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On the principal centers of a triangle

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Eight of the simplest triangle centers, namely H, I, J, L, M, N, O and I' (in the present notation) are shown to form a simple pattern of collinearities, which is completed by the addition of a ninth point K, a less well-known triangle center; see Figure 3.

Four of the well-known centers on the Euler line, namely L, O, M and O', are each a point of concurrence of four lines, each line containing one of the tritangent centers I_A, I_B, I_C and I and one other well-known triangle center. The sixteen other centers are: the Feuerbach points F_A, F_B, F_C and F; the Gergonne points G_A, G_B, G_C and G; the Nagel points N_A, N_B, N_C and N; and the countercenters J_A, J_B, J_C and J.

1 Introduction

A comprehensive publication by Kimberling [3] has listed many of the principal centers of a triangle ABC as well as the various collinearities among them. Most were discovered by accurate computer "drawing". Some but not all have received verification by formal proof.

Die nachfolgende Arbeit ist ein Beitrag zur Elementargeometrie, genauer zur Dreiecksgeometrie. Bekanntlich liegen die Schnittpunkte der Höhenlinien, der Seitenhalbierenden und der Mittelsenkrechten eines Dreiecks auf einer Geraden, der Euler-Geraden. Die Verbindung des Schnittpunkts der Winkelhalbierenden und des Schwerpunkts führt auf die sogenannte Nagel-Gerade, auf der weitere besondere Punkte des Dreiecks liegen. Der Autor stellt hier einen neuen interessanten Punkt vor, der die bekannten neun besonderen Punkte eines Dreiecks in harmonischer Weise zu einem regelmässigen Muster ergänzt. *jk*

In the present paper it will be shown that eight of the most elementary centers fall into a group which when augmented by the addition of one further point, not listed by Kimberling [3], form a remarkable family as shown schematically in Figure 3. Here

O is the circumcenter (meet of the perpendicular bisectors of the three sides of ABC),

I is the incenter (meet of the bisectors of the three angles),

M is the median point (meet of the three medians, each joining a vertex, say A, to the mid-point A' of the opposite side BC),

H is the orthocenter (meet of the three altitudes),

L is the Longchamps point (reflection of H in O),

N is the Nagel point (meet of the three lines AX_A, BY_B, CZ_C, where X_A is the point of contact of the excircle opposite A with the side BC),

J is the Yff point or countercenter (meet of the perpendiculars from the three excenters I_A, I_B, I_C to the corresponding sides of ABC).

Also

I' is the Spieker point (incenter of the median triangle $A'\,B'\,C'$),

J' is the countercenter of $A'\,B'\,C'$.

The new point is K. It may be defined by construction as the meet of NH and LI. M is here shown to be the median point of the triangle KLN, so that K also lies on JM. Other properties of K remain to be explored.

In addition to this family of centers which we may call *rational* because of the ratios of parallel segments, we also discuss the relationship of the Gergonne point G, and other associated points, to the scheme of Figure 3. In addition we show that each of the points $LOMO'$ on the Euler line is itself a 4-fold point of concurrence of lines joining other centers of the triangle ABC.

2 The Euler and Nagel lines

As is well-known, the points H, M, O and L all lie (with many other centers) on the Euler line of the triangle ABC, while the points I, M, I' and N all lie on a second line, which after Hofstadter [2] we may call the Nagel line. The Euler line $HMOL$ and the Nagel line $IMI'N$ thus intersect in the median point M, as shown in Figure 1. Moreover, the corresponding segments of these two lines are in proportion:

$$\left.\begin{aligned} HM : MO : OL &= 2:1:3 \\ IM\ : MI' : I'N &= 2:1:3 \end{aligned}\right\} \tag{2.1}$$

