Kleine Mitteilungen

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In Bild 5 ist dargestellt, wie man so aus dem *Ovoid* bzw. «falschen Keplerschen Ei» $m_{\frac{1}{3}}$ ([4], S. 247) mit der Gleichung

$$y_1^3 = E(y_1^2 + y_2^2)$$

die bekannte *Cayley-Sextik* $s_{\frac{1}{3}}$, festgelegt durch

$$E[4E(x_1^2 + x_2^2) - x_1] = 27(x_1^2 + x_2^2)^2,$$

erhält, welche auch als Ort aller Scheitel von Parabeln aufgefasst werden kann, die einen Kreis berühren und einen festen Punkt auf dessen Peripherie als Brennpunkt haben.

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Kleine Mitteilungen

Constructing the Neumann series – an example

Constructing the Neumann series is a way of finding the solution of the linear integral equation of the second kind with the parameter as a series in powers of that parameter [2, 3].

In this note, we shall construct the Neumann series for the Fredholm type integral equation

$$\theta(\xi) = \theta_0 + v \int_0^{\ell} \theta(\varepsilon) K(\xi, \varepsilon) \, d\varepsilon, \tag{1}$$

where ℓ is a real constant, $0 < \ell < +\infty$; θ_0 is a finite real (or complex) constant; v is a complex parameter; ξ and ε are the real variables

$$0 \leq \xi, \varepsilon \leq \ell; \tag{2}$$

$\theta(\xi)$ is the unknown function and

$$K(\xi, \varepsilon) = \{\xi \text{ if } \varepsilon \geq \xi;\ \varepsilon \text{ if } \varepsilon < \xi\}. \tag{3}$$

The integral equation (1) arises in the theory of the inviscid, incompressible, weightless planar jet flowing from a nozzle onto the surface of a heavy liquid; the reader is referred to the equation (2.3) and the Figures 1 and 2 of the reference [5] for details.
In the general form, the Neumann series for the equation (1) may be written as

$$\theta(\xi) = \theta_0 + \theta_0 \sum_{m=1}^{\infty} v^m \int_0^{\ell} K_m(\xi, \varepsilon)\, d\varepsilon, \tag{4}$$

where $K_1(\xi, \varepsilon) = K(\xi, \varepsilon)$ and $K_m(\xi, \varepsilon)$ $(m = 2, 3, \ldots)$ are the m-th iterated kernels

$$K_m(\xi, \varepsilon) = \int_0^{\ell} K(\xi, \tau) K_{m-1}(\tau, \varepsilon)\, d\tau, \quad m = 2, 3, \ldots, \tag{5}$$

[3]. {The solution of the integral equation (1) as the series (4) is obtained by using the method of successive approximations [3]. The series (4) converges, to the solution of (1), for all v such that $|v|$ is smaller than a certain positive number [2]}.
To «transform» the Neumann series (4) to its concrete form, we have to evaluate the integrals of (4).
For the integral of (4) with $m = 1$ we have

$$\int_0^{\ell} K_1(\xi, \varepsilon)\, d\varepsilon = -\tfrac{1}{2}(\xi^2 - 2\ell\xi). \tag{6}$$

Now, from (5) it follows that

$$\int_0^{\ell} K_m(\xi, \varepsilon)\, d\varepsilon = \int_0^{\ell} K(\xi, \tau)\, d\tau \int_0^{\ell} K_{m-1}(\tau, \varepsilon)\, d\varepsilon \quad \text{for} \quad m = 2, 3, \ldots. \tag{7}$$

Using (7), one obtains

$$\int_0^{\ell} K_2(\xi, \varepsilon)\, d\varepsilon = \frac{1}{24}(\xi^4 - 4\ell\xi^3 + 8\ell^3\xi) \tag{8}$$

and then

$$\int_0^{\ell} K_3(\xi, \varepsilon)\, d\varepsilon = -\frac{1}{720}(\xi^6 - 6\ell\xi^5 + 40\ell^3\xi^3 - 96\ell^5\xi). \tag{9}$$

