Archimedes was right. Part two

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Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **42 (1987)**

Heft 4

PDF erstellt am: **25.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-40036

ELEMENTE DER MATHEMATIK
Revue de mathématiques élémentaires – Rivista di matematica elementare
Zeitschrift zur Pflege der Mathematik
und zur Förderung des mathematisch-physikalischen Unterrichts

El. Math. Vol. 42 Nr. 4 Seiten 83–114 Basel, Juli 1987

Archimedes was right

(Part Two)

A definition of the sum of a sequence of numbers, inspired by the second method of Archimedes for calculating the area of a parabolic section, is presented in the first part of this note. The great advantage of this definition over the standard one is that it avoids the notion of limit. Archimedes' second method also suggests a definition of integral in which limits are not used either; they are replaced by sums of sequences. This definition, presented in this second part, is simpler than the usual one and, at the same time, more general.

It should be mentioned perhaps that the «Archimedean» definition of sum leads to absolutely summable sequences and, similarly, the «Archimedean» definition of integral produces absolutely integrable functions. If we want to deal with conditionally summable sequences or improperly integrable functions, then the use of limits becomes inevitable. However, such notions are necessary only in more advanced chapters of analysis dealing with questions whose complexity is not really increased by the involvement of limits.

Calculations of areas and volumes carried out by Archimedes are justly considered as the point of departure towards integral calculus. Apparently, the direction of the path leading there was influenced more by the first way in which Archimedes calculated the area of a parabolic section rather than the second way. To make the description of the first way shorter, we shall take only a special case and rearrange it as suggested by O. Toeplitz in his book [3]. Actually, the calculation was taken up and subsequently developed by later masters in a similarly rearranged form.

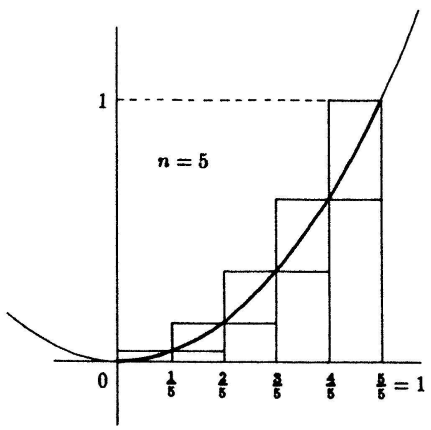

Figure 6.

Let us consider the parabola with equation $y=x^2$ and let us calculate the area, $\mu(S)$, of the part S of the unit square which lies «under» the parabola (figure 6). The base of the unit square is divided into n segments of equal length. With these segments as bases two sets of rectangles are erected. The rectangles of one set are just contained in S and the rectangles of the other set just contain the corresponding portion of S. The sum of the areas of the former is denoted by l_n and of the latter by u_n. Then

$$l_n = \frac{1}{n}\left(\frac{0}{n}\right)^2 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2,$$

$$u_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2 + \frac{1}{n}\left(\frac{n}{n}\right)^2,$$

for every $n=1,2,3,\ldots$ Archimedes was able to calculate these sums explicitly. Namely,

$$l_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 (0^2+1^2+2^2+\ldots+(n-1)^2) = \frac{1}{6}\frac{(n-1)\,n\,(2n-1)}{n^3},$$

$$u_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 (1^2+2^2+\ldots+(n-1)^2+n^2) = \frac{1}{6}\frac{n\,(n+1)\,(2n+1)}{n^3},$$

for every $n=1,2,3,\ldots$ It follows that

$$l_n < \frac{1}{3} < u_n$$

for every $n=1,2,3,\ldots$ But, because

$$u_n - l_n = \frac{1}{n}$$

for every $n=1,2,3,\ldots$, by Archimedes' lemma, 1/3 is the only number larger than l_n and less than u_n for every $n=1,2,3,\ldots$ On the other hand, $l_n < \mu(S) < u_n$, for every $n=1,2,3,\ldots$ Therefore,

$$\mu(S) = \frac{1}{3}.$$

Using precisely the same procedure, Bonaventura Cavalieri succeeded, 1880 years after Archimedes (!), in calculating the area «under» the curve $y=x^3$ and then «under» $y=x^k$ for $k=4,5,\ldots,9$. Altering the procedure somewhat, Pierre de Fermat calculated the area «under» the curve $y=x^k$, for any positive integer k around 1650.
It is not hard to see in this procedure a model for the notion of Riemann integral, especially in the formulation of Darboux, and for the related notion of Jordan measure. True enough, it took mathematicians another 200 years or so to arrive at a conscious and explicit formulation, but what is a mere 200 years compared with 2000?

