Some inequalities for the triangle

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Some inequalities for the triangle

Notation. a, b, c – sides BC, CA, AB of a triangle ABC; α, β, γ – its angles; s – semiperimeter; F – area; R – radius of circumcircle; r – radius of incircle; h_a, h_b, h_c – altitudes; m_a, m_b, m_c – medians and r_a, r_b, r_c – radii of excircles.

Theorem 1. *In every triangle the following equalities are valid:*

$$\frac{h_b+h_c}{r_a}+\frac{h_c+h_a}{r_b}+\frac{h_a+h_b}{r_c}=6 \tag{1}$$

$$\cos^3\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}+\cos^3\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2}+\cos^3\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}=\frac{3s}{4R}. \tag{2}$$

Proof. Since

$$h_a=\frac{2F}{a}, \qquad h_b=\frac{2F}{b}, \qquad h_c=\frac{2F}{c} \quad \text{and}$$

$$r_a=\frac{F}{s-a}, \qquad r_b=\frac{F}{s-b}, \qquad r_c=\frac{F}{s-c},$$

we have

$$\frac{h_b+h_c}{r_a}+\frac{h_c+h_a}{r_b}+\frac{h_a+h_b}{r_c}$$
$$=\frac{2}{abc}(a(s-a)(b+c)+b(s-b)(c+a)+c(s-c)(a+b))$$
$$=\frac{2}{abc}(a^3+b^3+c^3-2s(a^2+b^2+c^2)+2s(ab+bc+ca)).$$

From $abc=4Rrs$, $ab+bc+ca=r^2+s^2+4Rr$, $a^2+b^2+c^2=2(s^2-4Rr-r^2)$ and $a^3+b^3+c^3=2s(s^2-6Rr-3r^2)$, we get

$$\frac{h_b+h_c}{r_a}+\frac{h_c+h_a}{r_b}+\frac{h_a+h_b}{r_c}$$
$$=\frac{1}{2Rrs}(2s(r^2+s^2+4Rr)-2s\cdot 2(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2))=6\,,$$

i.e. (1) holds.
Moreover, since

$$h_b+h_c=8R\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2},\qquad h_c+h_a=8R\sin\frac{\beta}{2}\cos^2\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2},$$

$$h_a+h_b=8R\sin\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}\quad\text{and}$$

$$r_a=s\cdot\tan\frac{\alpha}{2},\qquad r_b=s\cdot\tan\frac{\beta}{2},\qquad r_c=s\cdot\tan\frac{\gamma}{2},$$

we obtain

$$\frac{h_b+h_c}{r_a}+\frac{h_c+h_a}{r_b}+\frac{h_a+h_b}{r_c}$$
$$=\frac{8R}{s}\left(\cos^3\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}+\cos^3\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2}+\cos^3\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}\right),$$

which is equivalent to

$$\cos^3\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}+\cos^3\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2}+\cos^3\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}=\frac{3}{4}\cdot\frac{s}{R},$$

i.e., (2).

Corollary 1. *In every triangle*

$$3\cdot 2^n\le\left(\frac{h_b+h_c}{r_a}\right)^n+\left(\frac{h_c+h_a}{r_b}\right)^n+\left(\frac{h_a+h_b}{r_c}\right)^n\le 6^n,\qquad n\ge 1\,. \tag{A}$$

Equality in (A) *holds if and only if the triangle is equilateral and* $n=1$.

Proof. The inequality (A) follows from (1) putting $x_1=\frac{h_b+h_c}{r_a}$, $x_2=\frac{h_c+h_a}{r_b}$ and $x_3=\frac{h_a+h_b}{r_c}$ in ([7], 47)

$$3\left(\sum_{i=1}^{3}\frac{x_i}{3}\right)^n \leq \sum_{i=1}^{3} x_i^n \leq \left(\sum_{i=1}^{3} x_i\right)^n, \qquad (n\geq 1,\ x_i>0).$$

Remark 1. a) The left part of (A) holds for $n\leq 0$. b) If $0<n<1$, then the inequality (A) is reversed.

Remark 2. For $n=-1$, we have the inequality 6.26 in [3]:

$$\frac{r_a}{h_b+h_c}+\frac{r_b}{h_c+h_a}+\frac{r_c}{h_a+h_b}\geq\frac{3}{2}.$$

Corollary 2. *The following inequality*

$$\cos^3\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}+\cos^3\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2}+\cos^3\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}\leq\frac{9\sqrt{3}}{8}$$

holds, with inequality only for $\alpha=\beta=\gamma$.

