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Autor: Kluvánek, Igor
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Archimedes was right

(Part One)

In all fairness I have to say that I don't know of anybody having said that Archimedes was factually wrong. It has been suggested, rather, that some of his arguments are not rigorous or not complete. By and large it was done by proposing ways to improve the arguments or to fill the gaps. But even with this understanding of «right» and «wrong» I do not want to argue that Archimedes was always right. My intention is to demonstrate instead that the conventional criticism of certain aspects of his methods is not warranted and to mention other aspects which seem not to have been criticised but, in my opinion, deserve to be.

So, I do not intend to present an apology for Archimedes. An apology is more often than not futile. Moreover, a mathematician who contributed to the treasure of our knowledge does not need an apology. What a mathematician may need is exegesis. For there may be ideas in the work of a great mathematician that remained unnoticed or not sufficiently explored, but nevertheless could trigger trains of thought leading to new discoveries, especially if they are combined with the knowledge accumulated since his times. I wish to take up some such ideas of Archimedes which arose in connection with his calculation of the area of a parabolic section.

Archimedes calculated the area of a parabolic section by two methods. After many centuries of neglect, mathematicians returned mainly to the first of them. The gradual development of the ideas inherent in this method led ultimately to the notion of the Riemann integral and through it to the modern notion of the Lebesgue integral. I would like to show how the second method could have led directly to the Lebesgue integral.

This proposal may seem just a futile exercise in the exploration of history which did not eventuate. But, I hope, it is not really the case. I wish to demonstrate that the ideas stemming from the second method can stream-line the presentation of the Lebesgue integral in the class-room to the extent that, by comparison, the Riemann integral would appear just a cumbersome museum exhibit. They can also contribute to our understanding of the «inner working» of the integration theory and thereby open it to new possibilities of development and applications.

Another lesson derived from the second method concerns the summation of sequences. Actually, we shall consider summation of sequences before integration. This order is the reverse of the order in which a majority of students encounter these themes. The conventional order was probably established as the result of historical accidents and the pressure put to bear on teachers to proceed as quickly as possible with differential and integral calculus. Such an order of presentation is made possible by treating summation of sequences and integration of functions independently of each other using limits of sequences for the introduction of both these notions. But it has some undesirable consequences.

For example, summation of sequences is used as a technique, at least implicitly in the form of infinite decimal expansions, if not otherwise, long before integration and, what is still worse, long before the necessary relevant concepts are introducted. Consequently, the usual treatment of real numbers is not only excessively sloppy but also excessively formalistic. Students and teachers alike see of course no point in wasting their time with definitions of notions which they have been using already for a long time. Also, even relatively sophisticated mathematicians sometimes do not see in the elementary integration and summation of sequences just two instances of general integration theory and that the latter is conceptually much simpler than the former.

On ther other hand, reflection about Archimedes' calculation of the area of a parabolic section combined with a fair amount of hind-sight shows that

(i) summable sequences and their sums can and should be introduced in a substantially simpler manner than by the traditional method, avoiding the sophisticated notion of the limit of a sequence; and

(ii) the Lebesgue integral can be introduced by a geometrically transparent method, based on summation of sequences rather than limits, which avoids the cumbersome machinery of partition of intervals and integral sums used for the introduction of the Riemann integral.

Let us then describe Archimedes' second method for the calculation of the area of a parabolic section.

The area of a planar figure F will be denoted by $\mu(F)$.

Let U and V be points on a parabola, $U \neq V$. Let us denote by S_{UV} the region bounded by the chord $\overline{UV}$ and the corresponding arc of the parabola (figure 1). Let us call the region S_{UV} the parabolic section determined by the chord $\overline{UV}$.

Our first aim is to find the area, $\mu(S_{UV})$, of the parabolic section determined by the chord $\overline{UV}$ following the second method of Archimedes.

