# Families of polyhedra

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**❙ Elemente der Mathematik**

# Families of polyhedra

G.C. Shephard

G.C. Shephard was awarded a doctorate at Cambridge University in 1951. He held po-
sitions at the University of Birmingham and the University of East Anglia (Norwich)
from which he retired as Professor Emeritus in 1987. He has written many papers on
convexity theory, polytopes, tessellations and theory of patterns.

## 1 Introduction

In the upper part of Figure 1(a) we show the net $N(P)$ of a polyhedron $P$. If this is cut
out of paper, folded along the (dashed and solid) lines, and edges with the same labels
$(a, b, c, \dots)$ joined together, we obtain a model of the surface $S(P)$ of a polyhedron $P$.
Here $P$ is a 4-*spindle* (a cube with two 4-pyramids adjoined to opposite faces). If, instead
of assembling $S(P)$ as described above, we cut the net into two pieces along the solid line
we get the regions shown in the lower part of Figure 1(a). These are the nets of two poly-
hedra $P_1$ and $P_2$. $P_1$, on the left of the diagram, is a 3-prism with a tetrahedron adjoined to
one of its triangular faces (known as an *elongated tetrahedron*), and $P_2$, on the right, is a
4-pyramid. Since $P_1$ and $P_2$ were derived from $P$, borrowing biological terminology, we
shall say that $P$ is a *parent* and $P_1$ and $P_2$ are its *offspring*. We write

$$P = P_1 \uplus P_2.$$

Wer ein Polyeder aus Papier oder Karton herstellt, geht dabei üblicherweise von einem
Netz aus. Das ebene Netz, ein Polygon, wird dabei ausgeschnitten, geeignet gefal-
tet und entlang der Kanten verklebt. Obwohl sich die Mathematik diesen Netzen seit
fast fünf Jahrhunderten widmet, sie tauchen etwa in den Werken Albrecht Dürers auf,
bleiben bis auf den heutigen Tag ungelöste Fragen offen. So ist zum Beispiel nicht
bekannt, ob jedes konvexe Polyeder ein Netz besitzt. Umgekehrt lässt sich die Frage,
wann ein gegebenes Polygon sich in ein konvexes Polyeder verkleben lässt (und in
wieviele verschiedene) inzwischen durch einen effizienten Algorithmus beantworten.
Die vorliegende Arbeit untersucht die Frage, wann das Netz eines Polyeders so in zwei
oder mehr Teile zerschnitten werden kann, dass jeder Teil das Netz eines Polyeders ist.

(a)



(b)

Fig. 1
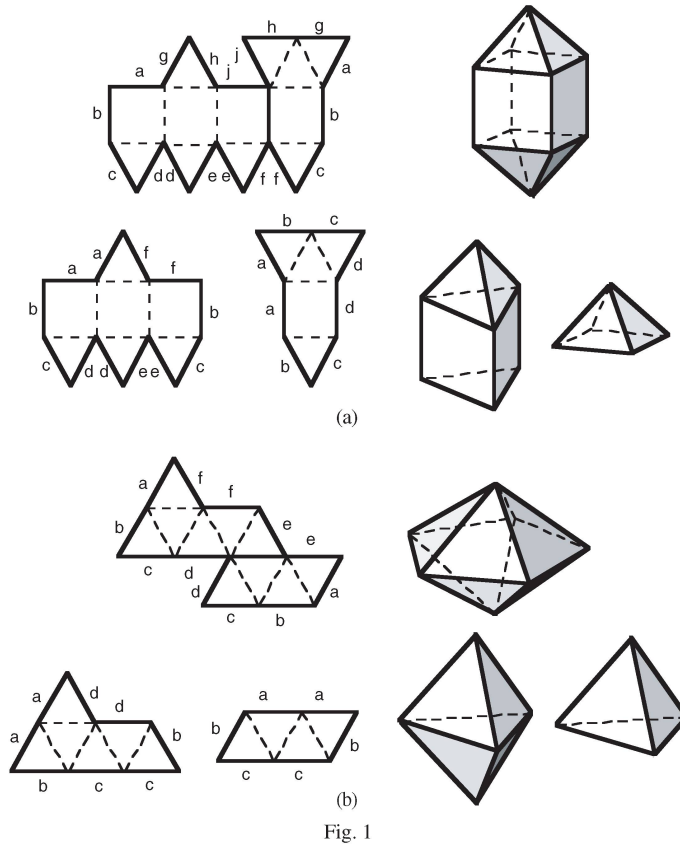
Note that this implies

$$S(P) = S(P_1) \cup S(P_2),$$

but not conversely. It certainly does not imply $P = P_1 \cup P_2$.

