Kleine Mitteilungen

Objekttyp: **Group**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **43 (1988)**

Heft 4

Kleine Mitteilungen

Positive derivatives and increasing functions

A function with positive derivative on an interval is increasing. While the generalization of this note is classical (cf. [3, p. 44] and [4, p. 364]), the proof offered below is simple enough to present in an honors calculus class and works for extensions of the derivative to which the usual proof, based on the mean-value theorem, does not apply. For a more complicated proof of a less general result, see Bourbaki [2, pp. 19–21].
Below, D will denote any of the following operators: the derivative, the right- (or left-) hand derivative, the right (or left) upper (or lower) Dini derivative, the symmetric derivative. Definition of these operators can be found in [3] and [4].

Theorem. *Let f be continuous on $[a,b]$ and suppose that $Df > 0$ on $(a,b)\setminus S$, where S is countable. Then $f(b) > f(a)$.*

Proof. First we show that $f(b) \geqq f(a)$. Suppose not; then $f(a) > f(b)$. Choose γ such that $f(a) > \gamma > f(b)$ and $\gamma \notin f(S)$; this is possible since $f(S)$ is countable and hence does not exhaust the interval $(f(b), f(a))$. Let $C = \{x \in [a,b] : f(x) > \gamma\}$. Since $a \in C$, C is nonempty. Set $c = \sup C$. From the definitions of C and c it follows that

(i) if $x > c$, then $f(x) \leqq \gamma$

(ii) for each $\delta > 0$ there exists x such that $c - \delta < x \leqq c$ and $f(x) > \gamma$.

Suppose $f(c) > \gamma$; then, since f is continuous, $f(c+\varepsilon) > \gamma$ for small $\varepsilon > 0$, contradicting (i). On the other hand, if $f(c) < \gamma$, then $f(c-\varepsilon) < \gamma$ for small $\varepsilon > 0$, which contradicts (ii). Thus $f(c) = \gamma$. From (i) and (ii) it is now evident that $Df(c) \leqq 0$, contradicting the assumption that $Df > 0$ on $(a,b)\setminus S$. Thus $f(b) \geqq f(a)$.
To complete the proof, take $x \in (a,b)$ and apply the result derived above to the intervals $[a,x]$ and $[x,b]$, obtaining $f(a) \leqq f(x) \leqq f(b)$. If $f(b) = f(a)$, f is constant on $[a,b]$; hence $f' \equiv 0$, contradicting $Df > 0$. Thus $f(b) > f(a)$, and f is strictly increasing on $[a,b]$.

Remarks. 1. The argument given above fails for the upper left Dini derivative D^-. To handle that case, consider instead $c' = \inf\{x \in [a,b] : f(x) < \gamma\}$ and observe that $D^- f(c') \leqq 0$.
2. When D is the derivative, the hypothesis that f is continuous is superfluous. Simple examples show that it is necessary in all other cases.
3. Let χ denote the usual Cantor function on $[0,1]$ and put $f(x) = x/2 + \chi(1-x)$. Then f is continuous and $f'(x) = 1/2$ a.e., yet $f(0) = 1 > 1/2 = f(1)$. This shows that «countable» cannot, in general, be enlarged to «measure zero.»
The subsequent development is standard. Suppose f is continuous on $[a,b]$. If $Df \geqq 0$ on $(a,b)\setminus S$, we may apply the theorem to $f(x) + \varepsilon(x-a)$ and let $\varepsilon \to 0+$ to conclude that f is nondecreasing on $[a,b]$. In case $Df < 0$ ($Df \leqq 0$), the previous results applied to $-f$ show that f is decreasing (nonincreasing). Finally, if $Df = 0$ on $(a,b)\setminus S$, then f is constant, since in this case it is both nondecreasing and nonincreasing on $[a,b]$.

Further generalizations, based on relaxing the hypothesis of continuity and involving the notion of measure zero, are surveyed in [3, pp. 44–46].

Lawrence Zalcman, Dep. of Mathematics,
Bar-Ilan University, Ramat-Gan, Israel

REFERENCES

1 Lipman B.: On avoiding the mean value theorem. Amer. Math. Monthly *74*, 583 (1967).
2 Bourbaki N.: Éléments de mathématique. IX. Première partie: Les structures fondamentales de l'analyse. Livre IV: Fonctions d'une variable réele (theoric élémentaire), Hermann et Cie., Paris 1949.
3 Bruckner A. M. and Leonard J. L.: Derivatives. Amer. Math. Monthly *73*, Part II, 24–56 (1966).
4 Hobson E. W.: The Theory of Functions of a Real Variable and the Theory of Fourier's Series vol. 1, 3rd edition. Cambridge University Press 1927.

 0013-6018/88/040120-02$1.50+0.20/0

Aufgaben

Aufgabe 965. Einer Parabel sei eine Folge von sie doppelt berührenden Kreisen $K_1, K_2, \ldots$ einbeschrieben, wobei K_1 der Scheitelkrümmungskreis sei und jeder der Kreise den folgenden berühre. Man ermittle für beliebiges $n \in \mathbb{N}$ das Verhältnis r_n/r_1 (r_i = Radius von K_i).

C. Bindschedler, Küsnacht

Lösung. Auf Grund der Ähnlichkeit genügt es, eine Parabel zu betrachten, z. B. die Parabel $y^2 = 2x$ mit der Subnormalen 1. Ist r_n der Radius des n-ten Kreises, so haben die Berührungspunkte mit der Parabel die Koordinaten $y = \pm\sqrt{r_n^2 - 1}$ und $x = (r_n^2 - 1)/2$. Demzufolge hat der Mittelpunkt des n-ten Kreises die Abszisse $(r_n^2 + 1)/2$. Für die Abszisse des Berührungspunktes mit dem $(n+1)$-ten Kreis gilt dann

$$(r_{n+1}^2 + 1)/2 - r_{n+1} = (r_n^2 + 1)/2 - r_n, \quad \text{d. h.}$$

$$(r_{n+1} - 1)^2/2 = (r_n + 1)^2/2,$$

woraus $r_{n+1} = r_n + 2$ folgt. Da, wie man leicht sieht, $r_1 = 1$ ist, gilt $r_n = 2n - 1$ und somit allgemein $r_n/r_1 = 2n - 1$.

J. M. Ebersold, Winterthur

Weitere Lösungen sandten J. Binz (Bolligen), W. Janous (Innsbruck, A), Klasse 7d (Kantonsschule Zug), L. Kuipers (Sierre), O. P. Lossers Jr. (Eindhoven, NL), V. Mascioni (Origlio), S. Nanba (Okayama, Japan), I. Paasche (Stockdorf, BRD; 2 Lösungen),