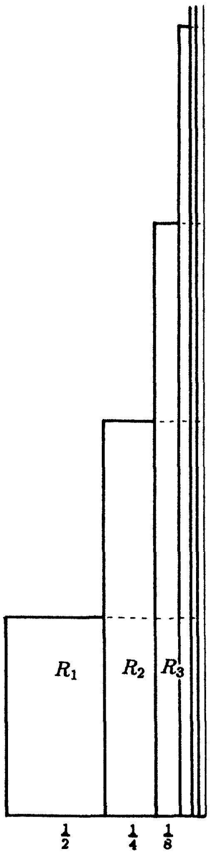

Figure 7 a.

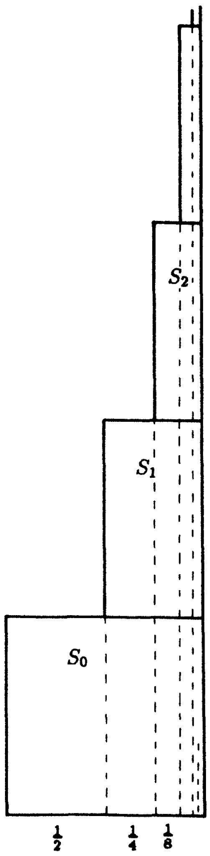

Figure 7 b.

And during all these years, the ideas latent in the second method of Archimedes remained by and large untapped and underdeveloped, although an isolated opportunity to develop them arose. Sometime in the 1350's, Nicole d'Oresme had shown that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\frac{5}{2^5}+\ldots=2. \tag{10}$$

This sum was already known but Oresme used a geometric method which is relevant for our discussion. He divided the same planar region into rectangles in two ways. Each rectangle R_n in figure 7 a has base of length $1/2^n$ and height n; therefore its area, $\mu(R_n)$, is equal to $n/2^n$, for every $n=1,2,3,\ldots$ Each rectangle S_n in figure 7 b has base length

$$\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\frac{1}{2^{n+4}}+\ldots,$$

which, by (8), is equal to $1/2^n$, for every $n=0,1,2,\ldots$ The height of each rectangle S_n is equal to 1 and, therefore, its area, $\mu(S_n)$, is equal to $1/2^n$, for every $n=0,1,2,\ldots$ Then

$$\mu(R_1)+\mu(R_2)+\mu(R_3)+\mu(R_4)+\ldots=\mu(S_0)+\mu(S_1)+\mu(S_2)+\mu(S_3)+\ldots$$

that is,

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\ldots=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$$

Because, by (8), the sum on the right is equal to 2, we have (10).
Archimedes' determination of the area of a parabolic section illustrated in figure 3, his finding the sum (2) using figure 4 and Oresme's finding the sum (10) are all based on the following property of area which is called σ-additivity: if a planar set S is equal to the union of sets $S_1, S_2, S_3, \ldots$, such that the common part of any two of them has the area equal to zero, then the area of S is equal to the sum of the areas of the sets $S_1, S_2, S_3, \ldots$ Archimedes' and Oresme's calculations show that this property can be used for the determination of areas of planar sets and also for finding sums of sequences. If we interpret the integral of a (positive) function as the area of the «region under its graph», then we can of course use this property for finding integrals.

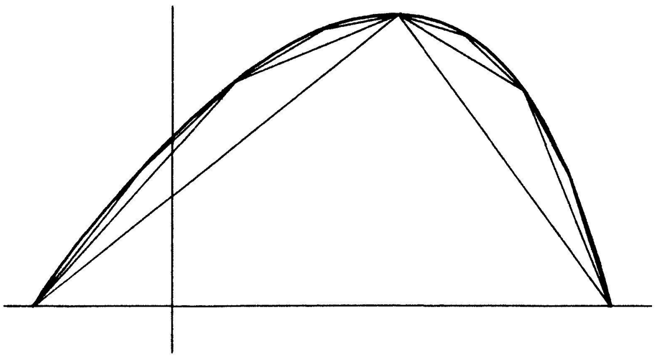

Figure 8.