Another example is shown in Figure 1(b). Here a 5-dipyramid is the parent of a 3-dipyramid and a tetrahedron. For simplicity and brevity we shall use the terms "5-dipyramid" instead of "pentagonal dipyramid", "3-prism" instead of "triangular prism", and so on, throughout.

The examples just given are atypical; cutting the net of a polyhedron into two pieces does not, in general, yield the nets of other polyhedra. For example, if any one of the eleven possible nets of the cube (Figure 2(b) and [1]) is cut into two parts in any way, neither part is the net of another polyhedron.

If the two offspring $P_1$, $P_2$ of a family $P = P_1 \uplus P_2$ are equal we call them *twins*. An example is shown in Figure 3. Here $P$ is the 5-antiprism with two regular pentagons and ten regular triangles as faces. Each of the twins $P_1$ and $P_2$ is a 5-pyramid with one regular pentagon and five regular triangles as faces.
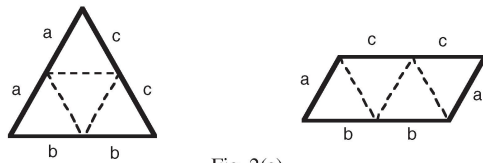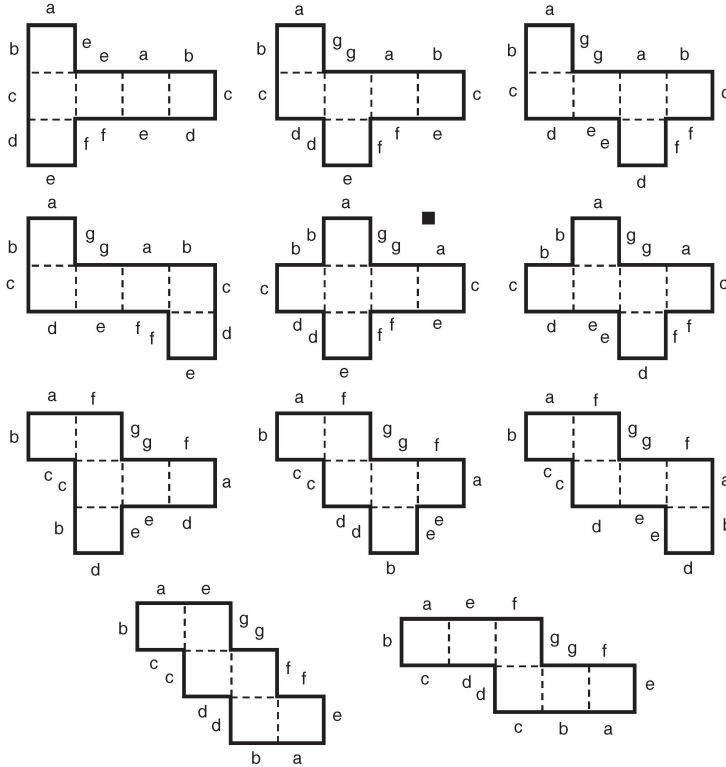
Fig. 2(a)



Fig. 2(b)