Next, we rewrite (6), (8) and (9) as

$$\int_0^{\ell} K_1(\xi, \varepsilon)\, d\varepsilon = -\frac{(2\ell)^2}{2!}\left[\left(\frac{\xi}{2\ell}\right)^2 - \left(\frac{\xi}{2\ell}\right)\right], \tag{10}$$

$$\int_0^{\ell} K_2(\xi,\varepsilon)\,d\varepsilon = \frac{(2\ell)^4}{4!}\left[\left(\frac{\xi}{2\ell}\right)^4 - 2\left(\frac{\xi}{2\ell}\right)^3 + \left(\frac{\xi}{2\ell}\right)\right], \tag{11}$$

$$\int_0^{\ell} K_3(\xi,\varepsilon)\,d\varepsilon = -\frac{(2\ell)^6}{6!}\left[\left(\frac{\xi}{2\ell}\right)^6 - 3\left(\frac{\xi}{2\ell}\right)^5 + 5\left(\frac{\xi}{2\ell}\right)^3 - 3\left(\frac{\xi}{2\ell}\right)\right]. \tag{12}$$

Now we can show that the integrals (10)–(12) may be expressed in terms of the Euler polynomials. (For the definition of the Euler polynomials and the description of their properties, the reader is referred to [1, 4].)
Namely, the expressions for the Euler polynomials $E_m(t)$ with $m = 2$, 4 and 6 are [4]

$$E_2(t) = t^2 - t, \quad E_4(t) = t^4 - 2t^3 + t, \quad E_6(t) = t^6 - 3t^5 + 5t^3 - 3t. \tag{13}$$

From (10)–(13), it follows that

$$\int_0^{\ell} K_m(\xi,\varepsilon)\,d\varepsilon = \frac{(-1)^m}{(2m)!}(2\ell)^{2m} E_{2m}\left(t = \frac{\xi}{2\ell}\right) \quad \text{for} \quad m = 1, 2 \text{ and } 3. \tag{14}$$

Now we shall use induction on m to prove that the result (14) is valid for any positive integer m.
Let us assume that (14) is valid for a certain positive integer m. Then, substituting (14) into the equation (7) with m replaced by $(m+1)$, one finds

$$\int_0^{\ell} K_{m+1}(\xi,\varepsilon)\,d\varepsilon = \frac{(-1)^m}{(2m)!}(2\ell)^{2m} I_m, \tag{15}$$

where

$$I_m = \int_0^{\ell} K(\xi,\tau)\,E_{2m}\left(t = \frac{\tau}{2\ell}\right)d\tau. \tag{16}$$

Taking into account (3), we rewrite the equation (16) as

$$I_m = (2\ell)^2 I_{m1} + 2\ell\,\xi\,I_{m2}, \tag{17}$$

where

$$I_{m1} = \int_0^{\xi/(2\ell)} t\,E_{2m}(t)\,dt, \quad I_{m2} = \int_{\xi/(2\ell)}^{1/2} E_{2m}(t)\,dt. \tag{18}$$

From the identity [1, 4]

$$E'_{m+1}(t) = (m+1)\,E_m(t), \quad m = 0, 1, 2, \ldots, \tag{19}$$

it follows that

$$E_{2m}(t) = (2m+1)^{-1} E'_{2m+1}(t), \quad m = 0, 1, 2, \ldots \tag{20}$$

After substituting the expression (20) for $E_{2m}(t)$ into the integrand of the integral I_{m1} and integrating by parts, we obtain

$$I_{m1}=\frac{1}{2m+1}\left[\frac{\xi}{2\ell}E_{2m+1}\left(t=\frac{\xi}{2l}\right)-\int_{0}^{\xi/(2\ell)}E_{2m+1}(t)\,dt\right]. \tag{21}$$

Further, from (19) we get

$$E_{2m+1}(t)=(2m+2)^{-1}E'_{2m+2}(t),\quad m=0,1,2,\ldots. \tag{22}$$

We also have [1, 4]

$$E_m(0)=-2(m+1)^{-1}(2^{m+1}-1)B_{m+1},\quad m=1,2,\ldots, \tag{23}$$

where B_{m+1} are the Bernoulli numbers.
Since all the Bernoulli numbers with odd indices greater than 1 are equal to zero [1, 4], we find

$$E_{2m+2}(0)=0,\quad m=1,2,\ldots. \tag{24}$$

After substituting into the integrand of the integral of (21) the expression for the polynomial $E_{2m+1}(t)$ from (22), integrating and taking into account (24), one obtains

$$I_{m1}=\frac{1}{(2m+1)}\frac{\xi}{2\ell}E_{2m+1}\left(t=\frac{\xi}{2\ell}\right)-\frac{1}{(2m+1)(2m+2)}E_{2m+2}\left(t=\frac{\xi}{2\ell}\right). \tag{25}$$

