On a diophantine equation (x2 - 1) (y2 - 1) = (z2 - 1)2

Autor(en): **Hirose, Shoichi**

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On a diophantine equation $(x^2-1)(y^2-1)=(z^2-1)^2$

1. Schinzel and Sierpiński [4] gave the complete solution of the equation

$$(x^2-1)(y^2-1)=\left(\left(\frac{y-x}{2}\right)^2-1\right)^2 \tag{1}$$

in positive integers $x, y, x<y$. Equation (1) is a special case of the equation

$$(x^2-1)(y^2-1)=(z^2-1)^2. \tag{2}$$

The only known solution of (2) which is not a solution of (1) is $(x, y, z)=(4, 31, 11)$ found by Szymiczek [5].
In this paper we give a method for finding positive integer solutions x, y, z of (2), and show two new solutions.

2. Let $1<x<y$, $x^2-1=\mathrm{D}\,v^2$, $y^2-1=\mathrm{D}'\,v'^2$, where D and D′ have no square factors. Then we get $\mathrm{D}=\mathrm{D}'$, because

$$(x^2-1)(y^2-1)=\mathrm{D}\,\mathrm{D}'(v\,v')^2=(z^2-1)^2.$$

If (u_1, v_1) is the minimal positive solution of the equation

$$u^2-\mathrm{D}\,v^2=1,$$

then all positive solutions are given by (u_k, v_k) for $k=1, 2, 3, \ldots$ where u_k and v_k are the integers defined by

$$u_k+v_k\sqrt{\mathrm{D}}=(u_1+v_1\sqrt{\mathrm{D}})^k. \tag{3}$$

Since we can put $x=u_i$, $v=v_i$, $y=u_j$, $v'=v_j$, $i<j$, we have

$$\mathrm{D}\,v_i v_j+1=z^2. \tag{4}$$

From (3), it follows that

$$u_{k+1}=u_1 u_k+\mathrm{D}\,v_1 v_k,$$

$$v_{k+1}=u_1 v_k+v_1 u_k.$$

Hence, by mathematical induction, $u_k = f_k(u_1)$ is a polynomial of degree k in u_1, and $v_k/v_1 = g_k(u_1)$ is a polynomial of degree $k-1$ in u_1. Indeed,

$$g_k(u_1) = \sum_{n=0}^{\left[\frac{k-1}{2}\right]} (-1)^n \binom{k-n-1}{n} (2u_1)^{k-2n-1}.$$

Thus, the left side of (4)

$$\begin{aligned} \mathrm{D}\, v_i v_j + 1 &= \mathrm{D}\, v_1 g_i(u_1) \cdot v_1 g_j(u_1) + 1 \\ &= (u_1^2 - 1)\, g_i(u_1)\, g_j(u_1) + 1 = \mathrm{F}_{ij}(u_1) \end{aligned}$$

is a polynomial of degree $i+j$ in u_1. If there is a solution (u_1, z), $u_1 > 1$ of the equation

$$\mathrm{F}_{ij}(u_1) = z^2,$$

then $(x, y, z) = (u_i, u_j, z)$ is a solution of (2).

(i) $i = 1$, $j = 2$. Since $v_2 = 2u_1 v_1$, we get $g_2(u_1) = 2u_1$, and so $\mathrm{F}_{12}(u_1) = (u_1^2 - 1) \cdot 2u_1 + 1 = 2u_1^3 - 2u_1 + 1$. $\mathrm{F}_{12}(u_1) = z^2$ has the following three solutions (u_1, z), $1 < u_1 < 10^5$:

$$(u_1, z) = (3, 7), (4, 11), (155, 2729).$$

Therefore, we obtain the following three solutions of (2), the first of which is a solution of (1), and the last of which is new.

$$(x, y, z) = (3, 17, 7), (4, 31, 11), (155, 48049, 2729).$$

(ii) $i = 1, j = 4$. Since $v_4 = v_1(8u_1^3 - 4u_1)$, we get

$$\mathrm{F}_{14}(u_1) = (u_1^2 - 1)(8u_1^3 - 4u_1) + 1 = 8u_1^5 - 12u_1^3 + 4u_1 + 1.$$

$\mathrm{F}_{14}(u_1) = z^2$ has the following solution (u_1, z), $1 < u_1 < 10^3$:

$$(u_1, z) = (2, 13).$$

Therefore, we obtain the following new solution of (2), where x is even, y and z are odd:

$$(x, y, z) = (2, 97, 13).$$

(iii) In the case of $3 \le i + j \le 9$, except above, we get only the following three solutions, all of which are solutions of (1).

i	j		u_1	x	y	z
2	3	$1 < u_1 < 10^3$	3	17	99	41
3	4	$1 < u_1 < 10^2$	3	99	577	239
4	5	$1 < u_1 < 35$	3	577	3363	1393

The equation (2) has no solutions other than above for all integers x and y, where $1 < x < y < 10^3$.
Though we cannot solve (2) completely, we conjecture that there are only three solutions of (2) which are not solutions of (1).

3. We applied our method to the equation

$$(x^2+1)(y^2+1) = (z^2+1)^2,$$

but we could not obtain other solutions than those Williams [6] found.
Our method cannot be applied to the equation

$$(x^2-e)(y^2-e) = (z^2-e)^2$$

where $e \neq \pm 1, \pm 4$. However, our method may be applied to the equation

$$(x^2-e)(y^2-e) = \{h(z)\}^2$$

where $e = \pm 1, \pm 4$ and $h(z)$ is a linear or quadratic expression in z.
For example, the equations

$$(x^2+1)(y^2+1) = (z^2+t^2)^2,$$
$$(x^2-1)(y^2-1) = (z^2-t^2)^2$$

have always integer solutions for all integers t, given by

$$(x, y, z) = (t, 4t^3+3t, 2t^2+1)$$
$$(x, y, z) = (2t^2 \pm 1, 8t^4 \pm 8t^2+1, 4t^3 \pm 3t),$$

respectively.

Shoichi Hirose, Mita High School, Tokyo, Japan

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