

# A further remark concerning Nagel cevians

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**Suggestions for further research.** We observe that in all cases presented above, as in the real case, the periodic continued fractions are symmetric. Namely, for period length  $k$  we have  $a_j = a_{k-j}$  and  $b_j = b_{k+1-j}$  for  $j = 1, 2, \dots, k-1$ . Further,  $a_k = 2 \cdot a_0$  appears to hold, as in the real case. We found no examples of odd period length.

There is also a connection to Pell-like equations. We illustrate this with the example  $\sqrt{376} \in \mathbb{Q}_5$ . Let  $p_n/q_n$  be the  $n$ th convergent of the continued fraction expansion, defined by

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = a_0, \quad p_n = a_n \cdot p_{n-1} + 5^{b_n} \cdot p_{n-2} \quad \text{for } n \geq 1, \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_n = a_n \cdot q_{n-1} + 5^{b_n} \cdot q_{n-2} \quad \text{for } n \geq 1. \end{aligned}$$

Put

$$p_n^2 - 376 \cdot q_n^2 = d_n \cdot 5^{c_n}, \quad c_n \in \mathbb{N}, \quad d_n \in \mathbb{Z}, \quad 5 \nmid d_n.$$

Then we find that  $c_n = \sum_{j=0}^n b_j$ , and the sequence  $\{d_n\}_{n=-1}^{\infty}$  is given by 1, -3, 17, -4, 17, -3, 1, -3, 17, ..., which is symmetric. The fifth convergent  $p_5/q_5 = 12\,103/603$ , for which  $d_5 = 1$ , gives rise to a sort of 5-adic «fundamental unit»  $12\,103 + 603 \cdot \sqrt{376}$ , in the sense that  $p_{i+6j} + q_{i+6j} \cdot \sqrt{376} = (p_i + q_i \cdot \sqrt{376}) \cdot (p_5 + q_5 \cdot \sqrt{376})^j$  for  $i = -1, 0, \dots, 4$ , and  $j = 0, 1, 2, \dots$ . It would be interesting to have a more general theory of these matters.

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## A further remark concerning Nagel cevians

*Dedicated to Professor D. S. Mitrinović on his 80th birthday*

In El. Math., vol. 41/5 and 42/4, R. H. Eddy and D. S. Milošević considered the cases  $L = 14$  and (the improvement)  $L = 10$ , resp., of

$$\sum n_a \leq 9r + L(R - 2r). \tag{1}$$

Here  $n_a, n_b, n_c$  are the Nagel cevians and  $R, r$  the circumradius and inradius, resp., of the given triangle.

In this note we show that the minimal  $L$  for (1) satisfies

$$6.258998 \dots \leq L_{\min} \leq 6.478487 \dots$$

The lower bound is  $\sup \{F(x), x > 1\}$  with  $F(x)$  as given below in i); the upper bound is the largest root of  $2x^3 - 3x^2 - 144x + 515 = 0$ .

**Proof.** Let  $N_a$  be the foot of  $n_a$  on side  $BC$ . Then,  $BN_a = s - c$ , and from the triangle  $ABN_a$  it follows via the law of cosines that

$$n_a^2 = c^2 + (s - c)^2 - 2c(s - c) \cos B.$$

After some algebraic manipulations this simplifies to

$$n_a = s \sqrt{(b - c)^2 + 4r^2} / a$$

where  $s$  represents the semiperimeter of the given triangle.

i) Considering triangles with  $c = 2$  and  $a = b = x, x > 1$ , we get  $r = \sqrt{(x - 1)/(x + 1)}$ ,  $R = x^2/2\sqrt{x^2 - 1}$ ,  $n_c = \sqrt{x^2 - 1}$  and  $n_a = n_b = \sqrt{x + 1} \sqrt{(x - 2)^2(x + 1) + 4(x - 1)}/x$ . Inserting this in (1) we get after some algebraic manipulations

$$L \geq F(x) := 2\sqrt{x - 1} \{ (2x + 2) \sqrt{(x - 2)^2(x + 1) + 4(x - 1)} + (x^2 - 8x) \sqrt{x - 1} \} / x(x - 2)^2.$$

Numerical calculations lead to  $L \geq F(4.699 \dots) = 6.258998 \dots =: m$ .

ii) The inequality between the arithmetic and square root means yields

$$\begin{aligned} \text{i.e. } \sum n_a &\leq \sqrt{3 \sum n_a^2}, \\ \sum n_a &\leq s \{ 3 \sum (b^2 + c^2 - 2bc + 4r^2) / a^2 \}^{1/2}. \end{aligned} \tag{2}$$

From  $\sum a^2 = 2s^2 - 8Rr - 2r^2$  and  $\sum bc = r^2 + s^2 + 4Rr$  we get  $b^2 + c^2 - 2bc + 4r^2 = 2ab + 2ac - a^2 - 16Rr$ . This and (2) yield

$$\sum n_a \leq s \{ 6 \sum (b + c) / a - 9 - 48Rr \sum 1/a^2 \}^{1/2}. \tag{3}$$