Now, since [1, 4]

$$E_m(1/2)=2^{-m}E_m,\quad m=0,1,2,\ldots, \tag{26}$$

where E_m are the Euler numbers, and since all the Euler numbers with odd indices are equal to zero [1], we find

$$E_{2m+1}(1/2)=0,\quad m=0,1,2,\ldots. \tag{27}$$

Taking into account (20) and (27), we get

$$I_{m2}=-\frac{1}{2m+1}E_{2m+1}\left(t=\frac{\xi}{2\ell}\right). \tag{28}$$

From (17), (25) and (28), it follows that

$$I_m=-\frac{(2\ell)^2}{(2m+1)(2m+2)}E_{2m+2}\left(t=\frac{\xi}{2\ell}\right) \tag{29}$$

and upon substituting this result for I_m into the equation (15), we get

$$\int_0^{\ell} K_{m+1}(\xi,\varepsilon)\,d\varepsilon = \frac{(-1)^{m+1}}{(2m+2)!}(2\ell)^{2m+2} E_{2m+2}\left(t = \frac{\xi}{2\ell}\right). \tag{30}$$

It is seen that (30) is (14) with m replaced by $m+1$; thus, we have proved that (14) is valid for any positive integer m.

We now substitute (14) into (4) and take into account that $E_0(t) \equiv 1$ [4] to find

$$\theta(\xi) = \theta_0 \sum_{m=0}^{\infty} (-1)^m \frac{(4\nu\ell^2)^m}{(2m)!} E_{2m}\left(t = \frac{\xi}{2\ell}\right). \tag{31}$$

The expression (31) is the Neumann series for the integral equation (1), in its concrete form.

With respect to (31), let us make two comments.

First, the series (31) may be summed. Namely, if we use the expansion 6.3.4.2 of [4], we find that

$$\theta(\xi) = \theta(\xi,\nu) = \theta_0 \sec\sqrt{\nu}\,\ell \cos\sqrt{\nu}\,(\ell - \xi). \tag{32}$$

It is possible to verify that the function $\theta(\xi,\nu)$, by (32), is the solution of the integral equation (1) for $\nu \neq \nu_j$, where

$$\nu_j = [(2j-1)\pi/(2\ell)]^2, \quad j = 1, 2, \ldots. \tag{33}$$

The values $\nu = \nu_j$ are the characteristic values of the kernel $K(\xi,\varepsilon)$. [The easiest way to find those values – or to find the solution $\theta(\xi) = \theta(\xi,\nu)$ of (1) – is, of course, to first reduce (1) to the boundary-value problem for the ordinary differential equation (of the second order) and then to consider that problem, instead of considering the integral equation (1) directly].

The Neumann series (31) may be restored, starting from (32), by writing out the series expansion of the function $\theta(\xi,\nu)$ in powers of ν.

Second, since the series (31) is the Maclaurin series with respect to ν of the function $\theta(\xi,\nu)$, it is evident that it converges [absolutely, to $\theta(\xi,\nu)$] if $|\nu| < \pi^2/(2\ell)^2 = \nu_1$ and that it diverges if $|\nu| > \nu_1$.

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Comment on curves in 2- and 3-dimensional Minkowski space

This article presents a correction and discussion of the results published by Graciela Silvia Birman in the paper «On L^2 and L^3» (see [2]). The result given there for L^3 is shown to be wrong. Furthermore, the kind of four-vertex-theorem proved there is trivial and it will be shown here that a more suitable version could be formulated, which is easy to see as well, but contains more information on the geometry of the curve under consideration.

Preliminaries

As in [2] we consider $\boldsymbol{R}^n$ for $n = 2, 3$ with the Minkowski or Lorentzian inner product

$$\langle \vec{x}, \vec{y} \rangle := -x_1 y_1 + \sum_{i=2}^{n} x_i y_i,$$

where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$. A vector $\vec{x}$ is called light-like or null, if $\langle \vec{x}, \vec{x} \rangle = 0$. Similarly it is called space-like respectively time-like if this quantity is > 0 respectively < 0. Considering simultaneously the standard Euclidean scalar product on $\boldsymbol{R}^n$, this can be described by the Euclidean angle between $\vec{x}$ and the hyperplane H spanned by the last $n-1$ coordinate axes: If this angle ranges in $(\frac{\pi}{4}, \frac{\pi}{2}]$ resp. $[0, \frac{\pi}{4})$, then $\vec{x}$ is time-like respectively space-like, if it is just $\frac{\pi}{4}$, then the vector is light-like.