Proof. This follows from (2) and $2s=a+b+c\leq 3R\sqrt{3}$ ([3], 5.3).

Theorem 2. *In every triangle*

$$\left(\frac{m_a}{r_a}\right)^n+\left(\frac{m_b}{r_b}\right)^n+\left(\frac{m_c}{r_c}\right)^n\geq 3^{1-3n/2}\cdot\left(\frac{s}{r}\right)^n\geq 3, \qquad n\geq 2/3, \tag{B}$$

and the equality is true if an only if the triangle is equilateral.

Proof. Since ([5], 156)

$$m_a\geq\sqrt{s(s-a)}, \qquad m_b\geq\sqrt{s(s-b)}, \qquad m_c\geq\sqrt{s(s-c)},$$

we have

$$\left(\frac{m_a}{r_a}\right)^n+\left(\frac{m_b}{r_b}\right)^n+\left(\frac{m_c}{r_c}\right)^n\geq\frac{1}{r^n s^{n/2}}\left((s-a)^{3n/2}+(s-b)^{3n/2}+(s-c)^{3n/2}\right).$$

Let us consider the function

$$f(x)=(s-x)^{3n/2}, \qquad (0<x<s,\ n\geq 2/3)$$

and its second derivative

$$f''(x) = \frac{3n}{4}(3n-2)(s-x)^{\frac{3n}{2}-2} \geq 0 .$$

Hence, the function f is convex, so that

$$\sum_{i=1}^{3} (s-x_i)^{3n/2} \geq 3^{1-3n/2}\left(3s - \sum_{i=1}^{3} x_i\right)^{3n/2}, \quad n \geq \frac{2}{3}. \tag{4}$$

Putting $x_1 = a$, $x_2 = b$ and $x_3 = c$ in (4), we get

$$(s-a)^{3n/2} + (s-b)^{3n/2} + (s-c)^{3n/2} \geq 3^{1-3n/2} \cdot s^{3n/2}. \tag{5}$$

Now (5) and (3) imply that

$$\left(\frac{m_a}{r_a}\right)^n + \left(\frac{m_b}{r_b}\right)^n + \left(\frac{m_c}{r_c}\right)^n \geq 3^{1-3n/2}\left(\frac{s}{r}\right)^n, \quad n \geq \frac{2}{3}. \tag{6}$$

From (6) and $s^2 \geq 27r^2$ ([3], 5.11), we obtain:

$$\left(\frac{m_a}{r_a}\right)^n + \left(\frac{m_b}{r_b}\right)^n + \left(\frac{m_c}{r_c}\right)^n \geq 3 .$$

Theorem 3. *In every triangle*

$$4\sqrt{3}(3R-4r) \leq \frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b} \leq \frac{2R\sqrt{3}}{r}(3R-4r). \tag{C}$$

Equality occurs if and only if the triangle is equilateral.

Proof. We have

$$\begin{aligned}&\frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b}\\ &= \frac{1}{F^2}(a^3(s-b)(s-c) + b^3(s-c)(s-a) + c^3(s-a)(s-b))\\ &= \frac{1}{F^2}(s(a^4+b^4+c^4) - s^2(a^3+b^3+c^3) + abc(a^2+b^2+c^2)).\end{aligned}$$

From $F = rs$, $abc = 4Rrs$, $a^2+b^2+c^2 = 2\cdot(s^2-4Rr-r^2)$, $a^3+b^3+c^3 = 2s(s^2-6Rr-3r^2)$ and $a^4+b^4+c^4 = 2(s^2-4Rr-r^2)^2 - 2(2rs)^2$, we obtain

$$\begin{aligned}\frac{a^3}{r_b r_c} + \frac{b^3}{r_c r_a} + \frac{c^3}{r_a r_b} &= \frac{2}{rs}(4Rr^2 + 2Rs^2 - 3rs^2 + r^3)\\ &= \frac{2}{F}(r\cdot r(4R+r) + s\cdot s(2R-3r)).\end{aligned} \tag{7}$$

Then, since ([3], 7.2, 5.11)

$$r(4R+r) \geq F\sqrt{3}$$

and

$$s^2 \geq 27r^2,$$

(7) implies

$$\frac{a^3}{r_b r_c}+\frac{b^3}{r_c r_a}+\frac{c^3}{r_a r_b} \geq \frac{2}{F}(r\cdot F\sqrt{3}+s\cdot 3r\sqrt{3}(2R-3r)) = 4\sqrt{3}(3R-4r),$$

i.e. the first part of the inequality (C).
By 5.3 and 5.6 in [3] and (7) we get the second part of (C).