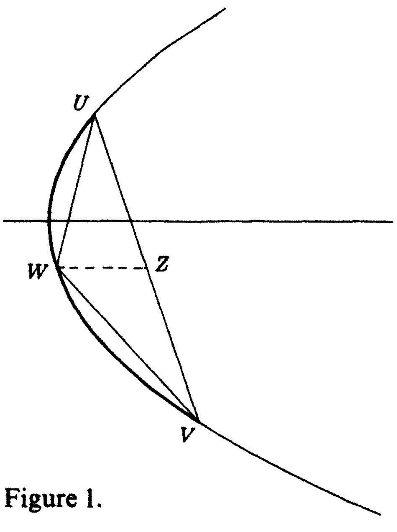


Figure 1.

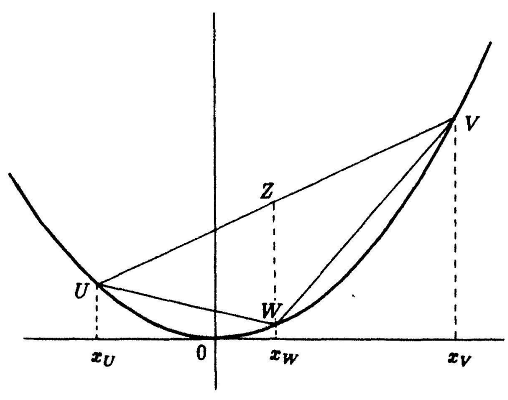


Figure 2.

Let Z be the mid-point of the chord $\overline{UV}$. Let W be the point in which the parabola meets the straight line parallel to the axis of the parabola through Z. The triangle with vertices U, V and W will be called the triangle determined by the chord $\overline{UV}$, and denoted by T_{UV}.

Archimedes first proved the following preliminary result:

If T is a triangle determined by a chord of the parabola and T_l and T_r the triangles determined by its other two sides, then

$$\mu(T_l) + \mu(T_r) = \tfrac{1}{4}\mu(T),$$

that is, the sum of the areas of the triangles T_l and T_r is equal to one quarter of the area of the triangle T.

The proof which Archimedes provided and the state of knowledge about parabolas on which it is based are lucidly and concisely presented on pages 35–39 of the book [1] by C. H. Edwards.

We present another proof which is more accessible to the modern audience. In it, we use a coordinate system. Coordinate systems were of course unknown to Archimedes.

We shall use the orthogonal system of coordinates, whose origin, O, is the point in which the parabola meets its axis, such that the axis of ordinates coincides with the axis of the parabola. The pair of coordinates of a point M in this system will be denoted by (x_M, y_M). The coordinate axes are assumed to be so oriented that $x_U < x_V$ and the equation of the parabola in this system is $y = ax^2$, where a is a positive number (figure 2). Modern students often meet the parabola for the first time in this set-up.

The area, $\mu(T_{UV})$, of the triangle T_{UV} is equal to the sum of the areas of the triangles with vertices, U, W, Z and W, V, Z, respectively. These triangles have the common base, $\overline{ZW}$, of length $y_Z - y_W$ and the same height equal to $\frac{1}{2}(x_V - x_U)$. Therefore,

$$\mu(T_{UV}) = \tfrac{1}{2}(y_Z - y_W)(x_V - x_U)$$

Because

$$y_Z = \frac{y_U + y_V}{2} = \frac{a x_U^2 + a x_V^2}{2} = \frac{a}{2}(x_U^2 + x_V^2)$$

and

$$y_W = a x_W^2 = a\left(\frac{x_U + x_V}{2}\right)^2 = \frac{a}{4}(x_U + x_V)^2,$$

we have

$$y_Z - y_W = \frac{a}{4}(2(x_U^2 + x_V^2) - (x_U + x_V)^2) = \frac{a}{4}(x_V - x_U)^2$$

and, hence,

$$\mu(T_{UV}) = \frac{a}{8}(x_V - x_U)^3 .$$

Similarly, replacing U and V first by U and W and then by W and V, we obtain

$$\mu(T_{UW}) = \frac{a}{8}(x_W - x_U)^3 \quad \text{and} \quad \mu(T_{WV}) = \frac{a}{8}(x_V - x_W)^3 .$$