Let us re-arrange figure 3 so that the parabolic section becomes the «region under a curve» as it is customary in calculus books. We obtain something like figure 8. Now, if the graph of our function is not a part of a parabola, then it is advantageous to replace the triangles by rectangles. For Archimedes' trick with triangles obviously would not work in general and, as we shall see, rectangles allow us to use differential calculus in its stead. We obtain a picture like that in figure 9.
Let us translate these geometric ideas into analytic language. Only a minimal amount of notation is needed to do it efficiently.
The length of a bounded interval J, that is, the absolute value of the difference of its end-points, is denoted by $\lambda(J)$. The characteristic function of an interval J is denoted by χ_J. That is, if x is a number belonging to J, then $\chi_J(x)=1$ and, if x is a number

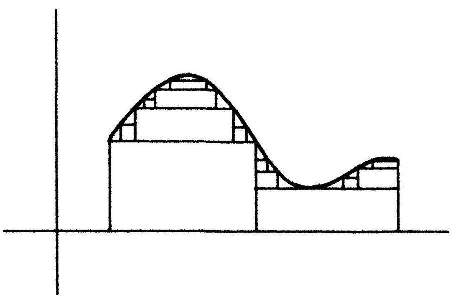

Figure 9.

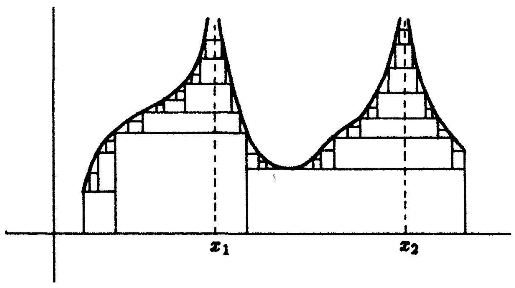

Figure 10.

which does not belong to J, then $\chi_J(x)=0$. So, if c is a positive number and J a bounded interval, then the «region under the graph» of the function $c\,\chi_J$ is a rectangle (the shaded region in figure 12) whose base has the length $\lambda(J)$, the height is c and, hence, the area is $c\,\lambda(J)$.

Using these conventions, which are more-or-less standard, we can formally describe the situation illustrated in figure 9. Namely, the «region under the graph» of a positive function f in an interval I can be expressed as a union of non-overlapping rectangles (with sides parallel to coordinate axes) if and only if there exist numbers c_j and intervals $J_j \subset I$, $j=1,2,3,\ldots$, such that

$$f(x)=c_1\chi_{J_1}(x)+c_2\chi_{J_2}(x)+c_3\chi_{J_3}(x)+c_4\chi_{J_4}(x)+\ldots \tag{11}$$

for every $x\in I$. The integral of the function f in the interval I (the area of the «region under the graph» of f in I) is then equal to the sum

$$c_1\lambda(J_1)+c_2\lambda(J_2)+c_3\lambda(J_3)+c_4\lambda(J_4)+\ldots \tag{12}$$

Although the class of such functions is already quite wide, Oresme's derivation of (10) suggest that it can be widened. For example, it is possible to include some unbounded functions and it is not necessary to assume that the underlying interval I be bounded. But of course, some precautions then have to be taken because the area of the «region under the graph» may be infinite and, besides, it may not be possible (or easy) to cover the «region under the graph» by rectangles without also covering some points off that region.

So, let us call Oresme integrable in an interval I any function f for which there exist numbers $c_j \geqq 0$ and bounded intervals $J_j\subset I$, $j=1,2,3,\ldots$, such that

(i) the sequence

$$c_1\lambda(J_1),\ c_2\lambda(J_2),\ c_3\lambda(J_3),\ c_4\lambda(J_4),\ldots \tag{13}$$

is summable; and

(ii) the equality (11) holds for every $x\in I$ for which the sequence

$$c_1\chi_{J_1}(x),\ c_2\chi_{J_2}(x),\ c_3\chi_{J_3}(x),\ c_4\chi_{J_4}(x),\ldots \tag{14}$$

is summable.