Theorem 4. *In every triangle*

$$\frac{4\sqrt{3}}{3}\cdot\frac{7R-2r}{9R-2r} \leq \frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b} \leq \sqrt{\frac{3R}{2r}}. \tag{D}$$

Equality holds if and only if the triangle is equilateral.

Proof. Since

$$h_a=\frac{2F}{a}, \quad h_b=\frac{2F}{b}, \quad h_c=\frac{2F}{c},$$

we have

$$\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}=\frac{abc}{2F}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$

$$=\frac{abc}{2F}\cdot\frac{(a^2+b^2+c^2+3(ab+bc+ca))}{((a+b+c)(ab+bc+ca)-abc)}.$$

From $abc=4Rrs$, $F=rs$, $a+b+c=2s$, $ab+bc+ca=r^2+s^2+4Rr$ and $a^2+b^2+c^2=2(s^2-4Rr-r^2)$, we get

$$\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}=\frac{R}{s}\cdot\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}$$

$$=\frac{R}{s}\left(5-\frac{2r(3R+2r)}{s^2+2Rr+r^2}\right).$$

Using ([3], 5.8, 5.3)

$$s^2 \geq r(16R - 5r)$$

and

$$2s = a + b + c \leq 3R\sqrt{3},$$

we obtain

$$\begin{aligned}\frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} &\geq \frac{R}{s}\left(5 - \frac{3R+2r}{9R-2r}\right)\\ &= \frac{R}{s} \cdot \frac{6(7R-2r)}{9R-2r}\\ &\geq \frac{4\sqrt{3}}{3} \cdot \frac{7R-2r}{9R-2r},\end{aligned}$$

i.e. the left part of the inequality (D).
From the inequalities

$$h_b + h_c \geq 2\sqrt{h_b h_c}, \quad h_c + h_a \geq 2\sqrt{h_c h_a}, \quad h_a + h_b \geq 2\sqrt{h_a h_b},$$

it follows that

$$\begin{aligned}\frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} &= 2F\left(\frac{1}{h_a(h_b+h_c)} + \frac{1}{h_b(h_c+h_a)} + \frac{1}{h_c(h_a+h_b)}\right)\\ &\leq \frac{F}{\sqrt{h_a h_b h_c}}\left(\frac{1}{\sqrt{h_a}} + \frac{1}{\sqrt{h_b}} + \frac{1}{\sqrt{h_c}}\right).\end{aligned} \tag{8}$$

If in the well-known inequality

$$(x+y+z)^2 \leq 3(x^2+y^2+z^2), \quad (x,y,z>0), \tag{9}$$

we put $x = \frac{1}{\sqrt{h_a}}$, $y = \frac{1}{\sqrt{h_b}}$ and $z = \frac{1}{\sqrt{h_c}}$, we obtain

$$\left(\frac{1}{\sqrt{h_a}} + \frac{1}{\sqrt{h_b}} + \frac{1}{\sqrt{h_c}}\right)^2 \leq 3\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right). \tag{10}$$

From $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$, $h_a h_b h_c = \frac{2F^2}{R}$, (8) and (10), we get

$$\frac{a}{h_b+h_c} + \frac{b}{h_c+h_a} + \frac{c}{h_a+h_b} \leq \frac{F}{\sqrt{h_a h_b h_c}}\sqrt{\frac{3}{r}} = \sqrt{\frac{3R}{2r}}.$$

This proves the right part of (D).

Remark 4. From (D) using the inequality 5.1 in [3] we have the inequality

$$\sqrt{3} \le \frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b} \le \frac{R}{2r}\sqrt{3},$$

shown in [4].

Theorem 5. *In every triangle*

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab} \le \frac{5}{4}+\frac{R}{2r}, \tag{E}$$

with equality holding if and only if the triangle is equilateral.