But $x_V - x_W = x_W - x_U = \frac{1}{2}(x_V - x_U)$. Therefore,

$$\mu(T_{UW}) + \mu(T_{WV}) = 2\frac{a}{8}\left(\frac{x_V - x_U}{2}\right)^3 = \frac{1}{4}\frac{a}{8}(x_V - x_U)^3 = \frac{1}{4}\mu(T_{UV}) .$$

Now, the preliminary result being proved, we can proceed as Archimedes did.
Let T_{11} and T_{12} be the triangles determined by the two sides of the triangle T_{UV} other than $\overline{UV}$, that is, $T_{11} = T_{UW}$ and $T_{12} = T_{WV}$ (figure 3).
The two triangles T_{11} and T_{12} have four new sides other than the chords determining them. These new sides are also the chords of the parabola and, therefore, determine four triangles, T_{21}, T_{22}, T_{23} and T_{24}.
In a similar manner, these four triangles have eight new sides other than the chords determining them. The new sides determine eight triangles, $T_{31}, T_{32}, \ldots, T_{37}$ and T_{38}.
And so on. Each lot of triangles produce in this manner twice as many new triangles.
Following Archimedes, we think of S_{UV} as the union of all triangles

$$T_{UV},\ T_{11},\ T_{12},\ T_{21},\ T_{22},\ T_{23},\ T_{24},\ T_{31},\ T_{32},\ \ldots, T_{37},\ T_{38},\ T_{41},\ T_{42},\ \ldots,\ T_{4,16},\ T_{51},\ \ldots$$

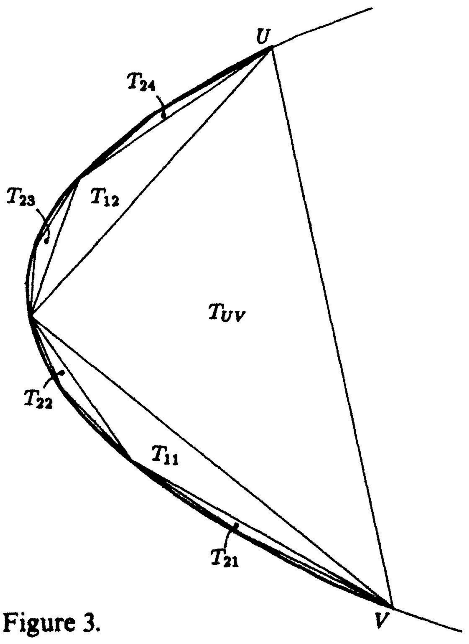


Figure 3.

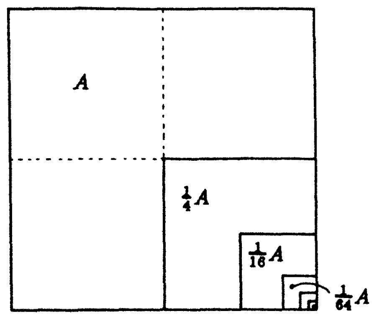


Figure 4.

so produced. Therefore,

$$\begin{aligned}\mu(S_{UV}) = \mu(T_{UV}) &+ (\mu(T_{11}) + \mu(T_{12})) \\ &+ (\mu(T_{21}) + \mu(T_{22}) + \mu(T_{23}) + \mu(T_{24})) \\ &+ (\mu(T_{31}) + \mu(T_{32}) + \ldots + \mu(T_{37}) + \mu(T_{38})) \\ &+ (\mu(T_{41}) + \mu(T_{42}) + \ldots + \mu(T_{4,15}) + \mu(T_{4,16})) + \ldots\end{aligned} \tag{1}$$

Let us denote $\mu(T_{UV}) = A$ and recall that, by the preliminary result just proved, the sum of the areas of triangles in each lot is equal to $\frac{1}{4}$ of the sum of the areas of triangles in the preceding lot. Therefore,

$$\mu(S_{UV}) = A + \frac{1}{4}A + \frac{1}{4^2}A + \frac{1}{4^3}A + \ldots$$

So, there just remains the problem of calculating this sum. Actually, it is not much of a problem for someone who knows the formula for the sum of a geometric sequence. But Archimedes did not have that formula. Therefore, he proceeded in the ingenious way illustrated in figure 4.

