

# A remark on a paper by A. Grytczuk

Autor(en): **Redmond, Don**

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As the result of this procedure, all the points of  $H'$  other than  $w$  have degree  $d$  and the degree  $r$  of the point  $w$  is odd, since the total deficiency  $s$  is odd. Finally we remove any  $(d-r)/2$  lines of the 1-factor  $F_{r+1}$  from  $H'$  and join  $w$  with the resulting  $d-r$  points of degree  $d-1$ . The graph so constructed is  $d$ -regular, contains  $G$ , and has order  $p+d+2$ .

We now show by a family of examples that  $d+2$  is best possible. Let  $G$  be obtained from the complete graph  $K_{p-1}$  with  $p \geq 5$  by subdividing just one line by the insertion of a new point of degree 2; the graph  $G_2$  in figure 1 illustrates  $p=5$ . Then we can readily see that if  $p$  is odd, at least  $d+2$  new points are needed to construct a  $d$ -regular supergraph of  $G$ .  $\square$

*Remark 1.* The smallest  $d$ -regular supergraph  $H$  will of course depend on the structure of  $G$  and its order can range between  $p$  and  $p+d+2$ .

*Remark 2.* When  $pr$  is even, the minimum order of an  $r$ -regular supergraph  $H$  will range between  $p$  and  $p+d+1$ . Thus the bound in the theorem is decreased by 1 in this case.

The strengthening of the theorem in the following statement is easily accomplished by a proof which we omit as it is entirely analogous.

**Corollary.** *Let  $G$  be a graph of order  $p$  with maximum degree  $d$ , and  $r$  be an integer such that  $d \leq r \leq p-2$ . Then  $G$  has an  $r$ -regular supergraph of order at most  $p+r+1$  or  $p+r+2$  if  $pr$  is even or odd, respectively.  $\square$*

Jin Akiyama, Hiroshi Era and Frank Harary,  
Nippon Ika University, Kawasaki, Japan; University of Michigan, Ann Arbor, USA

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## A remark on a paper by A. Grytczuk

In [3] Grytczuk showed that if  $c_k(n)$  denotes the Ramanujan trigonometric sum, then

$$\sum_{d|k} |c_d(n)| = 2^{\omega(k/(k,n))} (k, n), \tag{1}$$

where  $\omega(m)$  denotes the number of distinct prime divisors of  $m$ . In this note we prove a generalization of (1).

Let  $h$  be an arithmetic function and if  $m$  is a natural number, let

$$h_m(n) = \begin{cases} h(n) & \text{if } n|m \\ 0 & \text{else} \end{cases}$$

Define

$$H_m(n) = \sum_{d|(m,n)} \mu(m/d) h(d) = \sum_{d|m} \mu(m/d) h_n(d). \quad (2)$$

Note that  $H_1(n) = h(1)$  and if  $a \geq 1$ , then

$$H_{p^a}(n) = \begin{cases} h_n(p^a) - h_n(p^{a-1}) & \text{if } p^a | n \\ h(p^a) - h(p^{a-1}) & \text{if } p^{a-1} || n \\ 0 & \text{if } p^{a-1} \nmid n \end{cases}$$

We now show that  $H_m(n)$  is a multiplicative function of  $m$  if  $h(m)$  is also. To do this it will suffice to show that  $h_n(d)$  is a multiplicative function of  $d$ , since, by (2),  $H_m(n)$  is then a convolution of two multiplicative functions. If  $(m, n) = 1$  and  $k$  is a natural number, then  $mn|k$  if and only if  $m|k$  and  $n|k$ . Thus, by the definition of  $h_k(mn)$ , we have

$$\begin{aligned} h_k(mn) &= \begin{cases} h(mn) & \text{if } mn|k \\ 0 & \text{else} \end{cases} = \begin{cases} h(m)h(n) & \text{if } m|k \text{ and } n|k \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} h_k(m)h_k(n) & . \end{cases} \end{aligned}$$

Thus  $h_k(d)$  is a multiplicative function of  $d$  if  $h$  is multiplicative.