Proof. Using Bager's identity ([2], 39)

$$a m_a^2 + b m_b^2 + c m_c^2 = \frac{s}{2}(s^2+2Rr+5r^2),$$

we get

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab} = \frac{1}{abc}(a m_a^2 + b m_b^2 + c m_c^2)$$

$$= \frac{s}{2abc}(s^2+2Rr+5r^2).$$

Then, since ([3], 5.8, 5.1)

$$s^2 \le 4R^2+4Rr+3r^2$$

and

$$2r \le R,$$

we have

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab} \le \frac{1}{8Rr}(4R^2+6Rr+8r^2)$$

$$\le \frac{1}{8Rr}\left(4R^2+6Rr+8r\cdot\frac{R}{2}\right)$$

$$= \frac{5}{4}+\frac{R}{2r},$$

i.e. (E) holds.

Theorem 6. *For acute triangle the following inequality holds*

$$a^2 \tan\alpha + b^2 \tan\beta + c^2 \tan\gamma \geq \frac{4 r s^3}{2R^2 + r^2}. \tag{F}$$

Proof. If in Jensen's inequality for a convex function

$$f\left(\frac{\sum_{i=1}^{3} p_i}{\sum_{i=1}^{3} x_i}\right) \leq \frac{\sum_{i=1}^{3} x_i f\left(\frac{p_i}{x_i}\right)}{\sum_{i=1}^{3} x_i}, \quad p_i \text{ and } \frac{p_i}{x_i} \in \bar{I},$$

we put

$$x_1 = a, \quad x_2 = b, \quad x_3 = c, \quad p_1 = a^2, \quad p_2 = b^2 \quad \text{and} \quad p_3 = c^2,$$

then, for $f(x) = 1/x$, from ([1], 43; [3] 5.14)

$$a^2 + b^2 + c^2 = 4F(\cot\alpha + \cot\beta + \cot\gamma)$$

and

$$a^2 + b^2 + c^2 \leq 8R^2 + 4r^2,$$

we obtain (F).

Remark 5. This method shown in [6].

Theorem 7. *In every triangle*

$$3\sqrt{3r} \leq \sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \leq \left(1 + \frac{r}{R}\right)\sqrt{6R}. \tag{G}$$

Equality holds if and only if the triangle is equilateral.

Proof. By means of the arithmetic-geometric inequality, we get

$$\sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \geq 3\sqrt[6]{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}$$

$$= 3\sqrt[6]{\frac{2r^2 s^2}{R}}. \tag{11}$$

Then, since ([3], 5.12)

$$2s^2 \geq 27Rr,$$

(11) implies

$$\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c} \geq 3\sqrt{3r},$$

i.e. the left part of the inequality (G).

If in the inequality (9) we put $x=\frac{1}{\sqrt{a}}$, $y=\frac{1}{\sqrt{b}}$ and $z=\frac{1}{\sqrt{c}}$, then, from ([3], 5.17)

$$ab+bc+ca \leq 4(R+r)^2,$$

we obtain

$$\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c} \leq \sqrt{2F}\cdot\sqrt{3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

$$=\sqrt{\frac{3}{2R}(ab+bc+ca)}$$

$$\leq\left(1+\frac{r}{R}\right)\sqrt{6R},$$

i.e. the right part of (G).

Remark 6. The second part of the inequality (G) is more precise than

$$\sqrt{h_a}+\sqrt{h_b}+\sqrt{h_c} \leq \frac{3}{2}\sqrt{6R},$$

shown in [3].

Theorem 8. *In every triangle the following inequality is valid:*

$$\frac{r_b r_c}{a^2}+\frac{r_c r_a}{b^2}+\frac{r_a r_b}{c^2} \geq \frac{9}{4}. \tag{H}$$

Equality holds if and only if the triangle is equilateral.

Proof 1. Inequality (H) is equivalent to

$$\frac{s-a}{a^2}+\frac{s-b}{b^2}+\frac{s-c}{c^2} \geq \frac{9}{4s}. \tag{12}$$

The function

$$f(x)=\frac{s}{x^2}-\frac{1}{x} \quad (0<x<s)$$

is convex, so that

$$\sum_{i=1}^{3} f(x_i) \geq 3 \cdot f\left(\frac{\sum_{i=1}^{3} x_i}{3}\right), \quad x_i \in \bar{I}.$$

If we put $x_1 = a$, $x_2 = b$ and $x_3 = c$, we get the inequality (H).