from which it follows that HI, OI' and LN are all parallel and

$$HI = 2\,OI' = \frac{1}{2}LN. \tag{2.2}$$

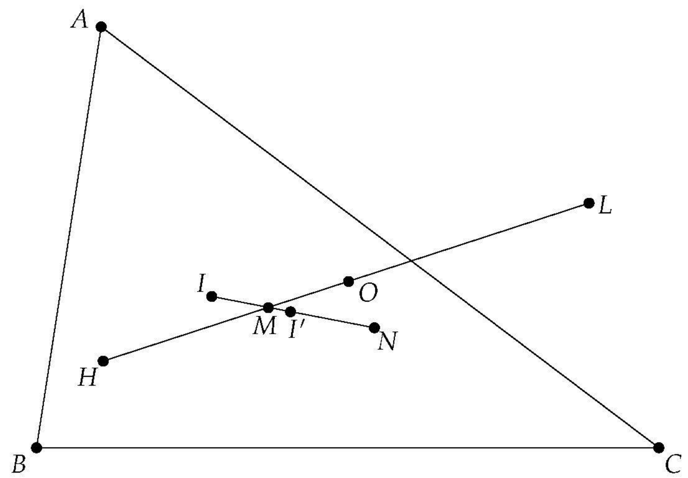

Fig. 1

3 The countercenter J

In the present note we are specially interested in another center listed by Kimberling [3] which has received no name, though certainly it is one of the simplest centers and, as we shall see, is intimately related to those just mentioned. For reasons that will appear, we name this point the *countercenter* and denote it by the letter J.

J may be defined as the point of concurrence of the three perpendiculars, each from one of the excenters I_A, I_B, I_C to the corresponding side of ABC. In Kimberling's [3] notation it is X_{40}. Apparently it was discovered by Peter Yff.

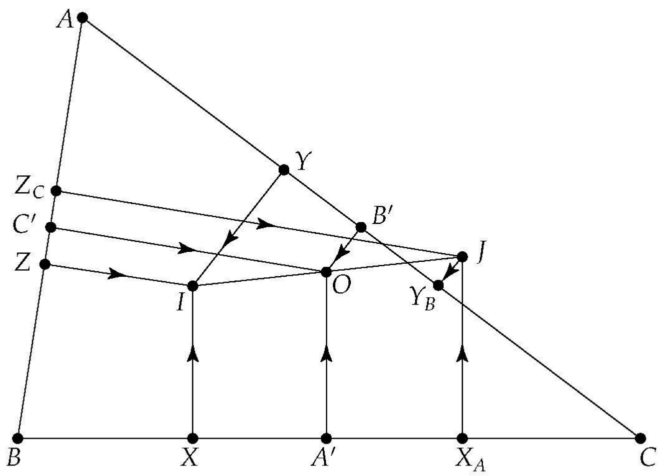

Fig. 2

One sees immediately from Figure 2 that O is the mid-point of IJ. For, the foot X_A of the perpendicular from J to the side BC is simply the point of contact of the excircle center I_A with the side BC. Similarly the foot of the perpendicular from I to BC is the point of contact X of the incircle with BC. But X and X_A are equidistant from the mid-point A' of BC, which is the foot of the perpendicular from the circumcenter O. Similarly for Y and Y_B, and Z and Z_C. Hence the result.

We see that J in a sense "counterbalances" the incenter I with respect to the circumcenter O.

The position of J relative to the other central points is clarified in Figure 3, where the Euler and Nagel lines have been artificially made more symmetric. Note that as O is the mid-point of both IJ and of HL, the figure $JLIH$ must be a parallelogram, hence JL is parallel to HI. But we previously saw that NL is parallel to HI. Therefore J lies on LH. Moreover JL equals HI. But we saw also that $NL = 2HI$. Therefore

$$J \text{ is the mid-point of } NL. \tag{3.1}$$

Incidentally J is also the circumcenter of the triangle $I_A I_B I_C$; see [1, Section 231].

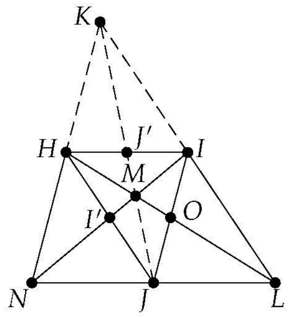

Fig. 3 Schematic diagram, showing collinearities.

4 Completion of the scheme

We can now complete the figure in a remarkable way (see Figure 3). Let us extend the lines NH and LI so as to meet in a point K. Since HI is parallel to NL, the triangles KHI and KNL are similar. But $NL = 2HI$, therefore $KN = 2KH$ and $KL = 2KI$. In other words, H and I are the mid-points of KN and KL respectively. Hence

$$M \quad \text{is the median point of the triangle } KLN. \tag{4.1}$$

From which it follows further that

$$J,\ K \text{ and } M \text{ are collinear} \tag{4.2}$$

and

$$KM = 2MJ. \tag{4.3}$$

We can state also the following theorem concerning the well-established centers H, I, J, L, M and N:

$$NH,\ JM \text{ and } LI \text{ are concurrent.} \tag{4.4}$$

This focuses attention on the point of concurrence K which, so far as is known to the author, has not been closely examined in the literature.

