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A point set everywhere dense in the plane

Dedicated to Professor L. Fejes-Tóth on his seventieth birthday

One day the famous Hungarian geometer László Fejes-Tóth could not find his compass. He searched through every nook and cranny of the house without success and, for it was a nice old piece given by his wife as a present many years ago, he felt very sad over the loss. However, a few days later his wife who was obviously better at searching found the compass and so she managed to present her husband with the same gift for the second time. (This is a very special feminine virtue.)

Professor Fejes-Tóth who has always been passionately interested in all sorts of properties of circle arrangements and did a lot of important work in this field, was quite delighted. He took the compass eargerly in his hand, opened its arms and started absent-mindedly drawing equal circles on a sheet of paper on his desk. He had to stop this action before long, because after a while no single spot of the paper had been left blank. Being a dedicated mathematician he immediately decided that this could not be just an accident. He scrutinized the picture and then he suddenly realized that he had been following a very simple and natural strategy of drawing:

(i) Take two points O_1, O_2 having mutual distance at most 2, and set $S_1 = \{O_1, O_2\}$.

(ii) If S_i $(i \geqq 1)$ has already been determined then add to the picture all unit circles whose centres belong to S_i and which have not been drawn in before. Further, let S_{i+1} be defined as the union of S_i and the set of all intersection points of units circles whose centres are in S_i.

If $|O_1O_2|$ (i.e. the distance of O_1 and O_2) is 1, $\sqrt{3}$ or 2 then, having taken a few steps, we obtain the following picture (see Fig. 1):

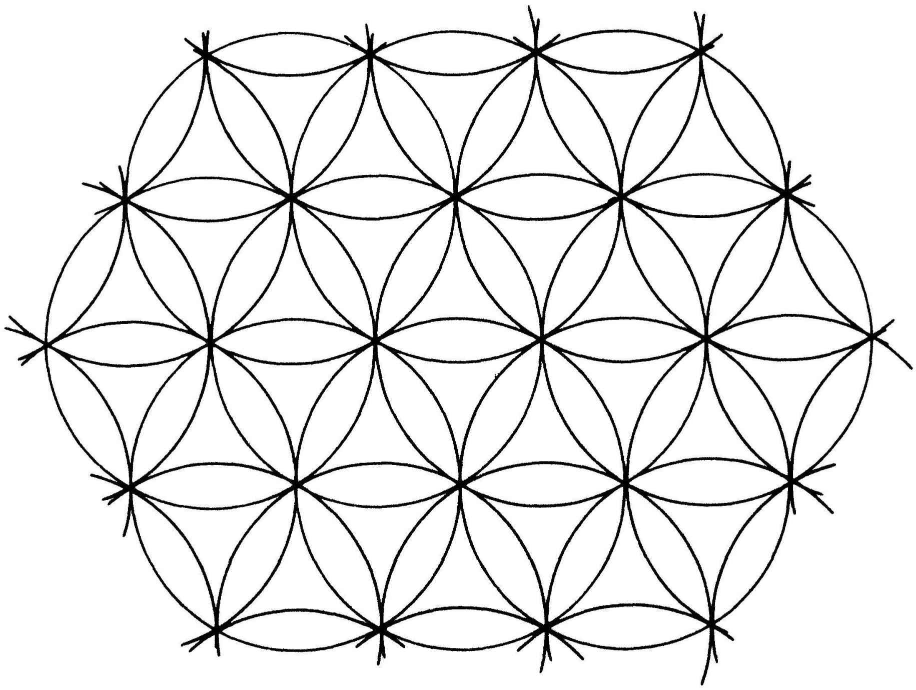

Figure 1

That is, in these cases $\bigcup_{i=1}^{\infty} S_i$ consists of all vertices of a *regular triangle-lattice* of side length 1.
Professor Fejes-Tóth who was not lucky enough to choose one of these three numbers for $|O_1O_2|$ got a somewhat uglier picture. In particular, his sets S_i filled up the space more and more 'densely' ($i \to \infty$).

Definition. *An infinite point set S is said to be everywhere dense in the plane if every circular disc contains at least one element of S.*

After a little meditation, L. Fejes-Tóth came up with the following conjecture:

Theorem 1. *Let the sets S_i ($i \geqq 1$) be defined as above. Then, $\bigcup_{i=1}^{\infty} S_i$ is either identical with the set of vertices of a regular triangle-lattice of side length 1, or it is everywhere dense in the plane.*

Our proof is based on the elementary

Lemma 1. *Let ABC be an equilateral triangle with side length at most 2, and suppose $A,B \in S_i$ for some $i \geqq 1$. Then, we have $C \in S_{i+3}$.*

Proof: It is obviously sufficient to prove the assertion for the case $|AB| < 2$.
Let us define four additional points (see Fig. 2) by
O: $|OA| = |OB| = 1$ and the line AB separates O and C;
A': $|A'A| = |A'O| = 1$ and the line AO does not separate A' and C;
B': $|B'B| = |B'O| = 1$ and the line BO does not separate B' and C;
B'': $|B''B| = |B''O| = 1$ and the line BO separates B'' and C.

