Winch curves

Let $z = (x - y)/\|x - y\|$. Then $\gamma_z(E) = D, \gamma_z(C) = C$ and $\gamma_z(C\backslash D) = C\backslash D$. Hence, by (1),

$$\begin{aligned}\mu(\gamma_z(M) \cap C) &= \mu(\gamma_z(M) \cap (C\backslash D)) + \mu(\gamma_z(M) \cap D) \\ &\geq \mu(M \cap (C\backslash D)) + \mu(M \cap E) = \mu(M \cap C) + \mu(M \cap E) > \mu(M \cap C).\end{aligned}$$

Since $\gamma_z(M) \in K$, this contradicts the choice of M, so the proof is complete. □

Let us remark that a slight variant of the proof above gives the following assertion. Let K be a non-empty closed subset of H which is also closed under the operators γ_z, i.e. which is such that $\gamma_z(A) \in K$ for all $A \in K$ and $z \in S^n$. Then K contains all caps of measure $m = \sup\{\mu(A): A \in K\}$.

Also, it is easily seen that the proof above implies various extensions of Theorem 3. For example, given finite sets $X, Y \subset S^n$ with $|X| = |Y|$, let us write $X \leq Y$ if for every $d > 0$, the number of pairs in X at distance at least d is not more than the number of pairs in Y at distance at least d. Furthermore, for sets $A, B \subset S^n$, let us write $A \leq B$ for the assertion that for every finite set $X \subset A$ there is a finite set $Y \subset B$ with $|Y| = |X|$ and $X \leq Y$. Then the following assertion holds. Let A be a non-empty closed subset of S^n and let C be a cap of measure $\mu(A)$. Then $C \leq A$.

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Winch curves

A taut rope connects a point in the origin of a rectangular coordinate system with a point in $R(a, 0)$. If the latter starts moving along the line $x = a$, it will trail the point in the origin. For each point P of the curve that is created in this way we have $PQ = a$, where Q is the intersection of the tangent to the curve in P with the line $x = a$. This curve, known as the tractrix, is represented by an equation that can be found as follows.
In the rectangular triangle PSQ (see fig. 1) we have

$$PQ = a, \quad PS = a - x, \quad SQ = (a - x)\,dy/dx.$$

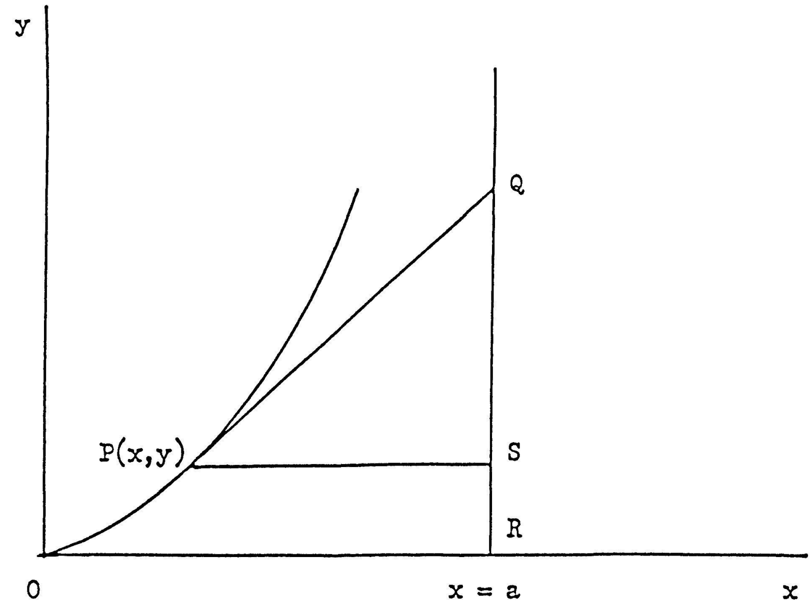

Figure 1. Winch curve.

Pythagoras' theorem leads to

$$a^2 = (a - x)^2 [1 + (dy/dx)^2].$$

Solving this differential equation for positive a we obtain

$$y = -\sqrt{a^2 - (a - x)^2} + a \ln C_1 \frac{a + \sqrt{a^2 - (a - x)^2}}{a - x}$$

as a solution in the first quadrant.
As the curve passes through the origin we have $y = 0$ for $x = 0$. This gives $C_1 = 1$, so the equation can be written

$$y = -\sqrt{a^2 - (a - x)^2} + a \ln \frac{a + \sqrt{a^2 - (a - x)^2}}{a - x} \tag{1}$$