K is itself a central point, though not a simple one, of the triangle ABC. The triangle $A_1B_1C_1$ formed by the parallels to the sides of ABC through the opposite vertices is called the *anticomplimentary* triangle to ABC. Clearly $A_1B_1C_1$ is in a 2 : 1 homotheticity with ABC, center M. Since in Figure 3 $JK = 2MJ$ we see that

K is the countercenter of the anticomplimentary triangle.

Note also that the point J' which is the mid-point of HI is the countercenter of the median triangle $A'B'C'$.

We see then that the ten points

$$H,\ I,\ J,\ K,\ L,\ M,\ N,\ O,\ I' \quad \text{and} \quad J' \tag{4.5}$$

form a family. The ratios of parallel segments are all rational numbers. This basic scheme we may call the *rational family* of triangle centers.

The nine-point center O', which is the meet of $I'J'$ with the Euler line $HMOL$, can also be considered as a member of this family.

Hofstadter [2] has proposed a somewhat different scheme that includes H, I, M, O, I' and also O', the circumcenter of $A'B'C'$, but excludes J and L. His scheme is completed by extending HI and NO to meet in a point T. However, T is not a central point of the triangle ABC.

5 Coordinates for K

From Figure 3 we can write down the following simple relations:

$$K + N = 2H, \quad K + L = 2I, \quad K + 2J = 3M, \tag{5.1}$$

also

$$I + J = 2O, \quad H + L = 2O, \quad N + 2I = 3M. \tag{5.2}$$

So to express the coordinates of K in terms of those of O, I and M we have

$$K = 3M - 2J = 3M - 2(2O - I) = 2I - 4O + 3M. \tag{5.3}$$

The trilinear coordinates of I, O and M are as follows:

$$\left.\begin{aligned} I &= r(1,1,1) \\ O &= R(\cos A, \cos B, \cos C) \\ M &= \frac{2\Delta}{3}\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \end{aligned}\right\} \tag{5.4}$$

where r is the inradius, R the circumradius and Δ the area of the triangle ABC, and a, b, c the lengths of the sides. Making use of the well-known relations

$$\left.\begin{aligned} \Delta &= abc/4R \\ (a,b,c) &= 2R(\sin A, \sin B, \sin C) \end{aligned}\right\} \tag{5.5}$$

and also (see [5])

$$r/R = \cos A + \cos B + \cos C - 1 \tag{5.6}$$

we deduce that $K = (\xi, \eta, \zeta)$ where

$$\xi = 2R(\sin B \sin C + \cos B + \cos C - \cos A - 1) \tag{5.7}$$

and similarly for η and ζ.

Since $\cos A = -\cos(B + C)$ we can also write

$$\begin{aligned}\xi &= 2R[\cos B \cos C + \cos B + \cos C - 1]\\ &= 2R[(\cos B + 1)(\cos C + 1) - 2]\\ &= 4R\left(2\cos^2 \frac{1}{2}B \cos^2 \frac{1}{2}C - 1\right)\end{aligned}$$

and similarly for η and ζ.

6 The Gergonne point and related centers

One of the simpler centers of a triangle ABC is the Gergonne point G, defined as the meet of the three lines joining the vertices A, B, C to the corresponding points of contact X, Y, Z of the incircle with the three sides; see Figure 2. Kimberling [3] found by computer that G lies on IL, and this was also discovered, and proved rigorously, by the present author [4]. From Figure 4 it is clear that by the 1 : 2 homotheticity center M, the Gergonne point G' of the median triangle $A'B'C'$ must lie on HJ. Thus G', I' and H are collinear, as was pointed out in [4].

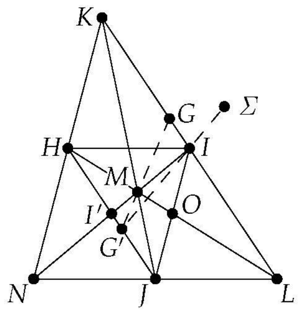

Fig. 4

Consider on the other hand the "symmedian point" Σ of the triangle ABC, that is to say the point of concurrence of the three symmedians, which are the lines obtained by reflecting the lines AA', BB', CC' in AI, BI, CI respectively. It is known [3] that Σ is collinear with I and the symmedian point of the excentral triangle $I_A I_B I_C$, which we denote by W. The latter point is Kimberling's X_9 and is known to lie on both MG and on HI' (see [3, p. 173]). Hence W coincides with G'. In other words,

G' is the symmedian point of the excentral triangle $I_A I_B I_C$.

Thus the Gergonne point G is, through G' and Σ, intimately connected with the *Lemoine geometry* of the triangle ABC; see [1, pp. 252–274].

