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Extension theorems for integral representations of solutions of a functional equation [*]

Summary. Integral representations $F(x)=\int_0^\infty e^{-t}\Phi(t,x-1)\,dt$ of solutions of the functional equation $F(x+1)=g(x)F(x)$, which were obtained in previous papers for the interval $(0,1)$, are extended to solutions on $\mathbb{R}_+$. Additional properties of the kernel function Φ are discussed in this context.

1. Introduction

We start with the following observations:

a) Let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$, $g\colon \mathbb{R}_+ \to \mathbb{R}$ be given. If $g = \exp\varphi$, then the solution set of the difference equation

$$G(x+1) - G(x) = \varphi(x) \tag{0}$$

is known, iff the set of positive solutions of

$$F(x+1) = g(x)F(x) \tag{1}$$

is known. In general, equations (0) and (1) are not equivalent.
b) The theory of equation (0), and in particular the theory of its *principal solutions* in the sense of Nörlund, Krull, Schroth (see [4], and [7] to [12]), is well developed. There is no obvious way to extend this concept of a principal solution to the general case of equation (1).
c) In the theory of equation (1) solutions of the form

$$F(x) = \int_0^\infty e^{-t}\Phi(t,x-1)\,dt \tag{2}$$

play a prominent role. Take as an example $F(x) = \Gamma(x)$, Euler's Gamma function.
In [1], [5] and [6] we obtained solutions of (1) which are of type (2), where the kernel satisfies the functional-differential equation

$$\Phi_t(t,x) = g(x)\Phi(t,x-1)\,. \tag{3}$$

For convenience of the reader we restate Theorem 1 of [1] in a slightly modified form:

Theorem 1. *Let* $\mathbb{U} \subset \mathbb{R}$ *and* $g\colon \mathbb{U} \to \mathbb{R}$. *Assume that* $\Phi\colon \mathbb{R}_+ \times (\mathbb{U} \cup \{u-1 : u \in \mathbb{U}\}) \to \mathbb{R}$ *has the following properties*:

h_1) $\Phi_t(t,x)$ *exists for every* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{U}$,

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h$_2$) $\lim_{t \to 0} \Phi(t,x) = 0$ *for every* $x \in \mathbb{U}$,

h$_3$) *for every* $x \in \mathbb{U}$ *there are* $p \in (0,1)$, $t_0 \in \mathbb{R}_+$, $K \in \mathbb{R}_+$ *such that*

$$e^{-pt}|\Phi(t,x)| \leq K \quad \textit{for every} \quad t \geq t_0 .$$

If Φ *satisfies the equation* (3) *for* $t \in \mathbb{R}_+$, $x \in \mathbb{U}$ *then the integrals*

$$F(x) = \int_0^\infty e^{-t}\Phi(t, x-1)\,dt \quad \textit{and} \quad F(x+1) = \int_0^\infty e^{-t}\Phi(t,x)\,dt$$

exist for every $x \in \mathbb{U}$ *and equation* (1) *holds for* $x \in \mathbb{U}$.
In [1] we used instead of h$_3$): $\lim_{t \to +\infty} e^{-pt}\Phi(t,x) = 0$ for every $x \in \mathbb{U}$ and every $p \in (0,1]$. It is readily checked that the theorem remains true with the weaker condition h$_3$).
In the following Theorem 2 of [2] we introduced another functional equation for the kernel Φ, motivated by the reflection formula for principal solutions:

$$\Phi(t, -x) = 1/\Phi(t,x). \tag{4}$$

Theorem 2. *Let* $\mathbb{U} = (0,1)$, $g\colon \mathbb{U} \to \mathbb{R}_+$, $\hat{\Phi}\colon \mathbb{R}_+ \times ((-1,0) \cup (0,1)) \to \mathbb{R}_+$ *and assume that* $\hat{\Phi}$ *satisfies* (4) *for* $t \in \mathbb{R}_+$, $x \in \mathbb{U}$. *Then* $\hat{\Phi}$ *satisfies* h$_1$), h$_2$) *and* h$_3$) *of* Theorem 1 *and* (3) *for* $t \in \mathbb{R}_+$, $x \in \mathbb{U}$, *iff*

$$\hat{\Phi}(t,x) = h(x)\hat{\gamma}(x)^{-\hat{\gamma}(x)} t^{\hat{\gamma}(x)}, \quad t \in \mathbb{R}_+, x \in \mathbb{U},$$

where $\hat{\gamma}(x) := g(x)/[g(1-x) + g(x)]$ *and* h: $\mathbb{U} \to \mathbb{R}_+$ *is solution of the reflection equation*

$$h(x)h(1-x) = g(x)^{\hat{\gamma}(x)} g(1-x)^{\hat{\gamma}(1-x)}.$$

This theorem shows that a kernel $\hat{\Phi}$ which satisfies (3) and (4) and some regularity conditions must be of *separation type*: $\hat{\Phi}(t,x) = \hat{G}(x)t^{\hat{\gamma}(x)}$ (Functions which will be later subject to an extension procedure are marked with a circumflex). The aim of this note is to extend these results for kernels defined on $\mathbb{R}_+ \times (-1, \infty)$ and to study the behaviour of kernels of separation type with respect to the equations (1), (2), (3), (4).