The integral of the function f in the interval I is then of course defined to be equal to the sum (12).

Condition (ii) does not restrict in any way the values of the function f at the points $x\in I$ for which the sequence (14) is not summable; the values of f at such points can be arbitrary.

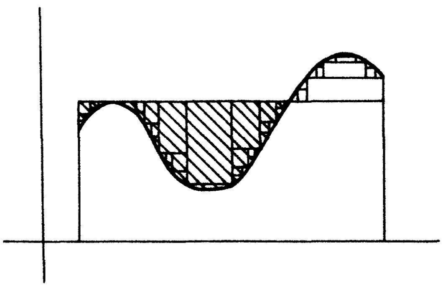

Figure 11.

The graph of an Oresme integrable function f is sketched in figure 10 together with a choice of rectangles covering the «region under its graph». These rectangles cover also some points which do not belong to that region, namely all points «above» the point $(x_1, f(x_1))$ and «above» the point $(x_2, f(x_2))$. The sequences $c_1 \chi_{J_1}(x_1)$, $c_2 \chi_{J_2}(x_1)$, $c_3 \chi_{J_3}(x_1), \ldots$ and $c_1 \chi_{J_1}(x_2)$, $c_2 \chi_{J_2}(x_2)$, $c_3 \chi_{J_3}(x_2), \ldots$ are not summable.

The class of Oresme integrable functions is still unnecessarily limited by our approach which is purely geometric in spite of the notation and terminology. To break out of this limitation, we should make full use of the possibilities present in the algebraic (calculational) properties of numbers. It is not difficult to do so. Surprisingly enough, it suffices to remove the restriction that the numbers $c_1, c_2, c_3, \ldots$, in the definition of an integrable function, be positive. Each integrable function will then be a difference of two Oresme integrable functions. Indeed, recall that a sequence is summable if and only if the sequence of its positive terms and the sequence of its negative terms are both summable. (We are speaking of the summability in the sense of the «Archimedean» definition!) Let us adopt the following definition.

A function f will be called Archimedes integrable in an interval I if there exist numbers c_j and bounded intervals $J_j \subset I, j = 1, 2, 3, \ldots$, such that

(i) the sequence (13) is summable; and

(ii) the equality (11) holds for every point $x \in I$ for which the sequence (14) is summable.

The integral of the function f over the interval I is then defined to be equal to the sum (12) and is denoted by

$$\int_I f \, d\lambda, \quad \int_I f, \quad \int_I f(x) \, dx,$$

or by any other symbol customarily used for this purpose.

Obviously, every Oresme integrable function is Archimedes integrable and Archimedes integrable functions include many negative functions and functions whose values are positive at some points and negative at others. But, compared with Oresme integrability, the definition of Archimedes integrability also widens the class of positive integrable functions. When we cover the region «under the graph» of a function, we do not have to stay with the rectangles strictly within that region. If we overshoot with some, we subtract the areas of rectangles covering the excess. In figure 11, the subtracted rectangles are shaded.

But there arises the question about the legitimacy of this definition. Because the numbers c_j and the intervals J_j, $j = 1, 2, 3, \ldots$, are not uniquely determined by the function f, the sum (12) which we called the integral of f appears not to be uniquely determined by f either.

This is of course a perfectly valid objection. What's more, it applies equally well to the calculations of Archimedes. If the given parabolic section were divided into triangles (or perhaps other polygonal figures) in a way different from that of Archimedes, would the sum of the areas of these triangles remain the same? Archimedes assumed that it would. He apparently assumed that there is «a number to be called the area of the parabolic section S», that is, no matter how S is represented as the union of a sequence of nonoverlapping triangles, the sum of the areas of such triangles is equal to the area of S. In other words, Archimedes assumed that the parabolic section is among the planar sets which do have area and that the area is σ-additive. But, because he did not prove this assumption, there is a gap in his argument. On the other hand, Archimedes was right in that this assumption is in fact correct; it has since been proved rigorously.

As to our situation, the following statement can be proved; differential calculus or more elementary methods can be used for this purpose, but either way the proof is technical.