7 4-fold points of concurrence

Just as the countercenter J can be constructed from the four tritangent circles (centers I, I_A, I_B and I_C) by specialising one center I, so we can construct three other countercenters J_A, J_B and J_C, by specialising each of I_A, I_B and I_C. Thus we may define J_A as the meet of the three perpendiculars, the first from I to the side BC, the second from I_B to AB and the third from I_C to AC. Corresponding definitions apply to J_B and J_C. Then it may easily be seen that O is also the mid-point of I_AJ_A, I_BJ_B and I_CJ_C. Thus O is the point of concurrence of the four lines IJ, I_AJ_A, I_BJ_B and I_CJ_C. The Figure $JJ_AJ_BJ_C$ is the reflexion of $II_AI_BI_C$ in the point O.

It has been seen previously [4] that two other points on the Euler line, namely L and M, are each 4-fold points of concurrence. For L is the meet of GI, G_AI_A, G_BI_B and G_CI_C, while M is the meet of IN, I_AN_A, I_BN_B and I_CN_C. Is there a fourth such point lying on the Euler line?

It appears that there is, namely the nine-point center O'. For, the Feuerbach point F, which is the point of contact of the incircle with the nine-point circle, must lie on the line IO'. Hence IF passes through O', see Figure 5. Also there are three analogous points F_A, F_B and F_C, in which the nine-point circle touches the three excircles. Hence I_AF_A, I_BF_B and I_CF_C also pass through O'.

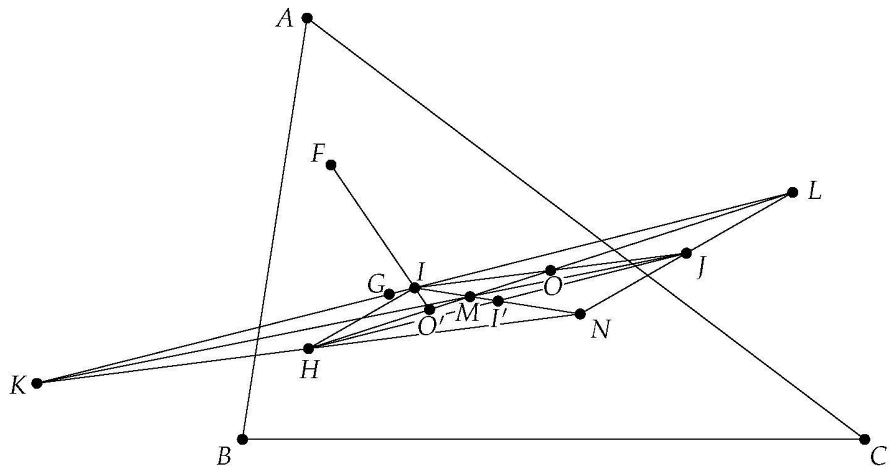

Fig. 5

Note that F can be constructed independently of O'; see Figure 6. For if $\bar{X}$ is the image point of X in the angle bisector AI as in Figure 6, then F lies on $A'\bar{X}$, where A' is the mid-point of BC (see [1, Section 215]). Similarly F lies on $B'\bar{Y}$ and $C'\bar{Z}$. F is therefore the meet of $A'\bar{X}, B'\bar{Y}$ and $C'\bar{Z}$.

Thus we have four 4-fold points of concurrence L, O, M and O' all lying on the Euler line of ABC. The centers I, I_A, I_B and I_C are also 4-fold points of concurrence of the system.

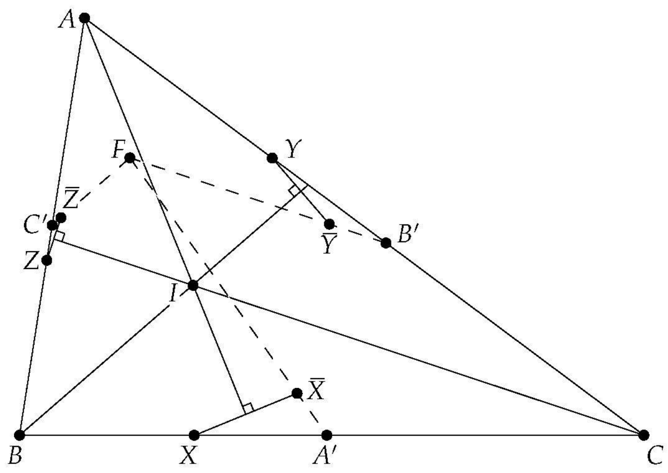

Fig. 6

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