The above notation is extended in the obvious way to families of more than two offspring. If we can cut the net of $P$ into three pieces so that these pieces are the nets of polyhedra $P_1$, $P_2$, $P_3$ then we write $P = P_1 \uplus P_2 \uplus P_3$, and so on. There are infinitely many families of polyhedra and so, to make the subject manageable we introduce some restrictions. One of these is to confine attention to convex polyhedra with regular polygons as faces. These will be called *regular-faced polyhedra*. It is known that the regular-faced polyhedra comprise the regular and archimedean polyhedra, the $n$-prisms and $n$-antiprisms ($n \geq 4$) and 92 other polyhedra known as the *Johnson solids*. The latter were discovered by Norman W. Johnson in 1966 [5] and the fact that Johnson's list of regular-faced polyhedra is complete was proved by Zalgaller in 1969 [8]. We shall refer to these polyhedra by the numbers J1–J92 assigned to them by Johnson. A list can be found in Johnson's original paper and
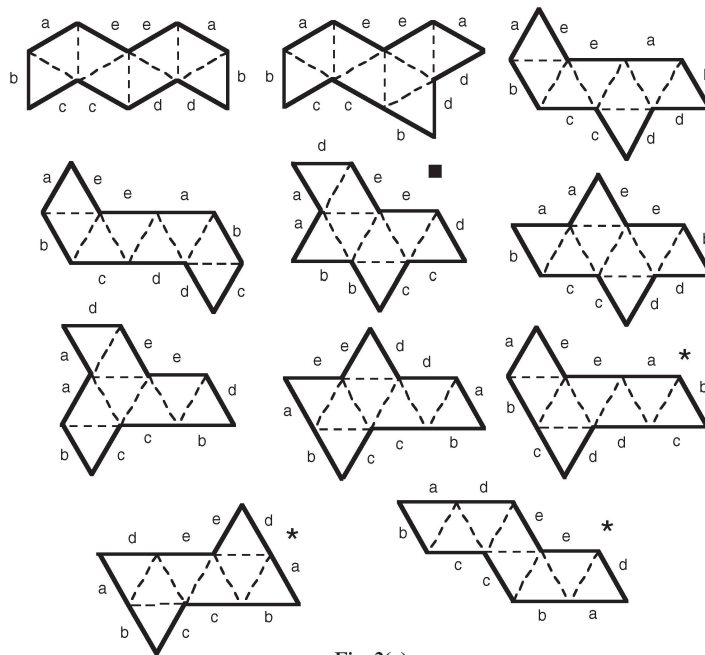
Fig. 2(c)
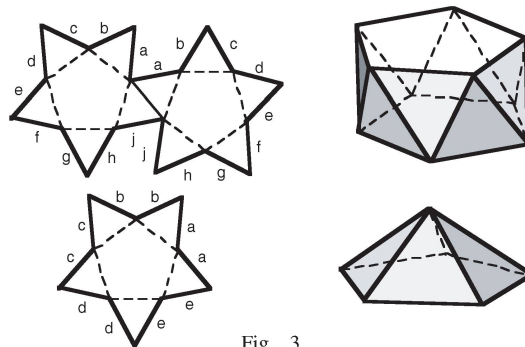


Fig. 3

also in Wikipedia. Even restricting attention to the regular-faced polyhedra, the number of families is in the thousands, hence we proceed as follows:

(a) In Section 2 we enumerate all families consisting of two twin polyhedra. Notice that we do not use the term "twin polyhedra" when the family contains more than two members, and

(b) in Section 3 we enumerate all families of polyhedra whose faces are regular (equilateral) triangles.

(c) Finally, in Section 4, we state some results of a general nature.

## 2 Regular-faced polyhedra

**Theorem 1.** *There exist* 11 *pairs of twin regular-faced polyhedra.*

*Proof.* For a polyhedron $P$ we write $f_n(P)$ $(3 \leq n)$ for the number of faces of $P$ which are $n$-gons. The vector $\mathbf{f}(P) = (f_3(P), f_4(P), \ldots)$ is called the $f$-vector of $P$. This vector is, of course, finite; in fact $f_n(P) = 0$ for all $n > 10$ if $P$ is any regular-faced polyhedron (apart from the prisms and antiprisms). If

$$P = P_1 \uplus P_2 \quad \text{then} \quad \mathbf{f}(P) = \mathbf{f}(P_1) + \mathbf{f}(P_2)$$

(but not conversely). Consequently, if a polyhedron $P$ is the parent of twins $P_1 = P_2$, then every entry in the $f$-vector $\mathbf{f}(P)$ must be even and $f_n(P_1) = f_n(P_2) = \frac{1}{2}f_n(P)$ for all $n \geq 3$. In this case we shall say that $\mathbf{f}(P)$ is an even vector and write $\mathbf{f}(P_1) = \mathbf{f}(P_2) = \frac{1}{2}\mathbf{f}(P)$. Since the $f$-vectors of all the regular-faced polyhedra are known, this yields a simple method of finding all the possible parents of twins.

Of the 92 Johnson solids and 18 Archimedean and regular polyhedra, 64 have even $f$-vectors $\mathbf{f}(P)$. Of these 52 can be eliminated immediately as parents of twins since in these cases $\frac{1}{2}\mathbf{f}(P)$ is not the $f$-vector of any regular faced polyhedron. For example, $\mathbf{f}(J34) = (20, 0, 12, 0, \ldots)$ but there is no regular regular-faced polyhedron with $f$-vector $(10, 0, 6, 0, \ldots)$. Of the remaining twelve, nine are the parents of twins, and there are two others namely the 4-antiprism and 5-antiprism bringing the total number to eleven (Theorem 1 and Table 1). This leaves three possibilities to be investigated, namely J27, J28

| Parent | Twin offspring | Diagram |
|---|---|---|
| octahedron | tetrahedra | Figure 5(a) |
| 4-antiprism | 4-pyramids | Figure 5(b) |
| cuboctahedron | Johnson solids J7 | Figure 5(c) |
| deltahedron $D_{16}$ (J17) | octahedra | Figure 5(d) |
| Johnson solid J29 | Johnson solids J8 | Figure 5(e) |
| Johnson solid J31 | Johnson solids J9 | Figure 5(f) |
| Johnson solid J62 | Johnson solids J2 (5-pyramids) | Figure 5(g) |
| Johnson solid J84 | Johnson solids J12 (3-dipyramids) | Figure 5(h) |
| Johnson solid J85 | Johnson solids J10 | Figure 5(j)[1] |
| 5-antiprism | 5-pyramids | Figure 3 |
| icosahedron | Johnson solids J13 (5-dipyramids) | Figure 4(b) |