In literature the tractrix is usually described as above.
In this article however we treat the trailing problem in a more general way. It will then turn out that the tractrix can be considered as a special case of a new family of curves. In order to achieve this we assume that the length of the rope PQ will be changed during trailing. In practice one could think of a vehicle Q trailing a load P and being equipped with a winch so that PQ can be shortened or lengthened during the trailing process. Usually Q will be moving at constant velocity and the length of PQ will be varied uniformly in time as well. We then have

$$RQ = y + (a - x)\, dy/dx = c_1 t \tag{2}$$

where t represents the time and c_1 stands for the velocity of Q along $x = a$. Furthermore we can write

$$PQ = a - c_2 t \tag{3}$$

where c_2 is the velocity with which the winch is wound up ($c_2 > 0$) or eased off ($c_2 < 0$). Elimination of t between equations (2) and (3) gives

$$PQ = a - b[y + (a - x)\,dy/dx]$$

with $b = c_2/c_1$.

It should be noted that the same result is obtained if the variation of the length of PQ and the velocity of Q are not constant in time on condition that the time dependencies in equations (2) and (3) are of the same form.

Writing $p = dy/dx$ and applying Pythagoras' theorem in the triangle PSQ we have

$$\{a - b[y + (a - x)p]\}^2 = (a - x)^2 (1 + p^2)$$

or in the first quadrant

$$a - b[y + (a - x)p] = (a - x)\sqrt{1 + p^2}. \tag{4}$$

This equation can be written in the form $y = g(p)x + f(p)$ and is called the differential equation of d'Alembert.

Differentiating with respect to x we obtain

$$-bp - b(a - x)\,dp/dx + bp = -\sqrt{1 + p^2} + (a - x)\frac{p}{\sqrt{1 + p^2}}\,dp/dx$$

or

$$dp/dx = \frac{\sqrt{1 + p^2}}{(a - x)\left[b + \dfrac{p}{\sqrt{1 + p^2}}\right]}. \tag{5}$$

Integration of (5) leads to

$$-\ln(a - x) = b\ln(p + \sqrt{1 + p^2}) + \tfrac{1}{2}\ln(1 + p^2) + C_2.$$

Now $p = 0$ for $x = 0$, yielding $C_2 = -\ln a$, so we can write

$$\ln\frac{a}{a - x} = \ln\sqrt{1 + p^2}\,\{p + \sqrt{1 + p^2}\}^b$$

or

$$x = a\left[1 - \frac{1}{\sqrt{1 + p^2}\,\{p + \sqrt{1 + p^2}\}^b}\right] \tag{6}$$

