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Triangles in squares

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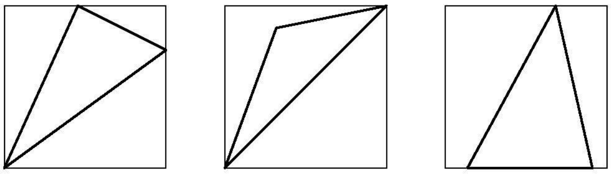
Introduction. In this note we determine precisely when a triangle fits in a square by finding necessary and sufficient conditions on the sides a, b, c, s for the triangle with sides a, b, c to fit into the square of side s. Our strategy is to find the side $s_{\min}(T)$ of the smallest square $S_{\min}$ that contains the given triangle T; then T fits into a square S of side s precisely when $s \geq s_{\min}(T)$. Scaling solves the equivalent dual problem: Find the largest triangle similar to a given triangle that fits in a given square.

Minimal squares about a triangle. By a *square* we sometimes mean the union of four line segments and sometimes the region they surround – the precise meaning will always be clear from the context. A triangle has *vertices* while a square has *corners*, and unless

Die Frage nach den Bedingungen, unter denen eine geometrische Figur in eine andere einbeschrieben werden kann, ist auch heute noch aktuell. Dies belegt zum Beispiel der Artikel von K.A. Post aus den neunziger Jahren, in dem eine Lösung zu einer alten Fragestellung von Steinhaus gegeben wird, notwendige und hinreichende Bedingungen für die sechs Seiten zweier Dreiecke zu finden, so dass das eine Dreieck in das andere einbeschrieben werden kann. In dem vorliegenden Artikel untersuchen die Autoren, wann ein Dreieck in ein Quadrat einbeschrieben werden kann. Dazu geben sie notwendige und hinreichende Bedingungen an, die die drei Dreiecksseiten und die Quadratseite zu erfüllen haben. Die duale, klassische Problemstellung, wann nämlich ein Quadrat in ein Dreieck einbeschrieben werden kann, hat einer der Autoren im stumpfwinkligen Fall vor kurzem studiert.

the contrary is explicitly stated, a *side* of either includes its ends. It is a consequence of compactness that among all squares that contain a given triangle T there is at least one whose side is as small as possible. We begin by considering how such a minimal square fits about the given triangle.



(a) Vertex square (b) Diagonal square (c) Side square

Fig. 1 Minimal squares

The possibilities are pictured in Fig. 1. A *vertex square* of T is a square containing T with exactly one vertex of T at a corner of the square and the other two vertices on the two (open) non-adjacent sides of the square (Fig. 1(a)). A *diagonal square* of T is a square containing T whose diagonal is a side of T (Fig. 1(b)). A *side square* of T is a non-diagonal square that contains T with two vertices on one side and the third vertex on the opposite side (Fig. 1(c)). We call all such squares *minimal.*

Lemma 1 *The smallest square* $S_{\min}$ *that contains a given triangle* T *is a vertex, a diagonal, or a side square of* T.

Proof. Suppose $S_{\min}$ is not a diagonal square of T. Then all three vertices of T must lie on $S_{\min}$ (otherwise a suitable rigid motion would move T entirely into the interior of $S_{\min}$, and it would lie in a square strictly smaller than $S_{\min}$). We examine the possibilities for the three vertices on the four sides of $S_{\min}$. If two vertices are on one side and the third is on an adjacent side, then T could be moved into the interior of $S_{\min}$ by a suitable rigid motion, a contradiction. If two vertices are on one side and the third is on the opposite side, then the square is a side square. If no two vertices of T are on the same side of $S_{\min}$ and no vertex lies at a corner of $S_{\min}$, then T could be moved into the interior of $S_{\min}$ by a suitable rigid motion, again a contradiction. The only remaining possibility is that $S_{\min}$ is a vertex square. □

We will find that if two angles of T are less than or equal to 45°, then the smallest square $S_{\min}$ is the diagonal square on the longest side, and in every other situation with just one exception, $S_{\min}$ is a vertex square. The exception, in which $S_{\min}$ is the side square on the shortest side of T, occurs when T is acute, the altitude to the shortest side is longer than that side, and the smallest angle is greater than 45°.