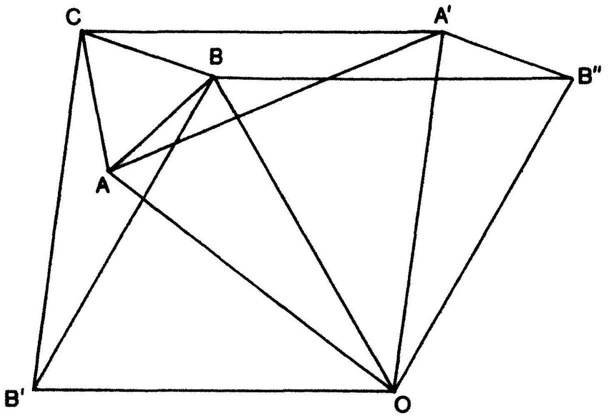

Figure 2

Observe now that $A'B''$ is the image of AB at a rotation about O with angle $\pi/3$. Consequently, $A'B''BC$ is a parallelogram. Then we have $|CA'| = |BB''| = 1$ and, similarly, $|CB'| = 1$.
Thus, C is an intersection point of the unit circles centred at A' and B', and using the fact that $O \in S_{i+1}$ and $A',B' \in S_{i+2}$ (by the definitions) we get $C \in S_{i+3}$.

Proof of theorem 1: First, we note that lemma 1 has the immediate consequence that, if A and B are two elements of $S = \bigcup_{i=1}^{\infty} S_i$ with $|AB| \leqq 2$, then S contains all vertices of the regular triangle-lattice determined by A and B. In view of this, we can clearly assume

$$\inf\{|AB| : A,B \in S\} = \delta > 0,$$

otherwise S is everywhere dense.
Let us now choose two points $A,B \in S$ such that $|AB| < (3/2)\delta$. Then S must be the set of vertices of the regular triangle-lattice determined by A and B; for if there were any additional point $P \in S$ in a triangular cell $A'B'C'$ of the lattice, then at least one of the distances $|PA'|$, $|PB'|$, $|PC'|$ would be shorter than $|A'B'|/\sqrt{3} = |AB|/\sqrt{3} < \delta$, a contradiction. This yields e.g. that $|AB| = \delta \leqq 1$.
It remains to prove that $\delta = 1$. Suppose to the contrary that $\delta < 1$, and let A^* denote the reflection of A about B. Further, let O and O^* be two points satisfying $O: |OA| = |OB| = 1$, $O^*: |O^*A| = |O^*A^*| = 1$ and the line AB does not separate O and O^*. It is easy to check now that $|OO^*| < \delta$, the desired contradiction.

Having seen our proof of theorem 1, L. Fejes-Tóth immediately asked whether or not a similar assertion is valid for arbitrary centrally symmetric closed convex curves, instead of circles. In what follows, we will settle this question in the affirmative.
Given any centrally symmetric strictly convex closed curve K and any point P in the plane, let $K(P)$ denote the translate of K *with centre* P.
(i′) Let O_1 and O_2 be two points such that $K(O_1)$ and $K(O_2)$ have at least one point in common, and set $S_1 = \{O_1, O_2\}$.
(i″) If S_i $(i \geqq 1)$ has already been determined then let S_{i+1} be defined as the set of all points which are either in S_i or in $K(P_1) \cap K(P_2)$ for some $P_1, P_2 \in S_i$.
If $O_2 \in K(O_1)$ then, following this procedure, we will find again that $\bigcup_{i=1}^{\infty} S_i$ is a *triangle-lattice* whose cells are congruent to the triangle $O_1O_2O_3$, where O_3 denotes one of the intersection points of $K(O_1)$ and $K(O_2)$. Any lattice obtained in this way is said to be *K-regular* (see Fig. 3).