Table 1 The eleven twin polyhedra with regular faces.

and J30. However, checking these cases is not straightforward as can be seen from the example in Figure 4. In Figure 4(a) we show the familiar net of the regular icosahedron with $f$-vector $(20, 0, \ldots)$. This can be cut into two pieces each of which is the union of ten triangles, but these are *not* nets of polyhedra.

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1. J85 is remarkable in that the net of one twin is the enantiomorph of the net of the other. In all other cases the nets are equal.
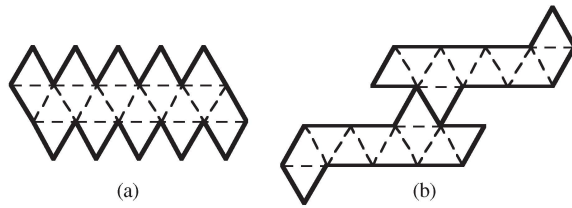
Fig. 4
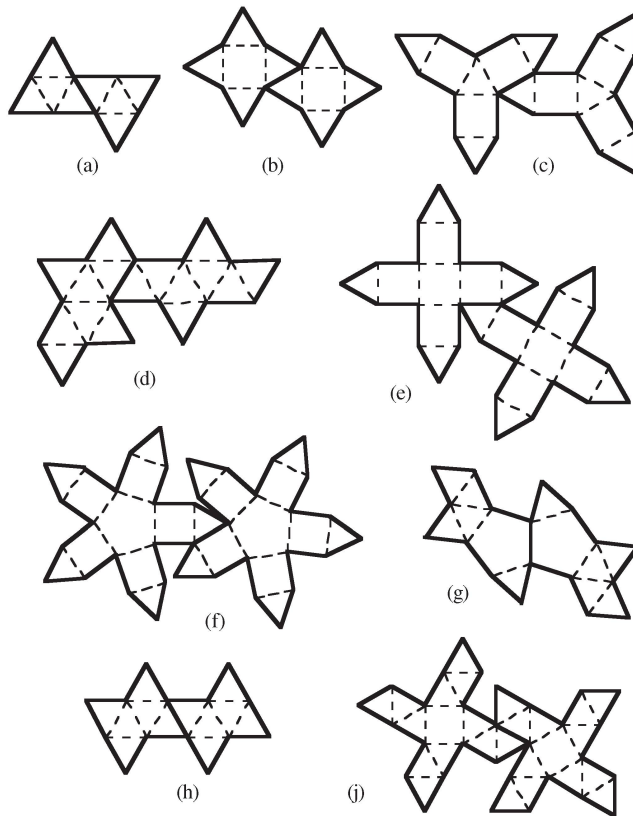


Fig. 5

However the less familiar net of the icosahedron shown in Figure 4(b) can be cut along the solid line into nets of twin 5-dipyramids (J2) as shown, each with $f$-vector $(10, 0, 0, \ldots)$. In Figure 4, and subsequent diagrams, the cut line is the solid line in the interior of the net, and edge labellings are omitted; no ambiguities can arise in the case of regular-faced polyhedra (in spite of the comments in [7]).

In establishing that the list of parents with twins in Theorem 1 is complete we have the difficulty which seems inherent in this type of problem. Whereas it is straightforward to

establish that a given polyhedron $P$ has twins simply by drawing a suitable diagram, it is not so easy to prove that the three polyhedra J27, J28, J30 are *not* the parents of twins. After investigation we believe that this is so, but we have no proof. In a similar situation (tessellation polyhedra) the analogous question was settled by computer, but it is difficult to see how a similar method can be applied in the present case.

It is natural to ask whether, in view of the existence of twins, triplets, quadruplets and quintuplets of regular-faced polyhedra can arise (clearly sextuplets are impossible). In fact all these are possible but are very few in number. Examples will be given in the next section. Note that the $f$-vector of the triaugmented dodecahedron (J61) is $(15, 0, 9, 0, \ldots)$ so one might expect it to have triplets each of which is the tridiminished icosahedron (J63) with $f$-vector $(5, 0, 3, 0, \ldots)$, but it appears that it does not do so.