The principal results. In this section we determine precisely when squares of each kind exist and which is smallest when there is more than one.

Let $T = ABC$, and write α, β, γ for $\angle A$, $\angle B$, $\angle C$, respectively. For $\{X, Y, Z\} = \{A, B, C\}$ we write S_X, S_{XY}, and S_x for the vertex square at the vertex X, the diagonal square on the side XY, and the side square on the side x opposite the vertex X, respectively, when they exist. We denote by h_x the altitude of T to the side x.

Vertex squares. A vertex square can exist only at a vertex of T where the angle is acute. We begin by determining precisely when there is a vertex square at such a vertex.

Lemma 2 *Suppose that $T = ABC$ with $\gamma < 90°$. Then there is a vertex square at C if and only if both*

$$h_a < a \qquad \text{and} \qquad h_b < b\,. \tag{1}$$

The square is unique when it exists, and its side s is given by the formula

$$s = \frac{ab\cos\gamma}{\sqrt{a^2 - 2ab\sin\gamma + b^2}}\,. \tag{2}$$

Proof. Suppose such a square $CPQR$ exists (Fig. 2), and let L be the foot of the perpendicular from B to CP. Suppose the altitude ray BE meets CP at Y. Then $\angle PCA = \angle LBY$, the right triangles CPA and BLY are congruent, and $BY = CA = b$. Since E lies between B and Y (because A lies between P and Q), $h_b = BE < BY = b$. A similar argument on the altitude ray AD shows that $h_a < a$, so that conditions (1) are necessary. When the vertex square $S_C = CPQR$ exists,

$$\tan\angle PCA = \frac{EY}{CE} = \frac{BY - BE}{CE} = \frac{b - a\sin\gamma}{a\cos\gamma}\,, \tag{3}$$

which establishes the uniqueness. Formula (2) for the side s follows from substituting (3) into $s = b\cos\angle PCA$.

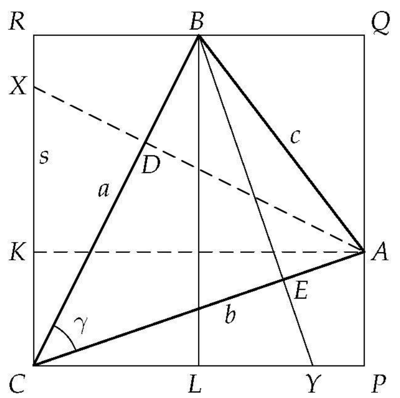


Fig. 2 Vertex square

To show the converse, take Y on the altitude ray BE so that $BY = b$, and let P be the foot of the perpendicular from A to the ray CY. Let L be the foot of the perpendicular from B to CY. Then $\angle PCA = \angle LBY$, triangles CPA and BLY are congruent, and $BL = CP$. Take points Q, R so that the figure $CPQR$ is a square. According to the hypothesis $h_b < b$, Y and B are on opposite sides of CA and A lies between P and Q. It remains only to show that B lies between Q and R. Suppose the altitude ray AD from

A to BC meets CR at the point X, and let K be the foot of the perpendicular from A to CR. Then $\angle XAK = \angle BCR$, and it follows that right triangles RCB and KAX are congruent and $AX = CB = a$. From the hypothesis $h_a < a$ we see that D lies between A and X, and consequently B lies between R and Q. □

The argument of the proof gives an elegant ruler and compass construction for the vertex square $CPQR$ when it exists.

If T has two vertex squares, say one at B and one at C, which is smaller? We show next that the smaller square is at the vertex whose angle is larger.