From a square one corner is removed. The removed corner is a square whose side is half of that of the original square and, hence, its area is $\frac{1}{4}$ of the area of the original square. The resulting L-shaped figure is assumed to have area equal to A so that the area of the original square is $(4/3)A$.

The same is done with the removed square in the corner, then with the square removed from this square and so on. The union of all the L-shaped figures so obtained is the whole of the original square. Furthermore, the area of each L-shaped figure is $\frac{1}{4}$ of the area of the preceding one. Therefore

$$A + \frac{1}{4}A + \frac{1}{4^2}A + \frac{1}{4^3}A + \ldots = \frac{4}{3}A, \tag{2}$$

and so,

$$\mu(S_{UV}) = \frac{4}{3}\mu(T_{UV}).$$

In words: The area of a section of parabola determined by a chord is equal to 4/3 of the area of the triangle determined by that chord.

The method by which Archimedes arrived at this formula is rather audacious and far reaching. In his time, the most convincing way of showing that the areas of two planar figures are equal was still to cut one of the figures into smaller pieces which could then be re-assembled to form the other figure. Such a procedure was also used for proving other similar statements. For example, there are several proofs of the theorem of Pythagoras executed by dividing and re-arranging planar figures. But the L-shaped figures and the square on figure 4 do not seem to have anything in common with the triangles and the parabolic section on figure 3. Nevertheless, Archimedes used them to show how to produce geometrically a square with the same area as the given parabolic section. For, in his time, a geometric construction «using a compass and a straight edge» of a square whose area is 4/3 times the area of a given triangle was well-known. This is remarkable because the parabolic section is not a figure delineated by straight segments and, consequently, it is not possible to divide a square by straight cuts into a finite number of pieces which would cover the given parabolic section without overlapping. Before Archimedes «squared» the parabolic section, only one figure bounded by curved lines that can be «squared» was known. Some 190 years earlier, Hippocrates had shown a construction of a square which has the same area as a certain figure bounded by two circular arcs (see, for example, [3], pp. 6–8). Unfortunately, an important step in the supporting argument is now lost.

To appreciate the import of the method of Archimedes, we should keep in mind that in his time the notion of the real numbers system as we know and use it nowadays did not exist. We use real numbers for «measuring» all sorts of quantities in their appropriate units and take the operations with them for granted. What's more, we also use freely «infinite» operations like taking limits or summing infinite sequences of numbers. The calculations of Archimedes, such as that of the area of a parabolic section, provided strong impetus for the development of real numbers. They also pointed to the properties which the real numbers must enjoy to become the ubiquituous facility they are in our times.

We should note, however, that the method of finding the area of a parabolic section as we presented it so far is surely not rigorous by our standards. In fact, we have only shown a clever picture. But Archimedes did not stop at the picture; he also offered a fine argument. What's more, he was aware of the logical prerequisities for the validity and completeness of that argument. Indeed, he was fist to formulate explicitly one of the pre-requisites, thus foreshadowing a property of numbers which now bears his name and which became an important principle of mathematical analysis. Because of its implicit abstractness, I may perhaps be allowed to present Archimedes' argument in the modern terms of real numbers.