Curves in Minkowski 3-space

In [2] a classification is proposed of regular curves in Minkowski 3-space which have light-like tangents only. The author states that these are exactly the light-like straight lines, i.e., the light-like geodesics of Minkowski 3-space. The proof of this statement must contain an error, because it is not true as can be seen from the following example: Let

$$\vec{\alpha}\colon \boldsymbol{R} \to \boldsymbol{R}^3, \quad \vec{\alpha}(t) := (t, \cos t, \sin t),$$

be a helix of constant (Euclidean) slope 1 with respect to the (x_2, x_3)-plane H. Then, interpreted in Minkowski geometry, this is a curve with light-like tangents everywhere that is different from a straight line.

Keeping in mind the Euclidean interpretation of light-like vectors different from $\vec{o}$, we get that the curves with light-like tangents everywhere are exactly the curves of constant slope 1 with respect to H. These curves have been studied very frequently in classical differential geometry (see e.g. [1], [4]).

Curves in the Minkowski plane

In [2] a formula for the Lorentzian curvature for a nowhere light-like curve in the Minkowski plane has been developed which should indicate that in the light-like limit its

curvature does not exist. But it is not immediately clear that the limit does not exist though the expression given there is not defined at points with light-like tangent. Furthermore the existence of at least four points with light-like tangent for a simply closed convex curve is presented in [2] as a Lorentzian version of the four-vertex-theorem.
The latter statement can easily be shown for every closed curve in the Minkowski plane, even if it has self-intersections: Let $\vec{\alpha}: S^1 \to \boldsymbol{R}^2$ be a closed regular curve. Then, according to our Euclidean picture, the points of $\vec{\alpha}$ having light-like tangents are exactly those where its Euclidean unit normals have the form $\varepsilon(1,1)$ respectively $\varepsilon(1,-1)$ with $\varepsilon \in \{-1,1\}$. These points are characterized by the property that the derivative of the function

$$h_1(t) := \alpha_1(t) + \alpha_2(t) \quad \text{respectively} \quad h_2(t) := \alpha_1(t) - \alpha_2(t)$$

vanishes. S^1 being compact we get at least two such points for each function, giving at least four light-like tangents of $\vec{\alpha}$, because the derivatives of h_1 and h_2 cannot vanish simultaneously.
For a strictly convex smooth curve in the Minkowski plane we have exactly four light-like tangents. If $t_1, \ldots, t_4 \in S^1$ are the corresponding parameter values, then each of the four arcs between t_1 and t_2, t_2 and t_3, t_3 and t_4, t_4 and t_1 is of constant causal character, i.e. time-like or space-like everywhere. The curvature is defined for each of these arcs and it will be shown that it attains an extremal value on each of them. This will give a Lorentzian analogue of the four-vertex-theorem.
Without loss of generality let $\vec{\alpha}: [a,b] \to \boldsymbol{R}^2$ be a strictly convex smooth regular curve having light-like tangents at the parameter values a and b and time-like tangents on (a,b). After a possible change of the orientation of the coordinate axes we may assume that the tangent at $\vec{\alpha}(a)$ is a positive multiple of $(1,-1)$ and that at $\vec{\alpha}(b)$ a positive multiple of $(1,1)$. Let $\vec{\beta}$ denote the reparametrization of $\vec{\alpha}|_{(a,b)}$ by time-like arc length s (proper time), i.e. $\vec{\alpha}(t) = \vec{\beta}(s(t))$. The Lorentzian angle $\gamma(s(t))$ of the tangent of $\vec{\alpha}$ with the first axis is defined by

$$\left.\frac{d\vec{\beta}}{ds}\right|_{s(t)} = (\cosh \gamma(s(t)), \sinh \gamma(s(t)))$$

(see [3]). Our assumptions on $\vec{\alpha}$ imply that its tangent image $\frac{d\vec{\beta}}{ds} \circ s|_{(a,b)}$ covers all of the branch of the Lorentzian unit circle which contains (1,0). Furthermore, the second component of this map is strictly monotonically increasing. This implies

$$\lim_{t \to a, t > a} \gamma(s(t)) = -\infty, \quad \text{and} \quad \lim_{t \to b, t < b} \gamma(s(t)) = \infty. \tag{1}$$

Like in the Euclidean case the curvature $\varkappa$ of $\vec{\alpha}$ is given by the change of the angle of its tangent with a given fixed direction, i.e.