2. Structure and Extension Theorems

We begin this section with a remark on kernels of the type $\Phi(t,x) = G(x)t^{\gamma(x)}$ and their connection with the functional equation (1). Euler's gamma function is involved.

Theorem 3. *Assume* $g\colon \mathbb{R}_+ \to \mathbb{R}$, $G\colon (-1,\infty) \to \mathbb{R}$, $\gamma\colon (-1,\infty) \to (-1,\infty)$ *and*

$$F(x) = \int_0^\infty e^{-t} G(x-1)t^{\gamma(x-1)}\,dt, \quad x \in \mathbb{R}_+ .$$

Then the following conditions are equivalent:

(i) *F satisfies* (1) *for* $x \in \mathbb{R}_+$,

(ii) $G(x)\Gamma(\gamma(x)+1) = g(x)G(x-1)\Gamma(\gamma(x-1)+1)$ *for* $x \in \mathbb{R}_+$.

Proof. For $x \in \mathbb{R}_+$, the following equations are equivalent:

$$\int_0^\infty e^{-t} G(x) t^{\gamma(x)}\, dt = g(x) \int_0^\infty e^{-t} G(x-1) t^{\gamma(x-1)}\, dt,$$

$$G(x) \int_0^\infty e^{-t} t^{\gamma(x)}\, dt = g(x) G(x-1) \int_0^\infty e^{-t} t^{\gamma(x-1)}\, dt$$

$$G(x)\Gamma(\gamma(x)+1) = g(x)G(x-1)\Gamma(\gamma(x-1)+1).$$ ■

The solutions of (3) which are of separation type now are characterized in the following

Theorem 4. *Assume* $g: \mathbb{R}_+ \to \mathbb{R}$, $G: (-1,\infty) \to \mathbb{R}$, $\gamma: (-1,\infty) \to \mathbb{R}$, *and* $\gamma(x)G(x) \neq 0$ *for every* $x \in \mathbb{R}_+$. *Then the following statements are equivalent:*

(j) $\Phi(t,x) = G(x)t^{\gamma(x)}$ *satisfies* (3) *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$,

(jj) $\gamma(x-1)+1 = \gamma(x)$ *and* $G(x)\gamma(x) = g(x)G(x-1)$ *for* $x \in \mathbb{R}_+$.

Proof. From (j) we deduce

$$\gamma(x)G(x)t^{\gamma(x)-1} = g(x)G(x-1)t^{\gamma(x-1)}, \quad x \in \mathbb{R}_+.$$

With $t = 1$ we get the second equation of (jj). Now $\gamma(x)G(x) \neq 0$ implies the first equation of (jj). Conversely, if $\Phi(t,x) = G(x)t^{\gamma(x)}$, the equations (jj) yield

$$\Phi_t(t,x) = \gamma(x)G(x)t^{\gamma(x)-1} = g(x)G(x-1)t^{\gamma(x-1)} = g(x)\Phi(t,x-1).$$

Here the assumption $\gamma(x)G(x) \neq 0$ was not used. ■

Now we show that a kernel

$$\hat{\Phi}(t,x) = \hat{G}(x)t^{\hat{\gamma}(x)} \quad t \in \mathbb{R}_+,\ x \in [0,1),$$

can be extended to

$$\Phi(t,x) = G(x)t^{\gamma(x)} \quad t \in \mathbb{R}_+,\ x \in (-1,\infty)$$

a solution of (3). More precisely:

Theorem 5. *Let* $g: \mathbb{R}_+ \to \mathbb{R}_+$, $\hat{\gamma}: [0,1) \to (-1,\infty)$, $\hat{G}: [0,1) \to \mathbb{R}$, *and assume*

$$\hat{\Phi}(t,x) = \hat{G}(x)t^{\hat{\gamma}(x)} \quad t \in \mathbb{R}_+,\ x \in [0,1).$$