Archimedes noted first that

$$A+\frac{1}{4}A+\frac{1}{4^2}A+\ldots+\frac{1}{4^n}A+\frac{1}{3}\frac{1}{4^n}A=\frac{4}{3}A, \tag{3}$$

for every $n=0,1,2,\ldots$ This equality can be «read» from figure 4. But his proof of (3) is based on the formula

$$\frac{1}{4^k}A+\frac{1}{3}\frac{1}{4^k}A=\frac{1}{3}\frac{1}{4^{k-1}}A,$$

which is valid for every $k=0,1,2,\ldots$ For the proof of (3), it suffices then to start with the left-hand side and to use this formula successively for $k=n$, $n-1$, $n-2, \ldots, 1$ and 0 to obtain the right-hand side.

Now, the equality (3) obviously implies that

$$A+\frac{1}{4}A+\frac{1}{4^2}A+\ldots+\frac{1}{4^n}<\frac{4}{3}A$$

for every $n=0,1,2,\ldots$ But Archimedes proved more; namely that $(4/3)A$ is the smallest of all numbers B such that

$$A+\frac{1}{4}A+\frac{1}{4^2}A+\ldots+\frac{1}{4^n}A<B \tag{4}$$

for every $n=0,1,2,\ldots$

It is for this purpose that Archimedes needed what became known as the Archimedean property of numbers, Archimedes' lemma, or the axiom of Archimedes. It asserts that, for any numbers $\varepsilon>0$ and K, there exists a natural number n such that $n\varepsilon>K$ or, which comes to the same thing, that

$$\frac{1}{n}K<\varepsilon. \tag{5}$$

Some quibbling has occured occasionally about whether Archimedes actually referred to «Archimedes' lemma» at this precise spot. But there should be no doubt whatever that he was fully aware of using it. In fact, he specifically singled it out in a letter accompanying the paper which contains his calculations. (See [2], pp. 233 and 234 of the Dover edition.)

As to the proof itself, if there were a number B less than $(4/3)A$ and satisfying (4) for every $n=0,1,2,\ldots$, we could take

$$\varepsilon=\frac{4}{3}A-B \quad\text{and}\quad K=\frac{1}{3}A.$$

As then $\varepsilon>0$, by Archimedes' lemma, there would exist a natural number n for which (5) holds. And because $4^n>n$, the inequality

$$\frac{1}{3}\frac{1}{4^n}A=\frac{1}{4^n}K<\frac{1}{n}K<\varepsilon=\frac{4}{3}A-B$$

would yield a contradiction with (3) and (4).

Some modern commentators (for example [2], Chapter VIII of the Introduction, pp. cxlii–cliv of the Dover edition, or perhaps [4], pp. 219 and 220) seem to suggest that it is more respectable, or at least tempting ([1], p. 39), to use the relation

$$\lim_{n\to\infty}\frac{1}{3}\frac{1}{4^n}A=0, \tag{6}$$

which of course produces (2) from (3). But to prove (6) one has to use the Archimedes lemma, or its equivalent, in the first place. So, the suggestion that the argument of Archimedes can be made rigorous by using limits would obviously be foolish.

Archimedes was not tempted to use limits because they were not yet invented. He was not even tempted to invent them himself. And perhaps, the main reason was not that the state of knowledge in his time had not made it possible for an idea of that sort to arise in the mind of even as creative mathematician as Archimedes, but rather that he had no need for it.

It may seem strange to claim that Archimedes had no need for the idea of limit. How then, we may ask, did he understand the equality (2)? What was his definition of the sum of a sequence of numbers? Actually, Archimedes did not formulate any such definition. But this is not a satisfactory answer because it avoids the issue. Even if Archimedes did not give a formal definition, we can still ask why he maintained that he had proved (2). How then did he prove (2) or, more accurately, what in fact did he prove?