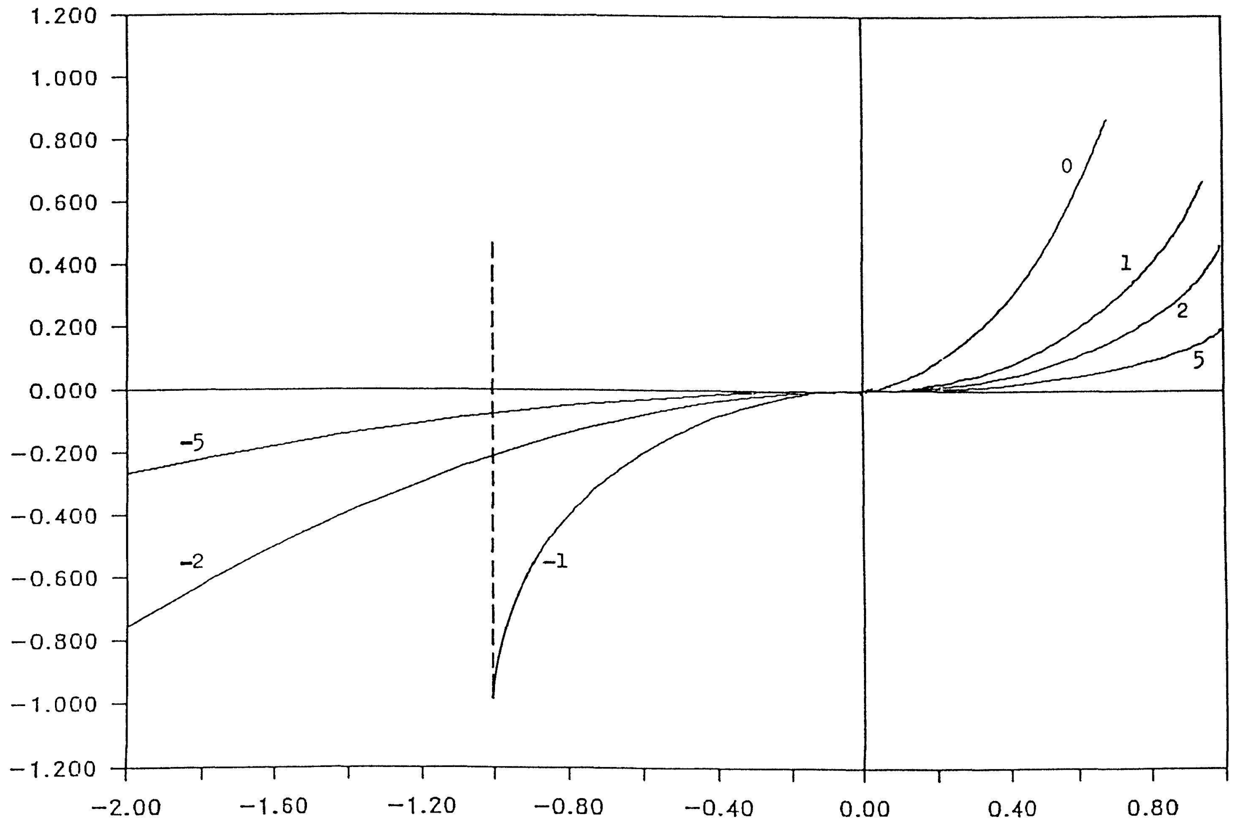

Figure 2. Winch curves for different values of b. The numbers along the framework are given in units a.

and from equation (4) we find for $b \neq 0$

$$y = \frac{a}{b}\left[1 - \frac{pb + \sqrt{1+p^2}}{\sqrt{1+p^2}\,\{p + \sqrt{1+p^2}\}^b}\right]. \tag{7}$$

Equations (6) and (7) may be considered as a parameter representation of the new family of *winch curves* with parameter $p = dy/dx$.
We will now check that equations (6) and (7) yield the tractrix for $b \to 0$. For $b \to 0$ equation (6) leads to

$$x = a - \frac{a}{\sqrt{1+p^2}}. \tag{8}$$

As equation (7) is of indeterminate form for $b \to 0$ we rewrite it as

$$y = a\left[\frac{\sqrt{1+p^2}\,\{p+\sqrt{1+p^2}\}^b - pb - \sqrt{1+p^2}}{b\sqrt{1+p^2}\,\{p+\sqrt{1+p^2}\}^b}\right].$$

Now De l'Hôpital's rule gives

$$y = \lim_{b\to 0} a\left[\frac{\sqrt{1+p^2}\,\{p+\sqrt{1+p^2}\}^b \ln(p+\sqrt{1+p^2}) - p}{\sqrt{1+p^2}\,\{p+\sqrt{1+p^2}\}^b + b\sqrt{1+p^2}\,\{p+\sqrt{1+p^2}\}^b \ln(p+\sqrt{1+p^2})}\right]$$