Lemma 3 *Suppose that T has vertex squares S_B at B and S_C at C. Then S_B is larger than, equal to, or smaller than S_C according as $\beta < \gamma$, $\beta = \gamma$, or $\beta > \gamma$.*

Proof. Write s, t for the sides of S_B, S_C, respectively, and let $\varphi = \angle PBC$ and $\psi = \angle BCZ$ (Fig. 3). Then $s = a\cos\varphi$ and $t = a\cos\psi$, so that $s < t$, $s = t$, or $s > t$ according as $\varphi > \psi$, $\varphi = \psi$, or $\varphi < \psi$. But $\tan\varphi = (a - c\sin\beta)/c\cos\beta$ and $\tan\psi = (a - b\sin\gamma)/b\cos\gamma$ (by (3)). Both parenthetical factors below being 1 by the law of sines, we see that

$$\frac{\tan\varphi}{\tan\psi} = \left(\frac{a - c\sin\beta}{a - b\sin\gamma}\right)\frac{b\cos\gamma}{c\cos\beta} = \frac{\tan\beta}{\tan\gamma}\left(\frac{b\sin\gamma}{c\sin\beta}\right) = \frac{\tan\beta}{\tan\gamma},$$

and the conclusion follows. □

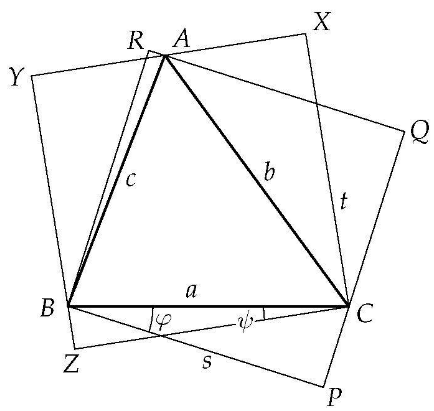


Fig. 3 Two vertex squares

Diagonal squares. The situation in which the smallest square is a diagonal square (Fig. 1(b)) is easy to characterize.

Lemma 4 *Suppose $T = ABC$ and $\beta, \gamma \le 45°$. Then the smallest square $S_{\min}$ that contains T is the diagonal square S_{BC} with side $a/\sqrt{2}$.*

Proof. Triangle T fits in S_{BC} as pictured in Fig. 1(b), and since $\operatorname{diam}(T) = a = \operatorname{diam}(S_{\min})$, no smaller square will do. □

Side squares. When T has a side square (Fig. 1(c)) on a side x, plainly $h_x \geq x$. It is convenient to introduce a term for this situation: We call a triangle T *tall* if it has a side x for which $h_x \geq x$. So a triangle has a side square only when it is tall.

Lemma 5 *If x, y are sides of a non-right triangle and $x \leq y$, then $h_y < y$. A right triangle is tall on its shorter leg, and tall on both legs only if it is isosceles.*

Proof. If θ is the angle between sides x and y and $\theta \neq 90^\circ$, then $h_y = x \sin\theta < x \leq y$. For the legs x, y of a right triangle we have $h_x = y$ and $h_y = x$, and the result follows. □

So the inequality $h_x \geq x$ is possible for a triangle that is not isosceles-right only when that triangle has a strictly shortest side, and x is that side.

It is convenient to employ notation that identifies the shortest side of T. We agree to label the vertices of T so that T is positively oriented (i.e., the sequence $A \to B \to C \to A$ is counterclockwise) and $\alpha \geq \beta \geq \gamma$, by replacing T by its mirror image if necessary. Then T can be realized in the coordinate plane with BC horizontal and A *above* the line BC; and because of the notational normalization, A lies in the circular triangle MDB bounded by the mediator MD of BC and the arc DB of the circle of radius a centered at C (Fig. 4).

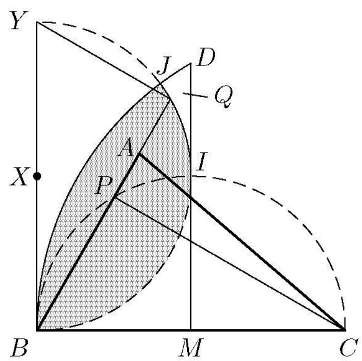


Fig. 4 The tall region

Suppose the semicircle with diameter BC meets the mediator MD at the point I. Let $BMIX$ be the square on BM, and suppose the circle with radius $a/2$ centered at X meets the arc BD at J and the ray BX at Y (Fig. 4). We call the lens BJ between the two circular arcs, shaded in Fig. 4, the *tall region* for T.