There are many examples of polyhedra $P$ which have offspring which are *siblings* $P_1$, $P_2$ but are *not* twins, that is $P = P_1 \uplus P_2$ but $P_1 \neq P_2$. A computer search for $f$-vectors of regular-faced polyhedra $P$, $P_1$, $P_2$ that satisfy equation (1) yielded 118 solutions. Many of these solutions lead to families of siblings, but some do not, and there are too many cases (even in the case of regular-faced polyhedra) to investigate in detail. However, if we make the further restriction that all the faces of the polyhedra are regular (equilateral) triangles, the investigation becomes tractable, as we shall see in the next section.

## 3    Polyhedra whose faces are regular triangles

For brevity we shall use the term $\Delta$-*polyhedron* for one whose faces are all regular (equilateral) triangles. There are eight such polyhedra, namely

- (i) the regular tetrahedron with $f_3 = 4$,
- (ii) the 3-dipyramid J12 with $f_3 = 6$,
- (iii) the regular octahedron with $f_3 = 8$,
- (iv) the 5-dipyramid J13 with $f_3 = 10$,
- (v) the deltahedron $D_{12}$ (J84) with $f_3 = 12$,
- (vi) the deltahedron $D_{14}$ (J51 ) with $f_3 = 14$,
- (vii) the deltahedron $D_{16}$ (J17 ) with $f_3 = 16$. and
- (viii) the regular icosahedron with $f_3 = 29$.

All the other components in the $f$-vectors are zero. As the $f$-vectors have only one non-zero component, relations between them are easy to find. In view of the comments in the previous section, the following result is unexpected.

**Theorem 2.** *For polyhedra $P$ with regular triangles as faces, every solution of the equation*

$$\mathbf{f}(P) = \mathbf{f}(P_1) + \mathbf{f}(P_2) + \mathbf{f}(P_3) + \ldots \tag{1}$$

*corresponds to the expression of two or more such polyhedra as offspring of $P$.*

We prove this theorem by examining all such expressions. There are thirteen in which $f(P)$ is the sum of two $f$-vectors, eight in which it is the sum of three $f$-vectors, three
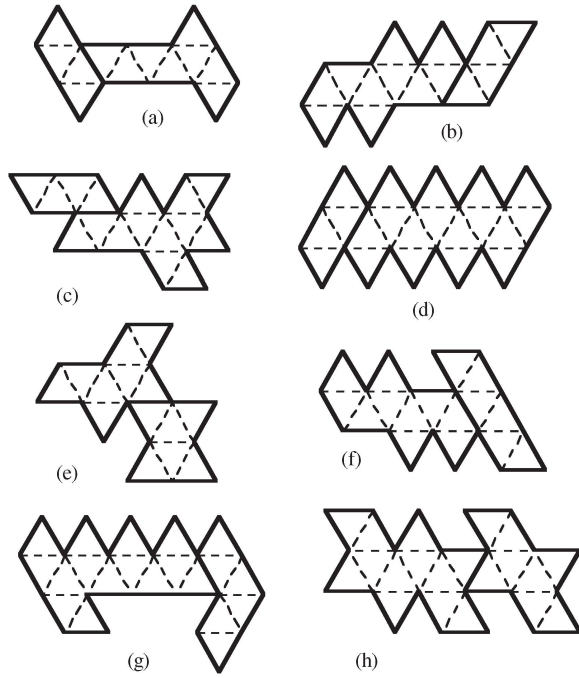
Fig. 6

| 8; 4, 4 | octahedron = tetrahedron ⊎ tetrahedron | Figure 5(a) |
|---|---|---|
| 10; 4, 6 | 5-dipyramid = tetrahedron ⊎ 3-dipyramid | Figure 1(b) |
| 12; 4, 8 | $D_{12}$ = tetrahedron ⊎ octahedron | Figure 6(a) |
| 14; 4, 10 | $D_{14}$ = tetrahedron ⊎ 5-dipyramid | Figure 6(b) |
| 16; 4, 12 | $D_{16}$ = tetrahedron ⊎ $D_{12}$ | Figure 6(c) |
| 20; 4, 16 | icosahedron = tetrahedron ⊎ $D_{16}$ | Figure 6(d) |
| 12; 6, 6 | $D_{12}$ = 3-dipyramid ⊎ 3-dipyramid | Figure 5(h) |
| 14; 6, 8 | $D_{14}$ = 3-dipyramid ⊎ octahedron | Figure 6(e) |
| 16; 6, 10 | $D_{16}$ = 3-dipyramid ⊎ 5-dipyramid | Figure 6(f) |
| 20; 6, 14 | icosahedron = 3-dipyramid ⊎ $D_{14}$ | Figure 6(g) |
| 16; 8, 8 | $D_{16}$ = octahedron ⊎ octahedron | Figure 5(d) |
| 20; 8, 12 | icosahedron = octahedron ⊎ $D_{12}$ | Figure 6(h) |
| 20; 19, 10 | icosahedron = 5-dipyramid ⊎ 5-dipyramid | Figure 4 |