Let c_j and d_j be numbers J_j and K_j bounded intervals, $j = 1, 2, 3, \ldots$, such that

$$\sum_{j=1}^{\infty} |c_j| \, \lambda(J_j) < \infty \,, \quad \sum_{j=1}^{\infty} |d_j| \, \lambda(K_j) < \infty$$

and

$$\sum_{j=1}^{\infty} c_j \, \chi_{J_j}(x) = \sum_{j=1}^{\infty} d_j \, \chi_{K_j}(x)$$

for every point x for which

$$\sum_{j=1}^{\infty} |c_j| \, \chi_{J_j}(x) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |d_j| \, \chi_{K_j}(x) < \infty \,.$$

Then

$$\sum_{j=1}^{\infty} c_j \, \lambda(J_j) = \sum_{j=1}^{\infty} d_j \, \lambda(K_j) \,.$$

This statement says that the Archimedes integral of a function f in fact does not depend on the choice of the numbers c_j and the intervals J_j, $j = 1, 2, 3, \ldots$, with the required properties (i) and (ii). So, it takes care of the objections raised to the definition.

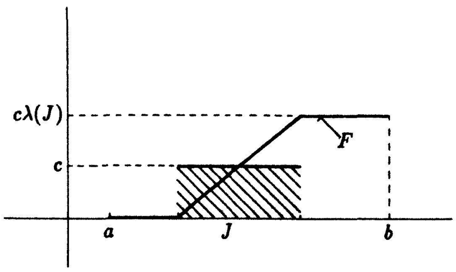

Figure 12.

Then there is the question of effectively calculating the integral of a given function. For a large class of functions this question can be answered satisfactorily.

In the definition, the type of the interval I is in no way restricted; it could be an open, closed, semi-open, bounded or unbounded interval. Let us now restrict our attention to the case $I = [a, b]$, where a and b are numbers such that $a < b$.

Suppose that $J \subset I$ is an interval, c a number and that $f = c\chi_J$ (see figure 12). It is easy to exhibit a function F, continuous in I, such that

$$F'(x) = f(x) \tag{15}$$

for every interior point x of the interval I except, possibly the endpoints of the interval J. It is then equally easy to show that

$$\int_I f\,\mathrm{d}\lambda = F(b) - F(a)\,. \tag{16}$$

By a step-function is usually understood a function f for which there exist numbers c_j and intervals $J_j, j = 1, 2, \ldots, n$, such that

$$f = \sum_{j=1}^{n} c_j \chi_{J_j}\,.$$

If f is such a step-function in the interval I, then, as above, there exists a function F, continuous in the interval I, such that (15) holds for all but finitely many points x of I and the equality (16) holds again.

Formula (16) is applicable to still more functions. If f is a function having a left limit at every point of the interval $(a, b]$ and a right limit at every point of the interval $[a, b)$ then the function f is Archimedes integrable in the interval $I = [a, b]$. Furthermore, there exists a function F, continuous in the interval I, such that the equality (15) holds for all points x belonging to I save countably many and the formula (16) holds.

So, we have a useful extension of the theorem known as the Newton-Leibniz formula or the fundamental theorem of calculus. Combined with other properties of the integral, such as its continuous dependence on the limits of integration or on the integrated function, this extended fundamental theorem provides a very wide range of possibilites to calculate integrals of specified functions in specified intervals.

The next question which naturally arises is one about the relation of the Archimedes integral to other notions of integrability and integral found in the literature. This question has a very interesting and perhaps surprising answer. It turns out that a function is Archimedes integrable if and only if it is Lebesgue integrable and the Archimedes integral of such a function coincides with its Lebesgue integral. So, the terms «Archimedes integrability» and «Archimedes integral» are used here only for the convenience in the exposition and refer to a specific method for introducing the Lebesgue integral but not to a new notion of integral. We shall not prove this fact; it is done in the book [1] of Jan Mikusiński where the «Archimedes» method of introducing the integral is used in a much wider context than here. The reader may also find it interesting to compare the «Archimedean» definition with one due to Fréderic Riesz presented in the book [2] by F. Riesz and B. Szökefalvi-Nagy. In sections 16–17 of this book, integration is treated on the same level of generality as here and it is not too difficult to use the Riesz method for showing that the Archimedes and the Lebesgue integrals coincide.