Lemma 6 *A triangle $T = ABC$ with $\alpha \geq \beta \geq \gamma$ is tall if and only if A lies in the tall region BJ; and then $h_c \geq c$, with equality precisely when A lies on the circular arc BIJ.*

Proof. Let A lie in the tall region BJ, and suppose the ray BA meets the semicircle CIB at P and the semicircle BIY at Q (Fig. 4). Then right triangles BCP and YBQ are congruent (indeed, a clockwise quarter-turn about I carries the first to the second), and it follows that $BQ = CP = h_c$. Consequently, $h_c > c$, $h_c = c$, or $h_c < c$ according as A lies on the (open) segment BQ, at Q, or beyond Q on the ray BP. □

Consequently a triangle (not isosceles right) has a side square if and only if it is tall. That side square is unique when it exists, it rests on the strictly shortest side of T, and its side is the longest altitude of T.

Although a tall obtuse triangle T has a side square S_c, that square can never be $S_{\min}$ because T can be moved into its interior by a suitable small motion. When T is tall and not obtuse, then in addition to a side square it also has a vertex square. We need to determine which of these two minimal squares is smaller.

Lemma 7 *Let $T = ABC$ be an acute or right triangle (but not isosceles right) with $h_c \geq c$. Then T has exactly two minimal squares, a side square S_c on c and a vertex square S_C at C, and S_C is smaller than, equal to, or larger than S_c precisely when $\gamma < 45°$, $\gamma = 45°$, or $\gamma > 45°$.*

Proof. Since $h_c \geq c$, we conclude from Lemma 5 that $h_a < a$ and $h_b < b$, so T has a side square S_c on c (whose side is h_c) and no other side squares. From Lemma 2 we see that T has a vertex square S_C at the vertex C and at no other vertex. So T has exactly two minimal squares, S_c and S_C.

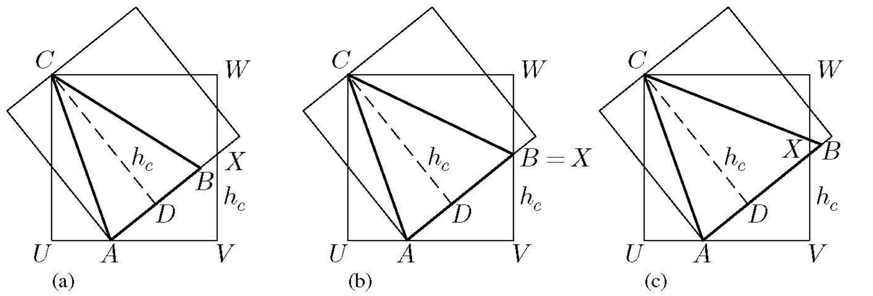


Fig. 5 Comparison of side and vertex squares

Let $CUVW$ be a square with side h_c positioned so that A lies on UV (Fig. 5). Let D be the foot of the altitude from C to AB, and let X be the point where the ray AB meets the side VW. Then since $\angle UCA = \angle ACD$ and $\angle DCX = \angle XCW$, $\angle ACX = \frac{1}{2}(2\angle ACD + 2\angle DCX) = \frac{1}{2}(90°) = 45°$. Consequently $\gamma < 45°$, $\gamma = 45°$, or $\gamma > 45°$ according as A-B-X, $B = X$, or A-X-B. If $\gamma < 45°$ so that A-B-X (Fig. 5(a)), then S_c is not the smallest square that contains T because a small motion would move T inside S_c; and consequently S_C must be smaller than S_c in this case. If $\alpha = 45°$, then $B = X$ (Fig. 5(b)), and $S_c = S_C$ by the uniqueness assertion of Lemma 2. And if $\alpha > 45°$ so that A-X-B (Fig. 5(c)) and if S_C were smaller than S_c, then T would fit in the larger square S_c with the vertex C at a corner, which it evidently does not do. □

The smallest square. Now it is easy to determine the smallest square that contains the given triangle. We continue to assume that the given triangle $T = ABC$ is positively oriented and $\alpha \geq \beta \geq \gamma$. In Fig. 6 (as in Fig. 4), MD is the mediator of the longest side BC of T, the circular arc DB has center C, the segment IK makes a 45° angle with BC, and the circular arc IJ has center at the point X for which $BMIX$ is a square.

