this line in the United States.

Informal and incidental classroom discussions are by their very nature transitory and seldom come to the knowledge of others than those who are actually present. It is accordingly difficult for me to say how much work of the first type is done in the United States, but I believe that there is a considerable amount of it. I am confident that substantially all University instructors in the United States are fully aware of the fact that the great majority of their students will in the future teach mathematics, and that they take this fact into account in such manner as they deem requisite. There are doubtless some who feel that if a student gets as thorough and extensive a grasp of the subject as possible, he will thus become best qualified to impart the mathematical material he may have to teach as it presents itself. But there are undoubtedly many others who feel that the future teacher needs some direct consideration of the material that he is to teach, and who accordingly in their own way and on appropriate occasion discuss the bearing of the work in hand in their University courses on the elementary field.

A good measure of the appreciation of the needs of teachers that is felt by those in the American Universities and their sympathy with efforts to meet these needs is found in the large number of University men connected with the work of the American branch of the International Commission on the Teaching of Mathematics, in the fact that almost none of those that were asked to participate in that work declined to undertake the tasks they were requested to perform, and that of the very few who found themselves obliged to decline, none did so on the ground of lack of sympathy for the work.

In conclusion, I select from official announcements, brief descriptions of a few courses given in American Universities, that in various ways aim to present portions of mathematics in a broad style and often have the needs of future teachers explicitly in view. These are simply specimens, and are in no sense a complete

or fully representative list of all the courses of this character that are given in the Universities of the United States.

The courses described are given at the following Universities: 1, 2, 3, 4, at Columbia University; 5, 6, at Cornell University; 7, 8, at Harvard University; 9, 10, 11, at The University of Chicago; 12, at The University of Pennsylvania.

1. *The Fundamental Concepts of Modern Mathematics.*

Pre-requisite or parallel: Calculus.

This course will include some of the results of recent investigation, especially in their bearing on elementary mathematics. Among the topics treated will be the following: Euclidean and higher constructions; dimensionality and co-ordinates; the geometry of motion, transformations, groups and invariants; the concepts of number and function; infinity and the theory of assemblages; the algebra of logic.

2. *Fundamental Concepts in Mathematics.*

This course, designed to give preliminary orientation in the modern higher mathematics, deals with such matters as the theory of aggregates, the distinction of finite and infinite, postulate systems, the notions of group, generalized number, function, space element, dimensionality, geometrical constructions, and the quadrature of the circle.

Pre-requisite: A year of calculus.

3. *Mathematics.*

This course is designed primarily for mature students who, though specializing in other fields and not intending to become mathematicians, yet desire to gain some understanding of the nature and spirit of mathematics regarded as a human enterprise and as a body of human achievements. Offered thus in the interest of general intelligence and not designed to give facility in or mastery of any specific mathematical branch or branches, the course will have for its principal aim to serve those whose interest in mathematics is avocational or cultural rather than vocational or professional. It will deal with such topics as: the history and rôle of postulate-systems; questions respecting simplicity, convenience, adequacy, arbitrariness, equivalence, independence, compatibility, categoricalness of systems; the nature and offices of primitive ideas and primitive propositions, of definition and demonstration; the character and limitations of incomplete and of mathematical induction; multiple interpretations of a given system; the service of such systems as foundations of various branches of geometry, algebra, analysis and logic, as metric, projective, and descriptive geometries, Euclidean, non-Euclidean, and multi-dimensional geometries, the concepts and theories of variables real and complex; the logical foundations of mathematics as a whole, the branch-foundations as parts of the superstructure, symbolic logic; the general function concept, its varieties; the notion of limit, the method of limits as a process of idealization and generalization; *Mengenlehre*, the distinctions of finite and infinite, discrete and continuous, denumerable and non-denumerable, orderless, ordered, well-ordered; the concepts of transformation, invariance, and group, their rôles in the history of thought; the back-ground, origin, and position of the differential and integral calculus, the notions of derivative and integral, the idea of differential equation, its solution and interpretation.

4 *Philosophy of Mathematics.* The first half will be devoted to the logical foundations of mathematics, especially to a study of the thesis that mathematics is a continuous prolongation of logic, the two sciences being essentially one having a basis in a small number of primitive ideas and primitive propositions. The second part will be concerned with the history and significance of central mathematical concepts, the aim being to view some of the more important mathematical notions and doctrines as contributions to thought in the larger sense of the term and especially to indicate their bearings on matters of psychology, philosophy and art.