Then there are extensions $\gamma: (-1,\infty) \to (-1,\infty)$, $G: (-1,\infty) \to \mathbb{R}$, $\Phi: \mathbb{R}_+ \times (-1,\infty) \to \mathbb{R}_+$ *of* $\hat{\gamma}, \hat{G}, \hat{\Phi}$ *such that* $\Phi(t,x) = G(x)\, t^{\gamma(x)}$ *satisfies* (3) *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$.
If $\hat{\gamma}(x)\hat{G}(x) \neq 0$ *for every* $x \in [0,1)$ *and* $\hat{c}_r(0) \neq 0$ *then* Φ *is uniquely determined.*

Proof. For the existence statement define $\gamma(x) := \hat{\gamma}(x-[x]) + [x]$ for $x \in (-1,\infty)$ and $G(x) := \hat{G}(x+1)\hat{\gamma}(x+1)/g(x+1)$ for $x \in (-1,0)$, $G(x) := \hat{G}(x-[x]) \prod_{k=0}^{[x]} \frac{g(x-k+1)}{\gamma(x)+k-[x]}$ for $x \geq 0$, with the usual convention about an empty product; $[x]$ denotes the greatest integer not exceeding x. Then γ and G are extensions of $\hat{\gamma}$ and $\hat{G}$ which satisfy equations (jj) of Theorem 4:

$$\gamma(x-1) + 1 = \hat{\gamma}(x-1-[x-1]) + [x-1] + 1 = \hat{\gamma}(x-[x]) + [x] = \gamma(x),$$

$$G(x)\gamma(x) = \hat{G}(x)\hat{\gamma}(x) = G(x-1)g(x) \quad \text{for} \quad x \in [0,1),$$

$$G(x)\gamma(x) = [\hat{G}(x-1)g(x)/\gamma(x)]\gamma(x) = G(x-1)g(x) \quad \text{for} \quad x \in [1,2),$$

$$\begin{aligned} G(x)\gamma(x) &= [\hat{G}(x-2)g(x)\, g(x-1)/(\gamma(x)-1)\gamma(x)]\gamma(x) \\ &= [\hat{G}(x-2)g(x-1)/\gamma(x-1)]g(x) = G(x-1)g(x) \quad \text{for} \quad x \in [2,3). \end{aligned}$$

etc. ....
Because of $\gamma(x) - [x] = \hat{\gamma}(x-[x])$ the denominators $\gamma(x) + k - [x]$ in the above products are positive when $[x] \geq 1$.
The final remark in the proof of Theorem 4 shows that $\Phi(t,x) = G(x)\, t^{\gamma(x)}$ is a solution of (3) for $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$, regardless if $\gamma(x)G(x) \neq 0$ for $x \in [0,1)$ or not.
According to the uniqueness statement we note for any $\Phi(t,x) = G(x)\, t^{\gamma(x)}$ satisfying (3), necessarily

$$\gamma(x)G(x)\, t^{\gamma(x)-1} = g(x)G(x-1)\, t^{\gamma(x-1)}, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}_+.$$

Hence

$$\gamma(x)G(x) = g(x)G(x-1) \quad \text{for} \quad x \in \mathbb{R}_+.$$

Now our assumption $\hat{\gamma}(x)\hat{G}(x) \neq 0$ for $x \in [0,1)$ implies: $\gamma(x-1) = \gamma(x) - 1$ for $x \in (0,1)$. For $x \in [1,2)$ we have $\gamma(x)G(x) = g(x)G(x-1) \neq 0$ because of $g(x) > 0$ and $G(x-1) \neq 0$ and again $\gamma(x-1) = \gamma(x) - 1$. By induction $\gamma(x-1) = \gamma(x) - 1$ for every $x \in \mathbb{R}_+$. Consequently in this case the extensions γ, G of $\hat{\gamma}$, $\hat{G}$ must be defined as given above: the function γ defined above is the unique solution of $\gamma(x) - \gamma(x-1) = 1$ which coincides on $[0,1)$ with $\hat{\gamma}$ and the function G defined above is the unique solution of $\gamma(x)G(x) = g(x)G(x-1)$ which coincides on $[0,1)$ with $\hat{G}$. ■
Finally we apply the previous results to extend the kernel $\hat{\Phi}$ which was obtained in Theorem 2 for $x \in (-1,0) \cup (0,1)$ to a kernel Φ, defined for $x \in (-1,\infty)$ under preserving at all its useful properties:

Theorem 6. *Let* $g: \mathbb{R}_+ \to \mathbb{R}_+$ *be given and assume* $\hat{\Phi}: \mathbb{R}_+ \times ((-1,0) \cup (0,1)) \to \mathbb{R}_+$ *satisfies conditions* h_1), h_2), h_3) *of* Theorem 1 *and equations* (3) *and* (4) *for* $t \in \mathbb{R}_+$, $x \in (0,1)$. *Then* *there is an extension* $\Phi\colon \mathbb{R}_+ \times (-1, \infty) \to \mathbb{R}_+$ *of* $\hat{\Phi}$, *which is of separation type and satisfies* (3) *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$.