$$= a\left[\ln(p+\sqrt{1+p^2}) - \frac{p}{\sqrt{1+p^2}}\right]. \tag{9}$$

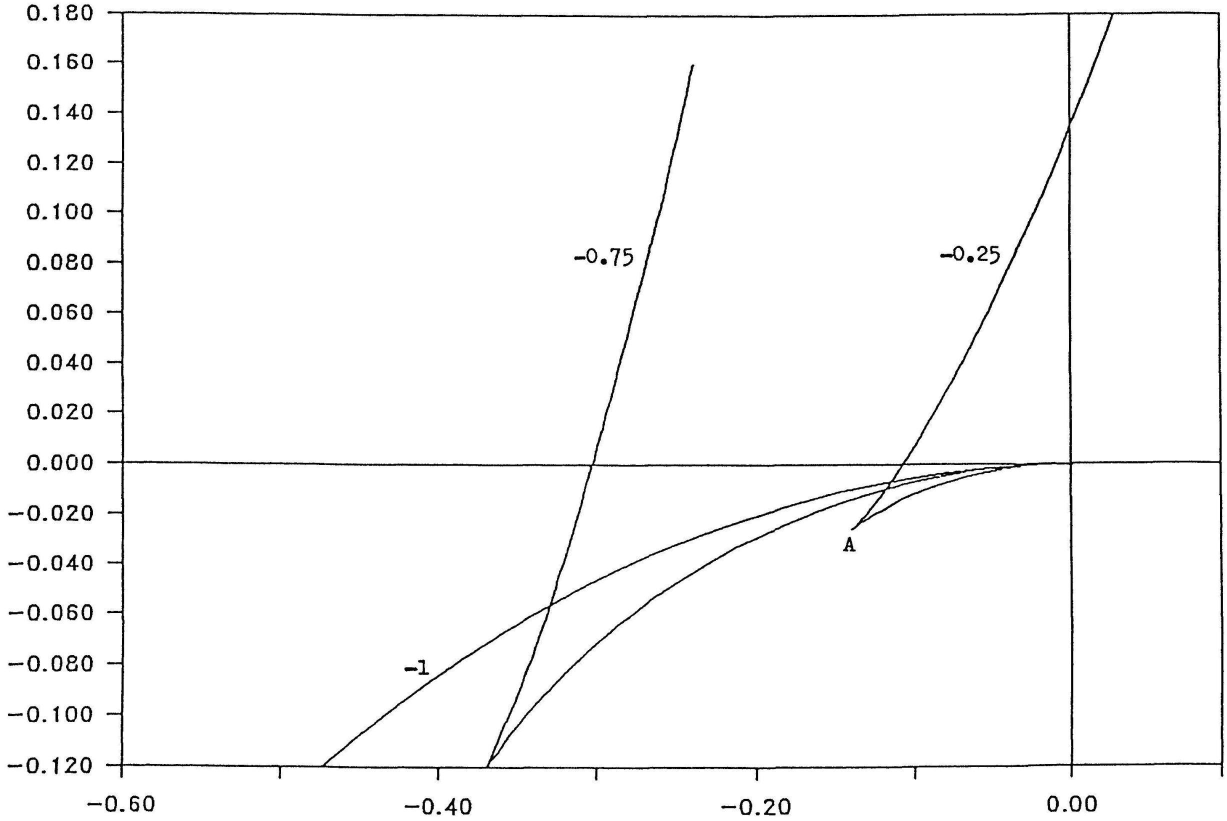

Figure 3. Cusped winch curves.

Eliminating p between (8) and (9) we arrive at equation (1). So the tractrix may indeed be considered as a winch curve with $b = 0$.

Figures 2 and 3 show a number of winch curves for different b.

It is assumed that the rope PQ remains tight even if it is eased off. As a consequence the curves will not or not exclusively be found in the first quadrant for $b < 0$.

In some of the curves a cusp A is shown. The coordinates of such a point can be found as follows.

Equation (6) shows that x is a continuous function of p. In A we have $dx/dp = 0$ so dp/dx goes to infinity. From equation (5) we then obtain

$$b + \frac{p_A}{\sqrt{1 + p_A^2}} = 0$$

or

$$p_A = \frac{-b}{\sqrt{1 - b^2}} \tag{10}$$

giving $-1 < b < 0$ for curves with a cusp since p_A is positive, real and finite.

Equation (10) substituted in (6) and (7) leads to the coordinates of A:

$$x_A = a\left[1 - \sqrt{(1-b)^{1-b}(1+b)^{1+b}}\right]$$

$$y_A = \frac{a}{b}\left[1 - \sqrt{(1-b)^{2-b}(1+b)^{2+b}}\right].$$

For $b \to -1$ point A approaches $(-a, -a)$ and P moves, after it has passed A, along a line that approaches the line $x = -a$.
For $b = -1$ the curve passes through $(-a, -a)$, here p is infinite.
For $b < -1$ the winch is eased off so quickly that P cannot be trailed anymore.
It will be clear that P and Q only meet each other for positive b. From (6) and (7) it can be derived that this will happen at point $(a, a/b)$.
Thanks are due to Mr H. J. de Vries for performing many calculations.

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Kleine Mitteilung

Über die Zahlenfolge $n! + k,\ 2 \leqq k \leqq n$

Fast jedes Buch über Zahlentheorie erwähnt die Tatsache, dass keine der Zahlen

$$n! + k \quad \text{mit} \quad 2 \leqq k \leqq n \tag{1}$$

eine Primzahl ist (es gibt also in der Folge der Primzahlen beliebig lange Lücken). Es scheint aber nicht allgemein bekannt zu sein, dass dieselbe Zahlenfolge auch die Unendlichkeit der Primzahl-Menge beherbergt. Dies entnimmt man dem folgenden

Satz 1. Für jedes $n > 1$ und $2 \leqq k \leqq n$ hat $n! + k$ entweder einen Primfaktor $> n$, oder aber k ist prim und grösser als $n/2$ und $n! + k$ ist eine Potenz von k.

Beweis. Sei $2 \leqq k \leqq n$. Für alle Primzahlen $p \leqq n$, welche $n! + k$ teilen, ist $p|k$. Falls $p < k$, also $p \in \{2, \ldots, k-1\}$ ist, gilt

$$p \,|\, n!/k \quad \text{und damit} \quad p \nmid (n!/k) + 1 = (n! + k)/k\,.$$

Die Zahl $(n! + k)/k$ und mit ihr $n! + k$ besitzt somit Primteiler $> n$. Hat also $n! + k$ nur Primteiler $\leqq n$, so ist k prim und dies ist der einzige Primteiler von $n! + k$.
Aus

$$n! + k = k^s, \quad s \geqq 2$$

folgt

$$k \nmid k^{s-1} - 1 = n!/k, \quad \text{also} \quad k > n/2\,.$$