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Planar Rectangular Sets and Steiner Symmetrization

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1 Introduction

Let K be a closed convex set in the plane. In **[1]**, Danzer establishes the following pretty result.

Theorem 1. *If no rectangle inscribed in* K *has exactly three of its vertices on the boundary of* K*, then* K *is a circular disk.*

We generalize Danzer's characterization in the following way. Let OX, OY be given, fixed orthogonal axes in the plane. We say that K is a *rectangular* set if no inscribed rectangle with edges parallel to the given axes has exactly three of its vertices on the boundary of K. Some anomalies can occur in this new setting. For example, if K has two adjacent perpendicular edges which are parallel to the axes, there is an infinite number of 'inscribed' rectangles having just three vertices on the boundary of K. We therefore interpret *inscribed* here to imply that the given rectangle is the largest in the family of homothetic rectangles having vertices on the boundary of K. This is the assumption we would make if talking about an incircle of K.

We now ask if it is possible to characterize in some way the family $\mathcal{R}$ of rectangular sets. We note that $\mathcal{R}$ contains sets which are symmetric about either or both of the axes.

Let K be a closed convex set in the plane, and OX, OY given, fixed orthogonal axes. We say that K is a *rectangular* set if no inscribed rectangle with edges parallel to the given axes has exactly three of its vertices on the boundary of K. We show that if S_X, S_Y denote Steiner symmetrizations about the axes OX, OY respectively, then K is a rectangular set (relative to these axes) if and only if $S_XS_Y(K) = S_YS_X(K)$. *psc*

It turns out that the family $\mathcal{R}$ has a nice characterization in terms of Steiner symmetrization, which we now define. Let OA be a given line – the *axis* l of symmetrization. For each point p on OA let $u(p)$ be the line through p which is perpendicular to l. The set $u(p) \cap K$ is either the empty set, a point, or a line segment. If it is the empty set, we define $B(p)$ to be the empty set. If it is a point, we define $B(p)$ to be the point p. If it is a line segment, we define $B(p)$ to be the segment of equal length whose midpoint is p and which lies on $u(p)$. We now define K_A by

$$K_A = \cup_{p \in l} B(p).$$

The process of obtaining K_A from K in this way is called *Steiner symmetrization* about the line OA. Properties of this well-known and useful form of symmetrization can be found, for example, in Eggleston **[2]**.

We shall establish the following connection between Steiner symmetrization and the family $\mathcal{R}$ of rectangular sets.

Theorem 2. *Let S_X, S_Y denote symmetrizations about the axes OX, OY respectively. Then K is a rectangular set (relative to these axes) if and only if*

$$S_X S_Y(K) = S_Y S_X(K).$$

2 Proof of Theorem 2

For consistency in naming in the proof, we drop the function notation used in the statement of the theorem, and use $S_X S_Y$, for example, to mean first apply S_X and then apply S_Y. We shall also use the words *horizontal* and *vertical* to describe lines which are parallel to OX, OY respectively.

First we suppose that K is a rectangular set. Let A be a point on the boundary of K. By assumption, A will be a vertex of a (perhaps degenerate) rectangle $ABCD$ whose four vertices lie on the boundary of K (see Figure 1).

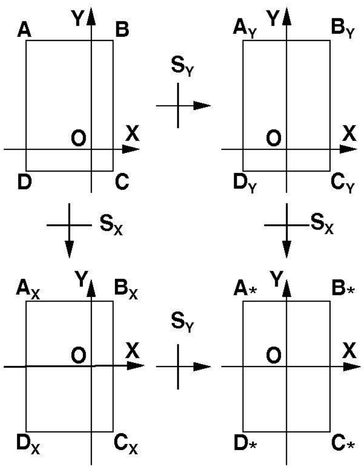

Fig. 1

Let $AB = 2x$ and $BC = 2y$. If we symmetrize K using S_Y to obtain a symmetrized set K_Y, then A will map to a point A_Y, a vertex of a rectangle $A_Y B_Y C_Y D_Y$, inscribed in K_Y, and congruent to $ABCD$. For, under the symmetrization, lengths AB, DC are preserved, and the image segments $A_Y B_Y, D_Y C_Y$ are centred on the axis OY. In particular, A_Y has x-coordinate x, and $A_Y D_Y = 2y$. If we now symmetrize K_Y using S_X to obtain set K_{YX}, then A_Y maps to a point A_{YX}, a vertex of a rectangle inscribed in K_{YX} and congruent to $ABCD$. For, under the symmetrization, lengths $A_Y D_Y, B_Y C_Y$ are preserved, and the image segments $A_Y D_Y, B_Y C_Y$ are centred on the axis OX. In particular, A_{YX} has x-coordinate x, and y-coordinate y.