**Theorem.** *If  $h$  is a multiplicative function such that*

$$h(p^a) - h(p^{a-1}) \geq 0 \quad (3)$$

*for all  $a \geq 1$  and primes  $p$ , then for all positive integers  $k$  and  $m$ , we have*

$$\sum_{d|k} |H_d(n)| = 2^{\omega(k/(k,n))} h((k, n)). \quad (4)$$

**Proof:** Since  $H_d(n)$  is multiplicative so is  $|H_d(n)|$  and thus so is the left hand side of (4) ([4], theorem 265). Since, for  $n$  fixed and  $(k, l) = 1$ , we have  $(kl, n) = (k, n)(l, n)$  (see [5], p. 17) and since  $h$  is multiplicative we see that the right hand side of (4) is also a multiplicative function of  $k$ . To prove the theorem it therefore suffices to show that (4) holds when  $k$  is a prime power,  $k = p^a$ .

Note that, by (3),  $h(p^a) \geq h(p^{a-1}) \geq \dots \geq h(1) \geq 0$ , since  $h$  multiplicative implies  $h(1) = 1$  or  $h(1) = 0$ .

Suppose  $p^b \parallel n$ . If  $0 \leq b < a$ , then

$$\begin{aligned} \sum_{j=0}^a |H_{p^j}(n)| &= h(1) + \sum_{j=1}^b (h(p^j) - h(p^{j-1})) + |H_{p^{b+1}}(n)| \\ &= h(1) + h(p^b) - h(1) + h(p^b) = 2h(p^b). \end{aligned} \tag{5}$$

If  $b \geq a$ , then

$$\sum_{j=0}^a |H_{p^j}(n)| = h(1) + \sum_{j=1}^a (h(p^j) - h(p^{j-1})) = h(p^a). \tag{6}$$

If we compare (5) and (6) with the right hand side of (4), in each of the two cases we see that they agree. This proves our theorem.

Examples:

1. Let  $r$  be a positive integer and

$$C_k^{(r)}(n) = \sum_{\substack{x_1, \dots, x_r \pmod k \\ (x_1, \dots, x_r, k) = 1}} \exp(2\pi i n (x_1 + \dots + x_r)/k).$$

In [2], theorem 1, Cohen proves that

$$C_k^{(r)}(n) = \sum_{d|(k,n)} d^r \mu(k/d).$$

Thus, if we take  $h(n) = n^r$ , we have, by the theorem,

$$\sum_{d|k} |C_d^{(r)}(n)| = 2^{\omega(k/(k,n))} (k,n)^r.$$

The case  $r = 1$  is (1), the result of Grytczuk.

2. Let  $h(n) = d(n)$ , the divisor function. Then  $d$  satisfies the hypotheses of the theorem and in this case

$$H_{p^a}(n) = \begin{cases} 1 & \text{if } p^a | n \\ -a & \text{if } p^{a-1} \parallel n. \\ 0 & \text{if } p^{a-1} \nmid n \end{cases}$$

The theorem states that in this case

$$\sum_{d|k} |H_d(n)| = 2^{\omega(k/(k,n))} d((k,n)).$$

It might be interesting to note that the evaluation of the sum  $\sum_{d|k} H_d(n)$  is much easier even if  $h$  is not known to be multiplicative. Indeed it follows immediately from the Möbius inversion formula ([4], theorem 266) that

$$H_m(n) = \sum_{d|(m,n)} \mu(m/d) h(d) = \sum_{d|m} \mu(m/d) h_n(d)$$

if and only if

$$\sum_{d|k} H_d(n) = h_n(k) = \begin{cases} h(k) & \text{if } k|n \\ 0 & \text{else} \end{cases}.$$

Finally we remark that it may be possible to generalize our result further to the class of functions considered by Anderson and Apostol in [1]. We hope to return to this in a later paper.

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Don Redmond, Southern Illinois University, Carbondale, USA

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## Nachtrag

**P. Läuchli: Das Gitterspiel, *El. Math.* 37, 109–113 (1982).**

Nach Drucklegung des Aufsatzes stellte ich aufgrund eines neuen Buches [2], dessen Manuskript mir freundlicherweise von den Autoren zur Verfügung gestellt wurde, fest, dass das «Gitterspiel» von C. Berge schon 1907 von Wythoff beschrieben wurde [3]. Ferner hat Coxeter [1] eine sehr elegante Verbindung zum goldenen Schnitt gezogen.

P. Läuchli

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