Polygonal Roulettes

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Polygonal Roulettes

If a curve rolls without slipping along another fixed curve, any point which moves with the rolling curve will describe a *roulette* [4, p. 139]. Many of the well-known plane curves are thus roulettes. For example, the cycloid is the roulette generated by a circle rolling along a straight line. The cardioid, nephroid, and other epicycloids are generated by rolling a circle along the exterior of a fixed circle; the deltoid, astroid, and other hypocloids are generated by rolling a circle along the interior of a fixed circle.

YATES [5] investigated the curves generated by a vertex of a regular polygon as it is rolled along either a straight line or else a similar polygon. Thus his curves consist of a sequence of contiguous circular arcs. Beginning with polygonal curves, however, it seems desirable to generate yet another polygonal curve. A natural way to accomplish this is to replace the circular arcs by their chords. This has recently been investigated in the cycloidal case by DETEMPLE [1], [2], and DETEMPLE and ENGQUIST [3], where many interesting properties of "polygonal cycloids" are described.

It still remains to examine the properties of other polygonal roulettes, and that is our purpose here. The polygonal epicycloid and polygonal hypocycloid will serve as examples.

Our calculations make frequent use of the following trigonometric identities:

$$\sum_{k=1}^{m-1} \sin(k\pi/m) = \cot(\pi/2m), \tag{1}$$

$$\sum_{k=1}^{m-1} \sin^2(k\pi/m) = m/2. \tag{2}$$

In addition we need the expression which gives the length r_k of a chord drawn from one vertex of a regular m-gon of circumradius R to a second vertex k sides away, namely

$$r_k = 2R\sin(k\pi/m). \tag{3}$$

1. Polygonal Epicycloids

Let q and n be positive integers with $qn \geq 3$, and suppose a regular qn-gon of circumradius R is rolled, without slipping, along the exterior of a regular qn-gon circumradius qR. A vertex of the rolling polygon, initially coincident with a vertex of the fixed polygon, traces a sequence of circular arcs. Replacing these arcs by their chords yields a curve we shall call the *(q,n)-polygonal epicycloid* (Fig. 1). We observe that every qth pivot is about a vertex of the fixed polygon, and here the rolling polygon turns through $4\pi/qn$ radians. At pivot points along the sides of the fixed polygon the rolling polygon turns through $2\pi/qn$. There are q points where the distinguished vertex coincides with a vertex of the fixed polygon, and so q represents the number of "cusps" of the (q,n)-polygonal epicycloid. There are n sides of the fixed polygon between successive cusps.

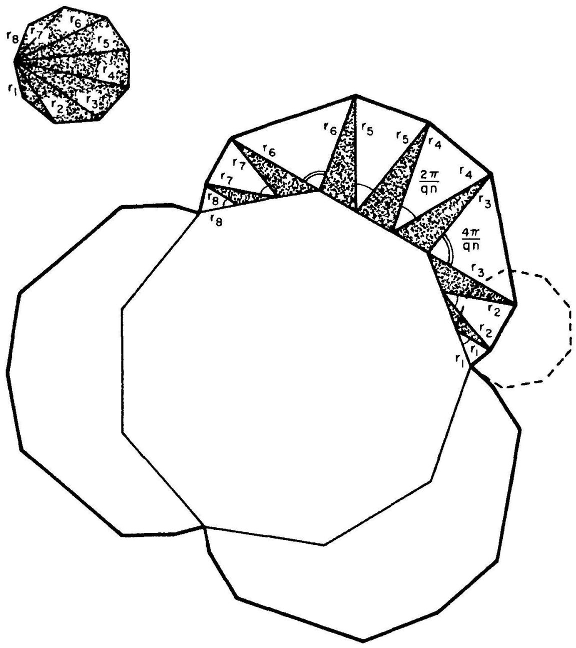

Figure 1. A polygonal epicycloid.

a. Length

Let $L_e(q,n)$ denote the length of the (q,n)-polygonal epicycloid. From Fig. 1 we have that

$$L_e(q,n)/q = \sum_{\substack{k=1 \\ q \nmid k}}^{qn-1} 2r_k \sin(\pi/qn) + \sum_{\substack{k=1 \\ q \mid k}}^{qn-1} 2r_k \sin(2\pi/qn), \tag{4}$$

where r_k is given by (3) with $m=qn$. For $n\geq 2$ we can rewrite (4) as

$$L_e(q,n)/q=2\sin(\pi/qn)\sum_{k=1}^{qn-1}2R\sin(k\pi/qn)$$
$$+[2\sin(2\pi/qn)-2\sin(\pi/qn)]\sum_{k=1}^{n-1}2R\sin(k\pi/n).$$

Evaluating the sums by means of the identity (1) leads to

$$L_e(q,n)/q=4R\sin(\pi/qn)\cot(\pi/2qn)$$
$$+4R[\sin(2\pi/qn)-\sin(\pi/qn)]\cot(\pi/2n),$$

which, except for rearrangement, gives the expression sought for $L_e(q,n)$ when $n\geq 2$.

For $n=1$ (4) becomes

$$L_e(q,1)/q=2\sin(\pi/q)\sum_{k=1}^{q-1}2R\sin(k\pi/q)$$
$$=4R\sin(\pi/q)\cot(\pi/2q)=4R(1+\cos(\pi/q))$$

where (1) and a half-angle formula for the cotangent have been employed. Of course in the $(q,1)$ case we really have a q-polygonal cycloid erected on each side of the fixed polygon. The right side of the equation just above thus gives the length of a q-polygonal cycloid, in agreement with [1] and [3].