The function $F(x) = \int_0^\infty e^{-t}\Phi(t, x-1)\,dt$ *is a solution of* (1) *on* $\mathbb{R}_+$.

Proof. By Theorem 2, $\hat{\Phi}(t,x) = h(x)\hat{\gamma}(x)^{-\hat{\gamma}(x)} t^{\hat{\gamma}(x)}$, for $t \in \mathbb{R}_+, x \in (0,1)$, where $\hat{\gamma}(x) = g(x)/[g(1-x) + g(x)]$ and h is a positive solution of the reflection equation

$$h(x)\,h(1-x) = g(x)^{\hat{\gamma}(x)} g(1-x)^{\hat{\gamma}(x-1)}, \quad x \in (0,1).$$

Define $\hat{G}(x) := h(x)\hat{\gamma}(x)^{-\hat{\gamma}(x)}$ for $x \in (0,1)$ and $\hat{G}(0) := \alpha$, $\hat{\gamma}(0) := \beta$ with $\alpha > 0$, $\beta \geq 0$ and $\hat{\Phi}(t,0) := \alpha t^{\beta}$. Then $\hat{G}\colon [0,1) \to \mathbb{R}_+$ and $\hat{\gamma}\colon [0,1) \to \mathbb{R}_+$.
By Theorem 5 there is a unique extension $\Phi\colon \mathbb{R}_+ \times (-1, \infty) \to \mathbb{R}_+$ of $\hat{\Phi}$ such that $\Phi(t,x) = G(x)\,t^{\gamma(x)}$, G and γ extensions of $\hat{G}$ and $\hat{\gamma}$ as defined in the proof of Theorem 5. Φ satisfies (3) for every $t \in \mathbb{R}_+$ and every $x \in \mathbb{R}_+$.
The conditions h_1), h_2), h_3) of Theorem 1 in the case $\mathbb{U} = \mathbb{R}_+$ are satisfied for our kernel Φ, hence

$$F(x) = \int_0^\infty e^{-t}\Phi(t, x-1)\,dt$$

defines a solution of (1) on $\mathbb{R}_+$.

Note that every collection $\{h, \alpha, \beta\}$, h a positive solution of the reflection formula above, $\alpha > 0, \beta \geq 0$, yields a kernel Φ; also note that the reflection equation has always a positive solution:

$$h(x) = g(x)^{\hat{\gamma}(x)}. \qquad \blacksquare$$

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A generalization of Nagel's middlespoint

In an 1836 paper, C. H. von Nagel defines the «Mittenpunkt» of a given triangle Even though this point is readily constructed, it seems not to have found its way into the modern geometry of the triangle. In this paper, we show that by looking at the point from a slightly different point of view, one can obtain an infinite family of such points. In addition, we generalize a set of three related points.

1. Introduction

In what seems to be a little-known and somewhat inaccessible paper [4], C. H. von Nagel defines the *middlespoint* (Mittenpunkt) of a given triangle $A_1A_2A_3$ in the following manner.

Definition. *Let* S_i, $i = 1, 2, 3$, *denote the midpoints respectively of the sides* $A_{i+1}A_{i+2}$ *and* I^i, *the excentre opposite* A_i *of the triangle* $A_1A_2A_3$, *then* $\bigcap_i S_iI^i = M$, *the middlespoint of the given triangle.*

The name probably derives from the fact that the point is obtained using *middles*, i.e., *centres* of circles and *midpoints* of line segments. Even though this point has a simple construction using well-known concepts associated with the triangle, it seems not to appear in the available literature. One good recent paper on the subject known to us is by Baptist [1].
In this paper, we show that there exists an infinite family of such points each being a centre of perspectivity of a pair of triangles one circumscribed, the other inscribed, with respect to a given triangle. We also generalize a set of three related points referred to as *«interior middlespoint»* by Nagel. For the convenience of the reader, we supply some background.

2. Some properties of the middlespoint

In [5], Nagel also proves that the points M, the centroid S, and the Gergonne point $G = \bigcap_i A_iG_i$, where G_i is the contact point of the incircle with side $A_{i+1}A_{i+2}$ are collinear. We add a further result in the form of a theorem which we have not previously seen.