A further remark concerning Nagel cevians

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Suggestions for further research. We observe that in all cases presented above, as in the real case, the periodic continued fractions are symmetric. Namely, for period length k we have $a_j = a_{k-j}$ and $b_j = b_{k+1-j}$ for $j = 1, 2, \ldots, k-1$. Further, $a_k = 2 \cdot a_0$ appears to hold, as in the real case. We found no examples of odd period length.
There is also a connection to Pell-like equations. We illustrate this with the example $\sqrt{376} \in \mathbb{Q}_5$. Let p_n/q_n be the nth convergent of the continued fraction expansion, defined by

$$p_{-1} = 1,\; p_0 = a_0,\; p_n = a_n \cdot p_{n-1} + 5^{b_n} \cdot p_{n-2} \quad \text{for } n \geqq 1,$$
$$q_{-1} = 0,\; q_0 = 1,\; q_n = a_n \cdot q_{n-1} + 5^{b_n} \cdot q_{n-2} \quad \text{for } n \geqq 1.$$

Put

$$p_n^2 - 376 \cdot q_n^2 = d_n \cdot 5^{c_n},\; c_n \in \mathbb{N},\; d_n \in \mathbb{Z},\; 5 \nmid d_n.$$

Then we find that $c_n = \sum_{j=0}^{n} b_j$, and the sequence $\{d_n\}_{n=-1}^{\infty}$ is given by 1, -3, 17, -4, 17, -3, 1, -3, 17, ..., which is symmetric. The fifth convergent $p_5/q_5 = 12\,103/603$, for which $d_5 = 1$, gives rise to a sort of 5-adic ‹fundamental unit› $12\,103 + 603 \cdot \sqrt{376}$, in the sense that $p_{i+6j} + q_{i+6j} \cdot \sqrt{376} = (p_i + q_i \cdot \sqrt{376}) \cdot (p_5 + q_5 \cdot \sqrt{376})^j$ for $i = -1, 0, \ldots, 4$, and $j = 0, 1, 2, \ldots$. It would be interesting to have a more general theory of these matters.

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A further remark concerning Nagel cevians

Dedicated to Professor D. S. Mitrinović on his 80th birthday

In El. Math., vol. 41/5 and 42/4, R. H. Eddy and D. S. Milošević considered the cases $L = 14$ and (the improvement) $L = 10$, resp., of

$$\sum n_a \leq 9r + L(R - 2r). \tag{1}$$

Here n_a, n_b, n_c are the Nagel cevians and R, r the circumradius and inradius, resp., of the given triangle.

In this note we show that the minimal L for (1) satisfies

$$6.258998\ldots \leq L_{\min} \leq 6.478487\ldots.$$

The lower bound is $\sup\{F(x), x > 1\}$ with $F(x)$ as given below in i); the upper bound is the largest root of $2x^3 - 3x^2 - 144x + 515 = 0$.

Proof. Let N_a be the foot of n_a on side BC. Then, $BN_a = s - c$, and from the triangle ABN_a it follows via the law of cosines that

$$n_a^2 = c^2 + (s-c)^2 - 2c(s-c)\cos B.$$

After some algebraic manipulations this simplifies to

$$n_a = s\sqrt{(b-c)^2 + 4r^2/a}$$

where s represents the semiperimeter of the given triangle.
i) Considering triangles with $c = 2$ and $a = b = x$, $x > 1$, we get $r = \sqrt{(x-1)/(x+1)}$, $R = x^2/2\sqrt{x^2-1}$, $n_c = \sqrt{x^2-1}$ and $n_a = n_b = \sqrt{x+1}\sqrt{(x-2)^2(x+1) + 4(x-1)/x}$. Inserting this in (1) we get after some algebraic manipulations

$$L \geq F(x) := 2\sqrt{x-1}\,\{(2x+2)\sqrt{(x-2)^2(x+1) + 4(x-1)} + (x^2 - 8x)\sqrt{x-1}\}/x(x-2)^2.$$

Numerical calculations lead to $L \geq F(4.699\ldots) = 6.258998\ldots =: m$.
ii) The inequality between the arithmetic and square root means yields

$$\sum n_a \leq \sqrt{3\sum n_a^2},$$

i.e.

$$\sum n_a \leq s\{3\sum(b^2 + c^2 - 2bc + 4r^2)/a^2\}^{1/2}. \tag{2}$$

From $\sum a^2 = 2s^2 - 8Rr - 2r^2$ and $\sum bc = r^2 + s^2 + 4Rr$ we get $b^2 + c^2 - 2bc + 4r^2 = 2ab + 2ac - a^2 - 16Rr$. This and (2) yield

$$\sum n_a \leq s\{6\sum(b+c)/a - 9 - 48Rr\sum 1/a^2\}^{1/2}. \tag{3}$$

We leave it as an exercise to the reader to derive the identities $\sum(b+c)/a = (s^2 + r^2)/2Rr - 1$ and $\sum 1/a^2 = \{(s^2 + 4Rr + r^2)/4Rrs\}^2 - 1/Rr$. Therefore (3) becomes

$$\sum n_a \leq \{(9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3)/R\}^{1/2}. \tag{4}$$