We already have the answer for the question so framed. As we have seen, Archimedes proved that $(4/3)A$ is the smallest of all numbers B such that the inequality (4) holds for every $n=0,1,2,\ldots$ So, his argument suggests the following definition:

A number s is called the sum of the sequence of positive numbers a_0, a_1, a_2, a_3, ... if s is the smallest of all the numbers b such that

$$a_0+a_1+a_2+\ldots+a_n \leqq b$$

for every $n=0,1,2,\ldots$ If s is the sum of the sequence of numbers a_0, a_1, a_2, a_3, ..., then we write

$$a_0+a_1+a_2+a_3+\ldots=s\,,$$

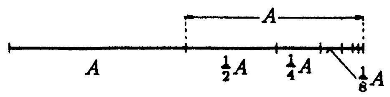
or, more formally,

$$\sum_{j=0}^{\infty} a_j = s\,.$$

If this definition is adopted, then the equality (2) becomes a true statement and Archimedes' proof of it is perfectly rigorous. The proof cannot be made more rigorous, nor can it be simplified by using limits. If the equality (2), or some more general statement of that sort, is not adopted for a basic assumption (axiom) of the theory of real numbers, then the step in the proof based on the Archimedes lemma has to be made at some stage. If it is not made in a direct proof of (2), then it has to be made elsewhere, for example, in the proof of (6). The step could be perhaps made simpler in the proof of (6) because the situation there is algebraically somewhat more transparent, but then the notion of limit would be drawn into the play and this notion is much more complex than that of the sum of a sequence of positive numbers. Indeed, a correct definition of the limit of a sequence requires the involvement of three «dummy» parameters; there is no escape from it, only a cover-up. For, recall that a number s is the limit of a sequence of numbers s_0, s_1, s_2, s_3, ... if and only if, for every $\varepsilon>0$, there exists a number δ such that the inequality $|s_n-s|<\varepsilon$ holds for every integer $n>\delta$. On the other hand, in the «Archimedean» definition of sum we got away with only two parameters (n and b). And it is not just the number of parameters that matters, but also their interplay.

The «Archimedean» definition of the sum of a sequence of positive numbers in intuitively so «obvious», it is so «natural», that Archimedes missed the necessity to formulate it explicitly. Actually, this ommission is rather common in high-school curricula. In this respect, there is a gap in his presentation. But, as we have seen, the gap can be closed without difficulty by analysing the context.
However, even all the previous discussion does not remove the feeling that there might be some point in the modern criticism of the methods and procedures of Archimedes. Indeed, it would be a rather odd situation if all the knowledge accumulated for more than two thousand years could not help us to improve them. For example, modern stream-lined terminology and notation can be used to improve Archimedes' presentation. In fact, we used them because by and large we are not familiar any more with the conventions which he employed in his writings. But, no matter how important they are, the notation and terminology are only external aspects of the method. As we have seen, as far as the requirements of accuracy go, the method of Archimedes is intrinsically flawless. So, unless the efforts of the generations of mathematicians who came after him have no import on our theme, we must have some other advantage over Archimedes.

Figure 5.



It is striking how high a degree of inventiveness was needed to discover the equality (2) by the means of figure 4. And yet this discovery does not help us, as it did not help Archimedes, to find other sums of similar nature. Perhaps, if we think of figure 4 as a two-dimensional analogue of figure 5 which allows us to see that

$$A+\frac{1}{2}A+\frac{1}{2^2}A+\frac{1}{2^3}A+\ldots=2A,$$

we can find the sum

$$A+\frac{1}{8}A+\frac{1}{8^2}A+\frac{1}{8^3}A+\ldots$$

and then possibly

$$A+\frac{1}{16}A+\frac{1}{16^2}A+\frac{1}{16^3}A+\ldots$$

and so on. But as to the sum,

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots,$$

not to speak of sums of non-geometric sequences, the method of Archimedes does not give us any clue. The problem with this method is that it depends on a geometric