5. *Teacher's Course in Mathematics.* A critical study of the methods of both algebra and geometry. Designed to give persons planning to teach in secondary schools a better perspective of these branches of mathematics. Proofs and assumptions particularly will be studied with a view to eliminating erroneous reasoning.

6. *Elementary Course in Higher Mathematics.* Intended for students who do not plan to take many courses in mathematics, but who nevertheless wish to become acquainted with the principal ideas of modern mathematics. The object will be to make clear the fundamental aims, methods, and results of a number of subjects, rather than to develop the technique of any one subject. The course will deal with theory of numbers, vector analysis, groups; advanced synthetic geometry, including some topics in projective, non-euclidean, and higher dimensional geometry; analytic geometry, calculus, famous problems of mathematics; algebra of logic, foundations of mathematics.

7. *The Subject-Matter of Elementary Mathematics.* The object of this course is to make a careful study, from the higher mathematical point of view, of certain fundamental topics in elementary mathematics. Special stress is laid upon the study of various sorts of numbers, integers, fractions, quadratic surds, general irrationals, and complex numbers, and of their different applications to algebra and geometry. The course is especially suitable for students who expect to teach lower mathematics in school or college, but is open only to those who have had calculus.

8. *The Fundamental Concepts of Mathematics.* This course is intended for students of Mathematics or Philosophy who desire a concise and systematic exposition of the fundamental concepts of mathematics in the light of recent researches in this field. No advanced technical knowledge of mathematics is required. The course may profitably be taken by Undergraduates who are interested in the exact logic of mathematics as well as by Graduates.

During the year 1916-17, the course will be devoted mainly to a comparative study of various types of serial order, including the continuum, and to the foundations of algebra.

9. *Graphical Methods in Algebra.* The cross-section paper as a mathemathical instrument for the graphical study of the notion of functionality.

10. *Critical Review of Secondary Mathematics.* A brief survey of the subject matter of secondary mathematics from the modern point of view, aiming both to organize the whole scientifically and to gather the products of this work for use in teaching.

11. *Selected Topics in Mathematics.* As many as feasible of the following topics will be taken up: the function concept and the three fundamental concepts of the calculus, the concept of the algebraic equation, the number field of algebra, certain elementary properties of integral numbers and their

application in the theory of the construction of regular polygons with ruler and compasses ; the foundations of algebra and of geometry, the theory of parallels, the non-Euclidean geometries, and certain phases of modern pure geometry. The course will give needed acquaintance with work that is closely related to the fields of secondary and early collegiate mathematics, but problems of teaching will not be taken up.

12. *The Fundamentals of Mathematical Study*. The object of the course is to give students opportunity to get the correct point of view for mathematical study, and correct ways of mathematical thinking.

II. — THE PRACTICAL SIDE

by Dav.-Eug. Smith (New-York).

My colleague, Professor J. W. A. Young, has already written upon this subject taking for his theme the theoretic side. It is, therefore, appropriate to consider what we are doing or failing to do on the practical side, that is, in the way of giving our prospective teachers some practical experience with classes before they are given a license to teach mathematics. Some such statement seems essential for the reason that our friends from Europe, when they visit our schools, commonly express some surprise that so little provision is made for what is certainly so desirable, namely, this same practical training.

In order clearly to understand this failure to provide that which, from the European standpoint, seems a *sine qua non*, it is necessary to consider briefly the economic conditions involved. Perhaps the question would not be worth considering at all, from the international standpoint, were the future conditions in Europe to be the same as those of the past. The results of the great conflict, however, will include some need for readjustment in the supply of teachers, and on this account it may be worth while to mention a few of the economic questions that are involved in the United States, the way in which these questions affect the training of our teachers, and the means which we take to minimize the difficulty of thoroughly preparing these teachers before they are officially employed in our secondary schools.

We have in the United States about 1 $^1/_2$ million students in these secondary schools. To teach these students we require some 75,000 teachers. Custom rather than necessity leads to paying these teachers a low salary in certains parts of the country and a satisfactory salary in other parts, but the average salary is only about $ 900 a year, no residence or other perquisites being given in addition. No man of fair intellectual capacity can afford to teach for such a sum where the opportunities for an adequate income are as good as they are here, for the sum is insufficient