Let $M = 2L - 9$. For (1) we now consider

$$9Rs^2 - 3rs^2 - 48R^2r - 24Rr^2 - 3r^3 \leq L^2R^3 - 2LMR^2r + M^2Rr^2,$$

i.e.

$$s^2 \leq \{L^2R^3 + (48 - 2LM)R^2r + (M^2 + 24)Rr^2 + 3r^3\}/(9R - 3r) =: A. \tag{5}$$

From [1], item 5.10, the inequality

$$s^2 \le 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} =: B$$

is known. Coupling it with (5) we now have to study $B \le A$, i.e.

$$6(R - 2r)(3R - r)\sqrt{R^2 - 2Rr} \le (L^2 - 18)R^3 - (36 + 2LM)R^2 r + (M^2 + 63)Rr^2$$

$$= R\{(L^2 - 18)R^2 - (4L^2 - 18L + 36)Rr + (4L^2 - 36L + 144)r^2\}.$$

Dividing the last inequality by $(R - 2r)\sqrt{R} \ge 0$ we arrive at

$$6(3R - r)\sqrt{R - 2r} \le \sqrt{R}\{(L^2 - 18)R - (2L^2 - 18L + 72)r\}. \tag{6}$$

Because of $R \ge 2r$ ([1], item 5.1) the right hand side of (6) is non-negative.

Squaring (6) we get upon letting $u = L^2 - 18$ and $v = 2L^2 - 18L + 72$

$$324R^3 - 864R^2 r + 468Rr^2 - 72r^3 \le u^2 R^3 - 2uvR^2 r + v^2 Rr^2,$$

i.e.

$$f(t) := (u^2 - 324)t^3 + (864 - 2uv)t^2 + (v^2 - 468)t + 72 \ge 0,$$

where we put $t = R/r$ (≥ 2).
It is easily checked that $f(2) = 2(v - 2u)^2 \ge 0$. Furthermore, f attains its local minimum at $t_m = (-q + \sqrt{q^2 - 3pr})/3p$, where we set $p = u^2 - 324$, $q = 864 - 2uv$ and $r = v^2 - 468$.
We claim: $q^2 - 3pr \ge 0$ for $L \ge m$.
Indeed,

$$q^2 - 3pr = u^2 v^2 + 1404u^2 + 972v^2 - 3456uv + 291\,600$$

$$= (uv - 540)^2 + 108u^2\{9(v/u)^2 - 22(v/u) + 13\}$$

$$\ge (uv - 540)^2 - 48u^2$$

where we used $g(t) := 9t^2 - 22t + 13 \ge -4/9$ for $t \in R$. Furthermore, from $v \ge 36$ we obtain $(uv - 540)^2 - 48u^2 \ge 48\{27(u - 15)^2 - u^2\} \ge 0$ if $u(3\sqrt{3} - 1) \ge 45\sqrt{3}$, i.e. $L \ge \{18 + 45\sqrt{3}/(3\sqrt{3} - 1)\}^{1/2} = 6.0477\ldots$.
As it seems hopeless to get bounds for L from the inequality $f(t_m) \ge 0$ we restrict ourselves to $t_m \le 2$ (and we are done because of $f(2) \ge 0$), i.e.

$$\sqrt{q^2 - 3pr} \le 6p + q. \tag{7}$$

In order to square (7) we have to assure ourselves of $6p + q \ge 0$, i.e.

$$6u^2 - 2uv - 1080 \ge 0, \text{ i.e. } (L^2 - 18)(L^2 + 18L - 126) \ge 540, \text{ i.e. } L \ge 6.25\ldots.$$

For these values of L (7) becomes

$$12p + 4q + r \geq 0, \text{ i.e. } 12u^2 - 8uv + v^2 \geq 900, \quad \text{i.e.}$$
$$(L-6)(2L^2 + 9L - 90) \geq 25, \quad \text{i.e.} \quad L \geq 6{,}47\ldots.$$

But this was to be shown.

Added in proof. There is now high numerical evidence for the conjecture:

$$L_{\min} = 6.258998\ldots.$$

Remarks. 1) It should be noted that in El. Math., vol. 35/5, R. H. Eddy proved the inequality

$$\sum n_a \geq \sum m_a$$

where m_a, m_b, m_c denote the medians of the given triangle. This and [1], item 8.3, i.e.

$$\sum m_a \geq 9r$$

yield the following converse of (1)

$$\sum n_a \geq 9r.$$

2) Inequality (4) and [1], item 5.9, i.e.

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

immediately lead to

$$\sum n_a < 6R - 2r.$$

3) It remains an open question to determine the precise value of $L_{\min}$.

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