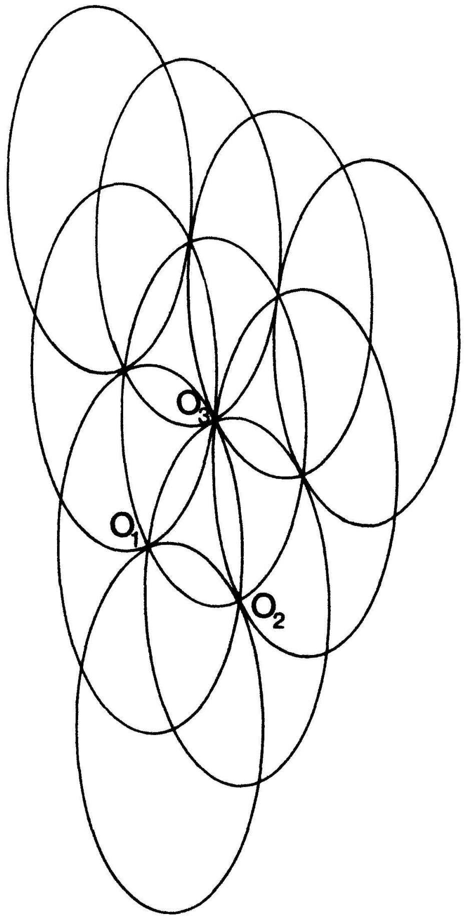

Figure 3

The following generalization of theorem 1 is true:

Theorem 2. *Let K be a centrally symmetric stricly convex closed curve in the plane, and let the sets S_i ($i \geqq 1$) be defined by (i') and (i''). Then $S = \bigcup_{i=1}^{\infty} S_i$ is either identical with the vertex set of a K-regular lattice, or it is everywhere dense in the plane.*

The core of the proof is contained in

Lemma 2. *Let A,B,C be three elements of S. Then all vertices of the triangle-lattice determined by these points belong to S.*

To see this we need three simple but useful observations.

Proposition 1. *Let $P,Q \in S$ such that $P \in K(Q)$. Then $Q + \overrightarrow{PQ}$, i.e. the reflection of P about Q, is an element of S.*

Proof: Let $R_1 \in K(P) \cap K(Q)$, and let R_2 denote the intersection point of $K(R_1)$ and $K(Q)$ distinct from P. Further, let R_3 be the intersection point of $K(R_2)$ and $K(Q)$ different from R_1. We obviously have $R_3 = Q + \overrightarrow{PQ} \in S$.

Proposition 2. *Let O_1 and O_2 denote, as above, the starting points of our algorithm (see (i')). Then $O_2 + \overrightarrow{O_1O_2} \in S$.*

Proof: Let $K(O_1) \cap K(O_2) = \{P_1, P_2\}$. By proposition 1, we have $P_1 + \overrightarrow{O_1P_1}$, $O_2 + \overrightarrow{P_2O_2} \in S$, hence, applying proposition 1 to these two latter points, we obtain $(O_2 + \overrightarrow{P_2O_2}) + \overrightarrow{O_1P_2} = O_2 + \overrightarrow{O_1O_2} \in S$.

Proposition 3. *Let O_1, O_2 be the starting points of our algorithm, and suppose $P,Q \in S$. Then $P \pm \overrightarrow{O_1Q}$ (i.e. the translates of P by the vectors $\overrightarrow{O_1Q}$ and $\overrightarrow{QO_1}$, resp.) are also elements of S.*

Proof: By proposition 2, we have $O_2 + \overrightarrow{O_1O_2} \in P$. Thus, starting our algorithm with the points $O_1' = O_2'$, $O_2' = O_2 + \overrightarrow{O_1O_2}$ (instead of O_1 and O_2), we obtain a set $S' \subseteq S$. Taking into account that $P \in S$, we get $P + \overrightarrow{O_1O_2} \in S'$.
Let us start next with $O_1'' = P$, $O_2'' = P + \overrightarrow{O_1O_2}$. Using the fact that $Q \in S$, we obtain now $Q \pm \overrightarrow{Q_1P} = P + \overrightarrow{O_1Q} \in S'' \subseteq S$. The relation $P - \overrightarrow{O_1Q} \in S$ can be established similarly.

Proof of lemma 2: It is sufficient to show e.g. that $A + \overrightarrow{BC} \in S$. But this is an immediate consequence of proposition 3 and the fact that $A + \overrightarrow{BC}$ can be written in the form $A + \overrightarrow{O_1C} - \overrightarrow{O_1B}$.

The proof of theorem 2 (based on lemma 2) is now a relatively easy exercise which can be left to the reader.
For those familiar with the notion of Minkowski spaces, we remark that the convex curve K in theorem 2 is, in fact, a *unit circle* in the corresponding Minkowski metrics. Using this terminology, steps (i') and (i'') of our 'drawing algorithm' reduce to (i) and (ii), resp.
Finally, note that similar assertions can be proved for circles in the hyperbolic plane and on the sphere.

Károly Bezdek and János Pach, Budapest

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