$$\varkappa(s(t)) = \left.\frac{d\gamma}{ds}\right|_{s(t)}.$$

As usual the chain rule implies

$$\left|\left\langle \frac{d\vec{\alpha}}{dt}, \frac{d\vec{\alpha}}{dt} \right\rangle\right|^{\frac{1}{2}} \varkappa(s(t)) = \frac{d(\gamma \circ s)}{dt}. \tag{2}$$

From (1) we get for arbitrary $c \in (a,b)$

$$\lim_{\tau \to a, \tau > a} \int_\tau^c \frac{d(\gamma \circ s)}{dt} dt = \infty, \quad \lim_{\tau \to b, \tau < b} \int_c^\tau \frac{d(\gamma \circ s)}{dt} dt = \infty.$$

Because $\gamma \circ s$ is monotonically increasing with t, this implies

$$\lim_{t \to a, t > a} \frac{d(\gamma \circ s)}{dt} dt = \infty, \quad \lim_{t \to b, t < b} \frac{d(\gamma \circ s)}{dt} = \infty.$$

Therefore, since the expression $\left| \left\langle \frac{d\vec{\alpha}}{dt}, \frac{d\vec{\alpha}}{dt} \right\rangle \right|^{\frac{1}{2}}$ is continuous on $[a, b]$ and hence bounded, we get from (2)

$$\lim_{t \to a, t > a} \varkappa(s(t)) = \infty, \quad \lim_{t \to b, t < b} \varkappa(s(t)) = \infty.$$

This implies that $\varkappa(s(t))$ must attain at least one minimum on (a, b) giving the desired vertex of $\vec{\alpha}$. The case of a strictly convex space-like arc with light-like tangents at its end points is handled similarly. Hence we have shown the following

Proposition: *Let $\vec{\alpha}$ be a strictly convex smooth arc in the Lorentzian plane having light-like tangents at its endpoints. Then there is at least one vertex on $\vec{\alpha}$.*

Remark 1: From the relativistic point of view the implication of the preceding proposition is clear. The curve $\vec{\alpha}$, as considered in the proof, describes the world line of a material particle moving on a straight line and being observed in the given inertial system from the inertial time a to b. At these boundary times it behaves like a photon but moves into opposite directions. The curvature of $\vec{\alpha}$ is a measure for the acceleration of the particle. This must be «infinite» to «pass» within finite inertial time from the photon state to the material state. This is the case at both boundary points. Hence the acceleration must attain a minimum somewhere between these two values. For the background of this interpretation see [3].
As already stated above, the proposition implies the desired four-vertex-theorem.

Corollary: *The curvature of a closed strictly convex smooth curve in the Lorentzian plane attains at least four extrema.*

Remark 2: The considerations above may be extended to nonconvex smooth curves which are simply closed. They also can be extended to closed convex curves in geodesically convex domains of Lorentzian 2-manifolds. If self-intersections are allowed, then there may be two extrema for the curvature only. As an example consider the figure «eight» having the coordinate axes as axes of symmetry and stretch it in a suitable way such that only four light-like tangents are possible. Taking into account the sign of the curvature, only two vertices will occur.

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Aufgaben

Aufgabe 1040. Durch

$$f(z) := \sum_{j=0}^{n} \binom{n}{j} \binom{m+n+z}{m+j}^{-1} \quad \text{und} \quad g(z) := \frac{m+n+1+z}{m+1+z} \binom{m+z}{m}^{-1}$$

sind für feste natürliche Zahlen m, n zwei komplexe Funktionen f, g gegeben; $\mathbb{D}$ sei der Durchschnitt ihrer Definitionsbereiche. Man zeige, dass

$$f(z) = g(z); \quad z \in \mathbb{D}.$$

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Solution. $f(z)$ can be rewritten as

$$f(z) = m!\,n!\,z!\,(m+n+z)!^{-1} \sum_{j=0}^{n} \binom{m+j}{j} \binom{z+n-j}{n-j}.$$

Note that for non-integral $\lambda, (\lambda)!$ is to be interpreted as $\Gamma(\lambda+1)$. Then using the known identity [1]

$$\sum_{j=0}^{n} \binom{m+j}{j} \binom{z+n-j}{n-j} = \binom{m+z+n+1}{n}$$

and $\Gamma(\lambda+1) = \lambda\,\Gamma(\lambda)$, it follows that $f(z) = g(z)$.

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