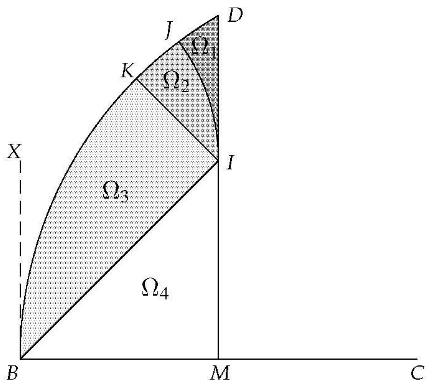


Fig. 6 The significant regions

The previous lemmas require four significant regions (Fig. 6). Recalling the normalization $a \geq b \geq c$, let

$$\begin{aligned} \Omega_1 &= \{A : \beta, \gamma > 45^\circ, \text{ and } AX \geq a/2\}, \\ \Omega_2 &= \{A : \beta > 45^\circ, \gamma \geq 45^\circ, \text{ and } AX \leq a/2\}, \\ \Omega_3 &= \{A \neq I : \beta \geq 45^\circ, \text{ and } \gamma \leq 45^\circ\}, \\ \Omega_4 &= \{A : \beta \leq 45^\circ\}, \end{aligned} \tag{4}$$

so that Ω_1 is the closed circular triangular region DJI minus the point I, Ω_2 is the closed circular triangular region KIJ minus the point I, Ω_3 is the closed circular triangular region BIK minus the point I, and Ω_4 is the closed triangular region BMI (including I). The point I, which represents the anomalous isosceles right triangle, is included in Ω_4 but excluded from Ω_1, Ω_2, and Ω_3.

Theorem 8 *Let $T = ABC$ be a given positively oriented triangle labeled in such a way that $\alpha \geq \beta \geq \gamma$, and let the four regions Ω_k be defined by* (4). *Then the smallest square $S_{\min}$ that contains T is*

$$S_{\min} = \begin{cases} S_A & \text{if } A \in \Omega_1, \\ S_c & \text{if } A \in \Omega_2, \\ S_C & \text{if } A \in \Omega_3, \\ S_{BC} & \text{if } A \in \Omega_4, \end{cases} \tag{5}$$

and T fits into a square S of side s if and only if $s \geq s_{\min}$, where the side $s_{\min}$ of $S_{\min}$ is given by

$$s_{\min} = \begin{cases} \dfrac{bc\cos\alpha}{\sqrt{b^2 - 2bc\sin\alpha + c^2}} & \textit{if } A \in \Omega_1, \\ a\sin\beta & \textit{if } A \in \Omega_2, \\ \dfrac{ab\cos\gamma}{\sqrt{a^2 - 2ab\sin\gamma + b^2}} & \textit{if } A \in \Omega_3, \\ \dfrac{a}{\sqrt{2}} & \textit{if } A \in \Omega_4. \end{cases} \tag{6}$$

Proof. See Fig. 6. If $A \in \Omega_1$, then T is not tall by Lemma 6; it has just three minimal squares, a vertex square at each vertex by Lemma 2; and the smallest of these is S_A by Lemma 3. The side of S_A is given by (2). If $A \in \Omega_2$ or Ω_3, then T is tall by Lemma 6. According to Lemma 7, it has just two minimal squares, a side square S_c and a vertex square S_C, and the smaller is S_c in Ω_2 and S_C in Ω_3. The side of S_c is $h_c = a\sin\beta$, and the side of S_C is given by (2). Finally, if $A \in \Omega_4$ the diagonal square S_{BC} is the only minimal square, and its side is $a/\sqrt{2}$. □

We have finally arrived at an answer to our original question: The triangle with sides a, b, c fits in the square of side s if and only if $s \geq s_{\min}$, where $s_{\min}$ is given by (6). Theorem 8 also provides a solution for the dual problem: Given a square S and a triangle T_0, find the largest triangle T similar to T_0 that fits in S. Indeed, if S has side s and T_0 has sides a_0, b_0, and c_0, then $s_0 = s_{\min}$ is the side of the smallest square that contains T_0, and the largest triangle T similar to T_0 that fits in S evidently has sides sa_0/s_0, sb_0/s_0, and sc_0/s_0.

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