construction. In fact, it is perhaps not appropriate to call his way of obtaining (2) a calculation.
Archimedes was handicapped in comparison with us by not having access to the facility of algebraic manipulation, that is calculation, with real numbers. The suggestion that, instead of following him we should use limits for the derivation of (2), can be seen as a suggestion to avail ourselves of that facility. Sums of sequences are commonly treated by the means of limits probably because the limits are considered to be a convenient means for calculation of and with the sums of sequences. However, such treatment leads to confusing terminology and inconsistency with integral calculus which is still generally maintained.
And yet it is possible to avoid limits in setting up and developing the calculus of summable sequences. Indeed, the «Archimedean» definition of the sum of a negative sequence is analoguous to one given for positive sequences. Alternatively, the definition of the sum of a negative sequence is reducible to that for a positive sequence in an obvious manner. A sequence of positive numbers or of negative numbers is then called summable if it has a sum.
The «Archimedean» definition of summability is extended to sequences of numbers of arbitrary sign by calling such a sequence summable if the sequence of positive terms and the sequence of negative terms are both summable. The sum of a summable sequence is defined by the formula

$$a_0+a_1+a_2+a_3+\ldots=(b_0+b_1+b_2+b_3+\ldots)+(c_0+c_1+c_2+c_3+\ldots),$$

where $b_0, b_1, b_2, b_3, \ldots$ are the positive terms and $c_0, c_1, c_2, c_3, \ldots$ the negative terms of the sequence $a_0, a_1, a_2, a_3, \ldots$ It is clear how the definition should be modified if a sequence has only a finite number of positive terms or a finite number of negative terms.
It turns out that the summability and sum of a sequence does not depend on the order of its terms (the extended commutative law holds) nor on the insertion of brackets (the extended associative law holds). Also, a sequence $a_0, a_1, a_2, a_3, \ldots$ is summable if and only if the sequence $|a_0|, |a_1|, |a_2|, |a_3|, \ldots$ is summable. Therefore, the sequences summable in the sense of the «Archimedean» definition are called absolutely summable. Also, the statement that a sequence $a_0, a_1, a_2, a_3, \ldots$ is summable is conveniently recorded by writing

$$\sum_{j=0}^{\infty}|a_j|<\infty.$$

The standard rules for calculation with summable sequences and for comparing their sums can easily be derived directly from the definition. It is not profitable to do it here, but it may be useful to show on a classical example how the rules for calculation can be used for finding sums of summable sequences.
If $r \neq 1$ is a number such that the sequence $1, r, r^2, r^3, \ldots$ is summable, let

$$s=1+r+r^2+r^3+\ldots$$

Then,

$$s=1+r+r^2+r^3+\ldots=1+r(1+r+r^2+r^3+\ldots)=1+rs.$$

Therefore,

$$s = \frac{1}{1-r}. \tag{7}$$

It is then not difficult to show, using the Archimedean property of numbers, that
(i) if $|r| \geqq 1$, then the sequence $1, r, r^2, r^3, \ldots$ is not summable; and
(ii) if $|r| < 1$, then this sequence is summable and, hence, its sum, s, is given by (7).
If we combine this result with the extended distributive law, we obtain that

$$A + Ar + Ar^2 + Ar^3 + Ar^4 + \ldots = A\frac{1}{1-r}, \tag{8}$$

for any number A and any number r such that $|r| < 1$. The choice of $r = 1/4$ gives (2).
It is very easy to show that, if s is the sum of the sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ in the sense of the «Archimedean» definition, then

$$s = \lim_{n \to \infty} s_n, \tag{9}$$

where

$$s_n = a_0 + a_1 + a_2 + \ldots + a_n,$$

for every $n = 0, 1, 2, \ldots$ Somewhat less easy but still not too difficult is to show that, if the limit (9) exists, then the sequence $a_0, a_1, a_2, a_3, \ldots$ is not necessarily summable in the sense of the «Archimedean» definition. It is appropriate to call such a sequence conditionally summable (or improperly summable) and the number s its conditional (improper) sum. This terminology is analoguous to the terminology concerning integrable and improperly integrable functions and their integrals. It enables us to avoid the conceptual difficulties associated with the use of the terms «convergent series» and «sum of series». These terms are of course superfluous; they do not have counterparts in basic integral calculus.

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Igor Kluvánek, Centre for Mathematical Analysis,
Australian National University, Canberra

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