It is clear from the symmetry of X and Y in this argument that the image of A under the product $S_X S_Y$ will be $A_{XY} = A_{YX} (= A_*$ in Figure 1). We deduce that $K_{XY} = K_{YX}$.

Now let us suppose that K is a set which has the same image under $S_Y S_X$ as it does under $S_X S_Y$. Thus $K_{YX} = K_{XY}$. We wish to show that K is a rectangular set. We observe that it will be sufficient to establish this result for the case when K is a polygon. The general case will then follow using a standard approximation argument. We may thus assume that the final symmetrized set $K_{XY} = K_{YX}$ is the convex hull of a finite family of rectangles having horizontal and vertical edges. If each of these rectangles occurs as the image of an inscribed rectangle in K, then K is a rectangular set, and there is nothing to prove. Suppose then that one of these rectangles, $R_{XY} = R_{YX}$ does not occur in this way. Let this rectangle have horizontal and vertical dimensions $2x$, $2y$ respectively. Suppose too that y is the largest number for which this happens.

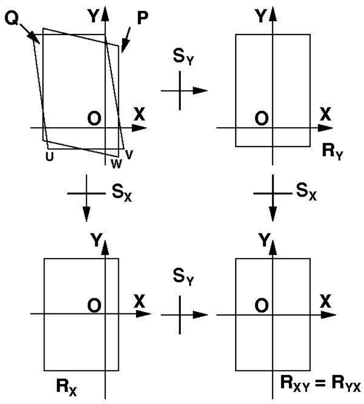

Fig. 2

Now R_{XY} is the image under S_Y of a set R_X (see Figure 2). In fact R_X is itself a rectangle, since it is inscribed in a set K_X which is symmetric about the X-axis. Further, R_X has horizontal and vertical dimensions $2x$, $2y$ respectively. Now rectangle R_X occurs as the image under symmetrization S_X of a set P inscribed in the original set K. By the properties of symmetrization, this set P must be a parallelogram having one pair of vertical parallel edges. The length of each of these parallel edges is $2y$, and the distance between them is $2x$. In the same way, R_{XY} occurs as the image under $S_Y S_X$ of a parallelogram Q inscribed in K having two horizontal parallel edges; the length of each of these parallel edges is $2x$, and the distance between them is $2y$.

If either of P, Q is a rectangle, then P, Q will coincide, as we have already seen that the image of a rectangle inscribed in K having horizontal and vertical edges is the same under the two successive symmetrizations, no matter which order of symmetrization is used. Hence parallelogram P extends strictly above or below the parallel horizontal edges of parallelogram Q. Inverting the figure if necessary, we may assume that P extends strictly below Q. Let UV denote the bottom horizontal edge of Q, labelled as in Figure 2, and W the vertex of P which lies below it. Then points U, W, V lie in an anti-clockwise order on the boundary of K. Since symmetrization is a continuous transformation, U, W, V will map under the successive symmetrizations S_Y, S_X to image points U^*, W^*, V^* lying in anti-clockwise order on the boundary of K_{XY}. But U^*V^* is the bottom edge of R_{XY}. It follows that W^* is the vertex of a rectangle inscribed in K_{XY} which does not arise as the image of a rectangle inscribed in K. Further, the vertical dimension of this rectangle exceeds the vertical dimension $2y$ of R_{XY} which was chosen to be maximal. This contradiction establishes the theorem.

3 Final Comment

The class of rectangular sets appears naturally here in terms of successive orthogonal symmetrizations; to my knowledge, this class does not occur elsewhere in the literature. It would be interesting to investigate whether this class of sets has other special properties.

References

[1] Danzer, L. W., "A characterization of the circle", *Proc. Symp. Pure. Maths* VII (1963), 99–100.

[2] Eggleston, H. G., Convexity, Cambridge Tract No. 47 (1963), Cambridge University Press.

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