Table 2  Δ-parents with two offspring.

in which it is the sum of four $f$-vectors, and one in which it is the sum of five $f$-vectors. These are tabulated below. The first column shows the value of $f_3$ for the parent followed (after the semicolon) by the values $f_3$ of its offspring, the second shows the relationships between the polyhedra, and the last gives a reference to a diagram showing the family.
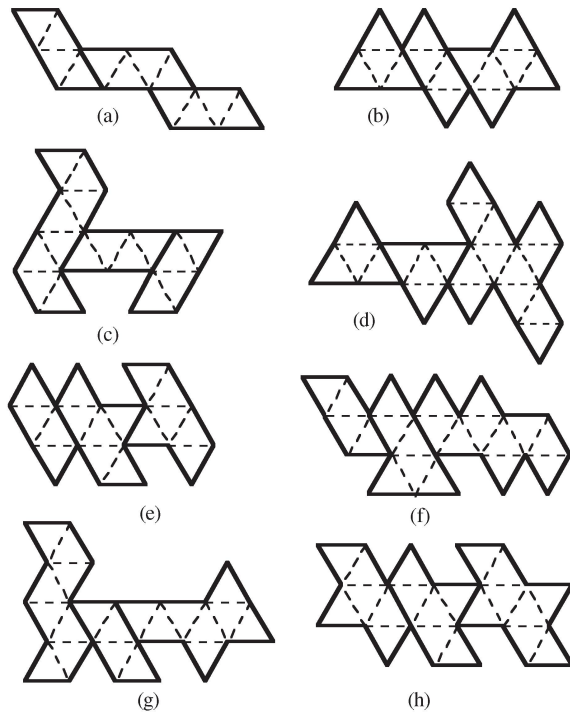
Fig. 7

| | | |
|---|---|---|
| 12; 4, 4, 4 | $D_{12}$ = tetrahedron ⊌ tetrahedron ⊌ tetrahedron | Figure 7(a) |
| 14; 4, 4, 6 | $D_{14}$ = tetrahedron⊌ tetrahedron ⊌ 3-dipyramid | Figure 7(b) |
| 16; 4, 4, 8 | $D_{16}$ = tetrahedron ⊌ tetrahedron ⊌ octahedron | Figure 7(c) |
| 20; 4, 4, 12 | icosahedron = tetrahedron ⊌ tetrahedron ⊌ $D_{12}$ | Figure 7(d) |
| 16; 4, 6, 6 | $D_{16}$ = tetrahedron ⊌ 3-dipyramid ⊌ 3-dipyramid | Figure 7(e) |
| 20; 4, 6, 10 | icosahedron = tetrahedron ⊌ 3-dipyramid ⊌ 5-dipyramid | Figure 7(f) |
| 20; 4, 8, 8 | icosahedron = tetrahedron ⊌ octahedron ⊌ octahedron | Figure 7(g) |
| 20; 6, 6, 8 | icosahedron = 3-dipyramid ⊌ 3-dipyramid ⊌ octahedron | Figure 7(h) |

Table 3  Δ-parents with three offspring.

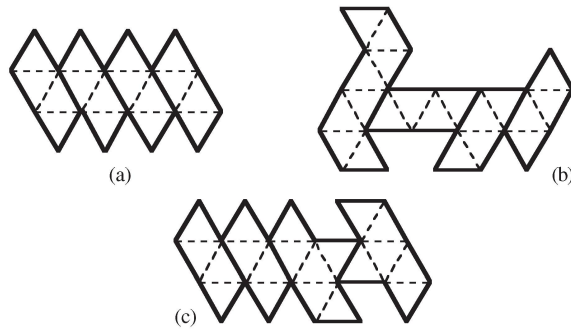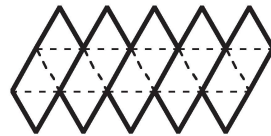| | | |
|---|---|---|
| 16; 4, 4, 4, 4 | $D_{16}$ = tetrahedron ⊌ tetrahedron ⊌ tetrahedron ⊌ tetrahedron | Figure 8(a) |
| 20; 4, 4, 4, 8 | icosahedron = tetrahedron ⊌ tetrahedron ⊌ tetrahedron ⊌ octahedron | Figure 8(b) |
| 20; 4, 4, 6, 6 | icosahedron = tetrahedron ⊌ tetrahedron ⊌ 3-dipyramid ⊌ 3-dipyramid | Figure 8(c) |

Table 4  Δ-parents with four offspring.