The important properties of the Archimedes integral, which make of it a powerful tool of analysis, can of course be proved much more directly than by showing first that it coincides with the Lebesgue integral. Hence, the «Archimedean» definition of integral is preferable to the Riemann integral in a basic course on integration not only because of its greater simplicity but also because it leads directly to important applications. This is apparent to a lesser degree in the cases when the application requires the integration of a particular function, although in such cases, the Archimedes (that is, the Lebesgue) integral represents an advantage, too, because the restrictions on the usage of methods of calculation are less stringent. But the real advantage of the Archimedes integral appears more clearly in cases where whole classes of functions are involved. Because families of functions defined in terms of the Archimedes integral are usually substantially richer than those defined in terms of the Riemann integral, there are important problems in mathematics and in its applications which are not solvable in terms of the Riemann integral but can be solved if the Archimedes/Lebesgue integral is used.

Ideas pertaining to whole classes of functions (or classes of geometric objects such as planar figures) transcend the boundaries of the circle of ideas accessible to Archimedes. So, it may be fitting to conclude by describing an example which illustrates our remarks about applications of the integral and, at the same time, indicates the direction in which mathematics advanced perhaps most markedly beyond the possibilities available to Archimedes.

A function f such that $|f|f$ is integrable in an interval I is said to be square-integrable in I. Let

$$e_n(t) = e^{\mathrm{i}nt},$$

for every real number t and every $n = 0, \pm 1, \pm 2, \ldots$

Now, if f is a 2π-periodic function, square-integrable according to Riemann or Archimedes and Lebesgue in an interval of length 2π, then the numbers,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e_n(-t)\, \mathrm{d}t, \tag{17}$$

are well-defined for every $n = 0, \pm 1, \pm 2, \ldots$ Moreover, the function f can be reconstructed from the functions $c_0 e_0$ and

$$c_n e_n + c_{-n} e_{-n}, \tag{18}$$

$n = 1, 2, 3, \ldots$, up to a possible difference which is completely immaterial from the point of view of applications.

In applications, the function f is interpreted, say, as an electric signal (current of varying intensity). The function (18) is $(2\pi/n)$-periodic which means that the frequency of the signal it represents is n times that of the signal f. It is called the n-th harmonic component of the signal f. The number

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2\, \mathrm{d}t$$

is (proportional to) the energy of the signal f in one period. The energy of the signal (18) is equal to $|c_n|^2 + |c_{-n}|^2$, $n = 1, 2, 3, \ldots$, and

$$\|f\|^2 = |c_0|^2 + (|c_1|^2 + |c_{-1}|^2) + (|c_2|^2 + |c_{-2}|^2) + (|c_3|^2 + |c_{-3}|^2) + \ldots,$$

which means that the energy of f is equal to the sum of energies of its harmonic components.

Somewhat more interesting from the engineering point of view is the following problem of synthesis. Given numbers c_n such that

$$|c_0|^2 + \sum_{n=1}^{\infty} (|c_n|^2 + |c_{-n}|^2) < \infty, \tag{19}$$

one wants to find a 2π-periodic function f, square-integrable in the interval $[-\pi, \pi]$, such that (17) holds for every $n = 0, \pm 1, \pm 2, \ldots$ In other words, one wishes to produce a signal with prescribed harmonic components. Because the total energy of the components is finite, it is natural to expect that such a signal would exist.

However, if only the Riemann integration were available, we would not always be able to produce such a signal. For it is not true that for any numbers c_n satisfying condition (19) there exists a 2π-periodic function f, Riemann square-integrable in the interval $[-\pi, \pi]$, such that (17) holds for every $n = 0, \pm 1, \pm 2, \ldots$ None the less, the stated problem of synthesis has a positive solution. Indeed, it can be proved that, if the c_n are arbitrary complex numbers satisfying condition (19) then there exists a 2π-periodic function f which is square-integrable in the interval $[-\pi, \pi]$ such that (17) holds for every $n = 0, \pm 1, \pm 2, \ldots$

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REFERENCES

1 Jan Mikusiński: The Bochner integral. Birkhäuser Verlag, Basel 1978.
2 F. Riesz and B. Sz-Nagy: Functional analysis, translated from 2nd French ed. by L. F. Baron. Blackie, London 1956.
3 Otto Toeplitz: The Calculus: a genetic approach. The University of Chicago Press, Chicago 1963.

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