We leave it as an exercise to the reader to derive the identities  $\sum (b + c) / a = (s^2 + r^2) / 2Rr - 1$  and  $\sum 1/a^2 = \{ (s^2 + 4Rr + r^2) / 4Rrs \}^2 - 1/Rr$ . Therefore (3) becomes

$$\sum n_a \leq \{ (9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3) / R \}^{1/2}. \tag{4}$$

Let  $M = 2L - 9$ . For (1) we now consider

$$\begin{aligned} \text{i.e. } 9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3 &\leq L^2R^3 - 2LMR^2r + M^2Rr^2, \\ s^2 &\leq \{ L^2R^3 + (48 - 2LM)R^2r + (M^2 + 24)Rr^2 + 3r^3 \} / (9R - 3r) =: A. \end{aligned} \tag{5}$$

From [1], item 5.10, the inequality

$$s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} =: B$$

is known. Coupling it with (5) we now have to study  $B \leq A$ , i.e.

$$\begin{aligned} 6(R - 2r)(3R - r)\sqrt{R^2 - 2Rr} &\leq (L^2 - 18)R^3 - (36 + 2LM)R^2r + (M^2 + 63)Rr^2 \\ &= R\{(L^2 - 18)R^2 - (4L^2 - 18L + 36)Rr + (4L^2 - 36L + 144)r^2\}. \end{aligned}$$

Dividing the last inequality by  $(R - 2r)\sqrt{R} \geq 0$  we arrive at

$$6(3R - r)\sqrt{R - 2r} \leq \sqrt{R}\{(L^2 - 18)R - (2L^2 - 18L + 72)r\}. \tag{6}$$

Because of  $R \geq 2r$  ([1], item 5.1) the right hand side of (6) is non-negative.

Squaring (6) we get upon letting  $u = L^2 - 18$  and  $v = 2L^2 - 18L + 72$

$$324R^3 - 864R^2r + 468Rr^2 - 72r^3 \leq u^2R^3 - 2uvR^2r + v^2Rr^2,$$

i.e.

$$f(t) := (u^2 - 324)t^3 + (864 - 2uv)t^2 + (v^2 - 468)t + 72 \geq 0,$$

where we put  $t = R/r (\geq 2)$ .

It is easily checked that  $f(2) = 2(v - 2u)^2 \geq 0$ . Furthermore,  $f$  attains its local minimum at  $t_m = (-q + \sqrt{q^2 - 3pr})/3p$ , where we set  $p = u^2 - 324$ ,  $q = 864 - 2uv$  and  $r = v^2 - 468$ .

We claim:  $q^2 - 3pr \geq 0$  for  $L \geq m$ .

Indeed,

$$\begin{aligned} q^2 - 3pr &= u^2v^2 + 1404u^2 + 972v^2 - 3456uv + 291600 \\ &= (uv - 540)^2 + 108u^2\{9(v/u)^2 - 22(v/u) + 13\} \\ &\geq (uv - 540)^2 - 48u^2 \end{aligned}$$

where we used  $g(t) := 9t^2 - 22t + 13 \geq -4/9$  for  $t \in \mathbb{R}$ . Furthermore, from  $v \geq 36$  we obtain  $(uv - 540)^2 - 48u^2 \geq 48\{27(u - 15)^2 - u^2\} \geq 0$  if  $u(3\sqrt{3} - 1) \geq 45\sqrt{3}$ , i.e.  $L \geq \{18 + 45\sqrt{3}/(3\sqrt{3} - 1)\}^{1/2} = 6.0477\dots$

As it seems hopeless to get bounds for  $L$  from the inequality  $f(t_m) \geq 0$  we restrict ourselves to  $t_m \leq 2$  (and we are done because of  $f(2) \geq 0$ ), i.e.

$$\sqrt{q^2 - 3pr} \leq 6p + q. \tag{7}$$

In order to square (7) we have to assure ourselves of  $6p + q \geq 0$ , i.e.

$$6u^2 - 2uv - 1080 \geq 0, \text{ i.e. } (L^2 - 18)(L^2 + 18L - 126) \geq 540, \text{ i.e. } L \geq 6.25\dots$$

For these values of  $L$  (7) becomes

$$12p + 4q + r \geq 0, \text{ i.e. } 12u^2 - 8uv + v^2 \geq 900, \text{ i.e.} \\ (L - 6)(2L^2 + 9L - 90) \geq 25, \text{ i.e. } L \geq 6,47\dots$$

But this was to be shown.

*Added in proof.* There is now high numerical evidence for the conjecture:

$$L_{\min} = 6.258998\dots$$

**Remarks.** 1) It should be noted that in El. Math., vol. 35/5, R. H. Eddy proved the inequality

$$\sum n_a \geq \sum m_a$$

where  $m_a, m_b, m_c$  denote the medians of the given triangle. This and [1], item 8.3, i.e.

$$\sum m_a \geq 9r$$

yield the following converse of (1)

$$\sum n_a \geq 9r.$$

2) Inequality (4) and [1], item 5.9, i.e.

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

immediately lead to

$$\sum n_a < 6R - 2r.$$

3) It remains an open question to determine the precise value of  $L_{\min}$ .

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