Fig. 8



Fig. 9

| 20; 4, 4, 4, 4, 4 | icosahedron = tetrahedron ⊎ tetrahedron ⊎ tetrahedron ⊎ tetrahedron ⊎ tetrahedron | Figure 9 |
|---|---|---|

Table 5  △-parents with five offspring.

## 4   General Remarks

Nets of polyhedra have been described by artists and others for nearly five centuries, and examples of nets of familiar polyhedra can be found, for example, in [3]. The subject of nets has largely been neglected by mathematicians, as one can see from the paucity of published papers. There are, however, a number of open problems which have so far defeated the efforts of competent mathematicians. The most noteworthy of these is: *Does every convex polyhedron have a net?* This was stated explicitly by the author in 1975 [7]. The question arises because, as is a common experience with makers of polyhedral models, thinking of the net as obtained by slitting a model of (the surface of) the polyhedron along edges and opening it out flat, it can certainly happen that one part of the intended net may overlap another, and so no net, as the term is used here (a region in the plane), is obtained. In the early nineteenth century J.D. Gergonne assumed the answer to this question is in the affirmative, but no proof is known. For a history of the problem and further information see Problem 9 of the Open Problems Project [6] where it is suggested that in spirit, at least, it goes back to Dürer (1525) [4]. As no counter-examples are known, it seems reasonable to assume the answer to this problem is in the affirmative.

We now make some general remarks on nets. Let $E$ be the set of edges of a polyhedron $P$. A net is obtained by slitting the surface $S(P)$ of $P$ along the edges of a subset $E' \subset E$. The

union of the edges in $E'$ must be a hamiltonian tree: it is a tree (a graph with no circuits) because the resulting net is connected, and it is hamiltonian (contains all the vertices of $P$) since otherwise it would not be possible to "open out flat" the surface $S(P)$.

Thus every net of $P$ arises from a hamiltonian edge-tree of $P$, but the converse is not true; there exist hamiltonian edge-nets of polyhedra which lead to overlaps. Hence, in general, the number of distinct nets of a polyhedron $P$ is less than the number of hamiltonian edge-trees on $P$.

For very few polyhedra is the number of distinct nets known. There are two for the regular tetrahedron, eleven for the cube and eleven for the regular octahedron, see Figure 2 and [1]. There are nine for the 3-dipyramid and the same number for the 3-prism. (Enantiomorphs are not counted as distinct.) All these have been determined empirically, and there seems to be no general method of determining these numbers. The number of distinct nets of, for example, the cuboctahedron seems to be unknown.

Associated with a polyhedron are the three numbers (integers) $f(P)$, $e(P)$, and $v(P)$, the numbers of faces, edges and vertices, respectively, which satisfy the well-known Euler equation

$$f(P) - e(P) + v(P) = 2. \tag{2}$$

For a net $N$ there are four numbers, namely $f(N)$ and $v(N)$ the numbers of faces and vertices of $N$, $e_b(N)$ the number of edges on the boundary of $N$, and $e_i(N)$ the number of edges in the interior of $N$. Euler's theorem for a planar region yields

$$f(N) - e_b(N) - e_i(N) + v(N) = 1. \tag{3}$$

**Theorem 3.** *The numbers $v(P)$, $e(P)$, $f(P)$, $v(N)$, $e_i(N)$, $e_b(N)$, and $f(N)$ satisfy the following relations:*

(i) $f(N) = f(P)$,

(ii) $e_b(N) = 2(v(P) - 1) = v(N)$,

(iii) $e_i(N) = f(P) - 1$.

Statement (i) is obvious. For (ii) we note that the hamiltonian tree T has $v(P) - 1$ edges, and each edge gives rise to two boundary edges of the net. The final part follows from (i), (ii), (2), or alternatively by observing that every two of the $f(P)$ polygons in the net is separated by an interior edge.

Theorem 3 has the unexpected consequence that all nets $N$ of a given polyhedron $P$ have the same values of $e_b(N)$ and $e_i(N)$. Thus both nets of the tetrahedron (Figure 2(a)) have $e_b(N) = 6$ and $e_i(N) = 3$, and all eleven nets of the cube (Figure 2(b)) have $e_b(N) = 14$ and $e_i(N) = 5$.