Proof 2. As

$$\frac{r_b r_c}{a^2} + \frac{r_c r_a}{b^2} + \frac{r_a r_b}{c^2}$$

$$= \frac{s(s-a)}{a^2} + \frac{s(s-b)}{b^2} + \frac{s(s-c)}{c^2}$$

$$= \frac{s^2}{a^2 b^2 c^2}(b^2 c^2 + c^2 a^2 + a^2 b^2) - \frac{s}{a b c}(a b + b c + c a)$$

$$= \frac{1}{16 R^2 r^2}(b^2 c^2 + c^2 a^2 + a^2 b^2 - 4 R r (a b + b c + c a)) . \quad (13)$$

From ([3], 5.17)

$$a b + b c + c a \geq 4 r (5 R - r),$$

$$(b c + a b + c a)^2 \leq 3 (b^2 c^2 + c^2 a^2 + a^2 b^2)$$

and (13), we have

$$\frac{r_b r_c}{a^2} + \frac{r_c r_a}{b^2} + \frac{r_a r_b}{c^2} \geq \frac{a b + b c + c a}{48 R^2 r^2}(a b + b c + c a - 12 R r)$$

$$\geq \frac{(5R - r)(2R - r)}{3R^2} . \quad (14)$$

By (14) and the inequality 5.1 in [3], we obtain (H).

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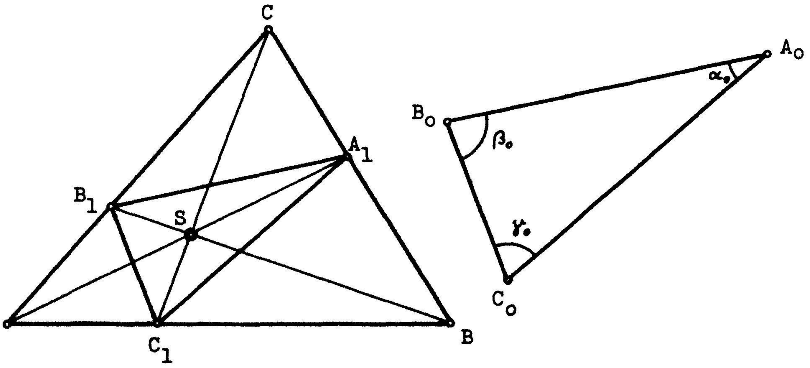
Ceva-Dreiecke

Ein einem Dreieck ABC einbeschriebenes Dreieck $A_1B_1C_1$ (Fig. 1) möge *Ceva-Dreieck* heissen, wenn sich die drei Transversalen AA_1, BB_1, CC_1 in einem Punkt S schneiden. Nach dem Satz von *Ceva* tritt dies genau dann ein, wenn die drei Teilverhältnisse

$$[ABC_1] = -\tau < 0\,; \quad [BCA_1] = -\varrho < 0\,; \quad [CAB_1] = -\sigma < 0 \tag{1}$$

der Gleichung genügen

$$\varrho \cdot \sigma \cdot \tau = 1\,. \tag{2}$$



Figur 1.

Im folgenden soll gezeigt werden:

Satz 1: *Gegeben sei ein beliebiges Dreieck ABC und ein ebenfalls beliebiges Dreieck $A_0B_0C_0$ (Fig. 1). Dann gibt es genau ein dem Dreieck ABC einbeschriebenes Ceva-Dreieck $A_1B_1C_1$, das dem Dreieck $A_0B_0C_0$ ähnlich ist.*

Beweis: Zunächst weisen wir die *Existenz* eines solchen Dreiecks $A_1B_1C_1$ nach. Dazu führen wir ein Koordinatensystem ein, das dem vorgegebenen Dreieck ABC nach Figur 2 angepasst ist.

Aus der Figur können die Bezeichnungen für die Masszahlen der Winkel (im Bogenmass) sowie die Koordinaten der Punkte (in Abhängigkeit von den Teilverhältnissen ϱ, σ, τ) abgelesen werden. Dabei dürfen wir ohne Einschränkung der Allgemeinheit voraussetzen, dass

$$b, c, b-a \in \mathbb{R}^+\,; \quad a \in \mathbb{R}\,; \quad \alpha_1, \beta_1, \gamma_1 \in \mathbb{R}^+\,; \quad \alpha_1 + \beta_1 + \gamma_1 = \pi\,. \tag{3}$$