Let $P'$ denote the dual of the polyhedron $P$. As the polyhedra $P$ we are considering are convex, a dual $P'$ can be defined as (the combinatorial type of) the polar reciprocal of $P$ with respect to any interior point. Clearly $f(P) = v(P')$, $e(P) = e(P')$, and $v(P) = f(P')$. Further, if $N'$ is a net of $P'$ then

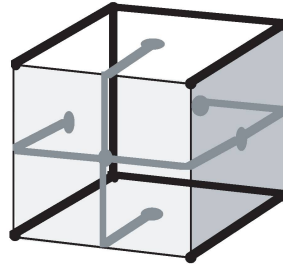**Theorem 4.** $e_b(N') = 2e_i(N)$ *and* $2e_i(N') = e_b(N)$.

Fig. 10

This follows because $e_b(N') = 2(v(P') - 1) = 2(f(P) - 1) = 2e_i(N)$, and $2e_i(N') = 2(f(P') - 1) = 2(v(P) - 1) = e_b(N)$.

Thus all the nets $N'$ of the octahedron $P'$, shown in Figure 2(c) have $e_b(N') = 2e_i(N) = 10$ and $e_i(N') = e_b(N)/2 = 7$ as is easily verified.

However, we can say more. Suppose the net $N$ of $P$ was constructed using the hamiltonian tree $T$. Then construct a new tree $T''$ as follows. The nodes of $T''$ are the polygons in $T$ and two such nodes are joined by an arc if the corresponding polygons have an edge in common. It is clear that this is a tree, and since the faces of $P$ correspond to the vertices of $P'$, $T''$ corresponds to a hamiltonian net $T'$ on $P'$, and thus to a net $N'$, which we shall refer to as the dual of $N$. Hence the number of distinct nets of a polyhedron $P$ and of its dual $P'$ are equal. (In Figure 10 we show a cube with two dual trees marked on it. One, indicated by solid thick lines shows the tree that leads to the net marked by a black square (■) in Figure 2(b), and the dual net, indicated by grey lines leads to the net of the octahedron marked by a black square in Figure 2(c).)

Clearly dual nets on the surface of a polyhedron are disjoint, and moreover are *maximal* in an obvious sense. Thus duality of polyhedra implies a duality between their nets. In Figures 2(b) and (c) dual nets of the cube and octahedron appear in the same relative positions in the diagrams. However, we cannot assert that just because a tree $T$ of $P$ leads to a net of $P$, the tree $T''$ necessarily leads to a net of $P'$. Moreover, the property of having a family does not dualise. We have remarked that the cube does not have any offspring, but three of the nets of the octahedron (marked by an asterisk in Figure 2(c)) have twin offspring – in every case tetrahedra. The idea of dual trees is not new (see [2, Fig 1.4A] for dual trees of the regular icosahedron and dodecahedron). However the concept of dual nets seems to be introduced here for the first time.

We conclude with a simple observation about the nets of polyhedra in a family.

**Theorem 5.** *Suppose*

$$P = P_1 \uplus P_2 \uplus \ldots \uplus P_n.$$

*Then*

(i) $\sum_{j=1}^{n} f(P_j) = f(P)$,

(ii) $\sum_{j=1}^{n} e(P_j) = e(P)$,

(iii) $\sum_{j=1}^{n} v(P_j) = v(P) + 2(n - 1)$.

Statement (i) is clear and (iii) follows since every two adjacent nets $N_j$ have two vertices in common and there are $n - 1$ adjacencies. Statement (ii) then follows from Euler's equation (2).

## Acknowledgement

## References

[1] Akiyama, J.: Tile-makers and semi-tile makers. *Amer. Math. Monthly* 114 (2007), 602–609.

[2] Coxeter, H.S.M.: *Regular Polytopes*. Dover Publications, third edition 1973.

[3] Cundy, M.M.; Rollett A.P.: *Mathematical Models*. Oxford University Press, second edition 1961.

[4] Dürer, A.: *Underweysung der Messung mit dem Zirckel und Richtscheyt*. Heironymus Andreaea, 1525. English translation by Strauss, W.L.: *The Painters' Manual*. Abaris Books, New York 1977.

[5] Johnson, N.W.: Convex Solids with Regular Faces. *Canad. J. Math.* 18 (1966), 169–200.

[6] The Open Problem Project. Edited by Demaine, E.D.; Mitchell, J.S.B.; O'Rourke, J. at
http://maven.smith.edu/~orourke/TOPP

[7] Shephard, G.C.: Convex Polytopes with Convex Nets. *Math. Proc. Cambridge Philos. Soc.* 78 (1975), 389–403.

[8] Zalgaller, V.A.: Convex Polyhedra with Regular Faces. *Consultants Bureau* (1969)

G.C. Shephard
University of East Anglia
Norwich, NR4 7TJ
England, U.K.
e-mail: g.c.shephard@uea.ac.uk