Theorem 1. Let $L_e(q,n)$ be the length of the (q,n)-polygonal epicycloid generated by a regular qn-gon of circumradius R rolling upon a regular qn-gon of circumradius qR. Then

$$L_e(q,n)=4qR[1+\cos(\pi/qn)-\sin(\pi/qn)\cot(\pi/2n)+\sin(2\pi/qn)\cot(\pi/2n],\quad n\geq 2, \tag{5}$$

$$L_e(q,1)=4qR[1+\cos(\pi/q)]. \tag{6}$$

From (5) it is readily checked that $\lim_{n\to\infty} L_e(q,n)=8R(q+1)$. Thus the length of the epicycloid of q cusps is $8(q+1)$ times the radius of the rolling circle.

Formula (5) greatly simplifies in the cases of the polygonal cardioid ($q=1$) and polygonal nephroid ($q=2$). If one uses the half-angle formula for the cotangent and double-angle formula for the sine we obtain

$$L_e(1,n)=8R(\cos(\pi/n)+\cos^2(\pi/n)).$$

That is, the length of the polygonal cardioid generated by rolling a regular n-gon of circumradius R about an equal polygon is 16 times the average of the inradius and circumradius of the n-gon whose vertices are the midpoints of the fixed (or rolling) polygon (see Fig. 2).

Similarly one can show

$$L_e(2,n)=8R(2+\cos(\pi/n)) \tag{7}$$

is the length of the polygonal nephroid.

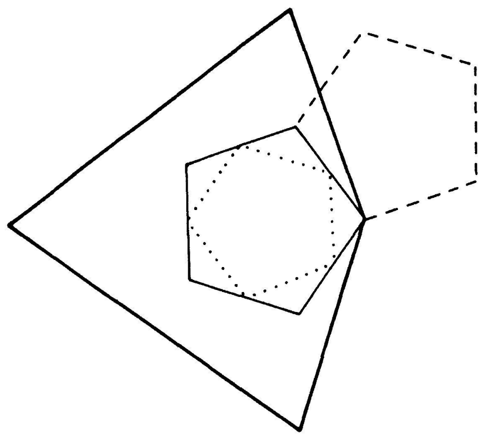

Figure 2. The polygonal cardioid generated by a pentagon.

b. Area

Let $A_e(q,n)$ denote the area enclosed by the (q,n)-polygonal epicycloid. From Fig. 1 we first note that the $qn-2$ shaded triangles between successive cusps have the area of the rolling polygon, namely $(1/2)\,qnR^2\sin(2\pi/qn)$. The unshaded $qn-1$ triangles between successive cusps have total area

$$\sum_{\substack{k=1\\ q\nmid k}}^{qn-1} \frac{1}{2} r_k^2 \sin(2\pi/qn) + \sum_{\substack{k=1\\ q\mid k}}^{qn-1} \frac{1}{2} r_k^2 \sin(4\pi/qn), \tag{8}$$

where $r_k = 2R\sin(k\pi/qn)$. The quantity in (8) is simplified much as was done with equation (4), although now it is identity (2) which is used. For $n \geq 2$ the area given by (8) reduces to

$$\frac{1}{2} nR^2 \sin(2\pi/qn)\,(2q+2-8\sin^2(\pi/qn))$$

and for $n=1$ it is $qR^2\sin(2\pi/q)$. Thus each of the q regions between cusps which are within the (q,n)-polygonal epicycloid but exterior to the fixed polygon has area

$$\frac{1}{2} nR^2 \sin(2\pi/qn)\,(3q+2-8\sin^2(\pi/qn)),\ n \geq 2, \tag{9}$$

$$\frac{3}{2} qR^2 \sin(2\pi/q),\ n=1.$$

The second formula just above shows us that the area of a polygonal cycloid is 3 times the area of the rolling polygon used to generate it.

To now obtain $A_e(q,n)$ we multiply (9) by q and then add it to $(1/2)\,qn\,(qR)^2\sin(2\pi/qn)$, which is the area of the fixed qn-gon of circumradius qR.

Theorem 2. Let $A_e(q,n)$ be the area of a (q,n)-polygonal epicycloid generated by a regular qn-gon of circumradius R rolling upon a fixed regular qn-gon of circumradius qR. Then

$$A_e(q,n) = \frac{1}{2} qnR^2 \sin(2\pi/qn)\,[(q+1)(q+2) - 8\sin^2(\pi/qn)],\ n \geq 2 \tag{10}$$

$$A_e(q,1) = \frac{1}{2} qR^2 \sin(2\pi/q)\, q\,(q+3),\ n=1. \tag{11}$$

The factor $(1/2)\,qnR^2\sin(2\pi/qn)$ is, we recall, the area of the rolling qn-gon. Thus taking the limit $n\to\infty$ in (10) we see that the area of the epicycloid of q cusps is $(q+1)(q+2)$ times the area of the rolling circle which generates it.

For the polygonal cardioid ($q=1$), the area expression can be put into the form

$$A_e(1,n)=8\left[\frac{1}{2}n(R\cos(\pi/n))^2\sin(2\pi/n)\right]-2\left[\frac{1}{2}nR^2\sin(2\pi/n)\right].$$

Thus the area of the polygonal cardioid is eight times the area of the n-gon with vertices at the midpoints of the rolling polygon, less twice the area of the rolling polygon (see again Fig. 2).

The situation is even simpler for the polygonal cardioids generated by rolling a polygon of $n=2m$ sides. Indeed

$$A_e(1,2m)=2\left[\frac{1}{2}(2m)R^2\sin(2\pi/2m)\right]+4\left[\frac{1}{2}mR^2\sin(2\pi/m)\right]$$

and so the area of the polygonal cardioid is twice the area of the rolling (or fixed) $2m$-gon plus four times the area of the regular m-gon of the same radius. (See Fig. 3.)

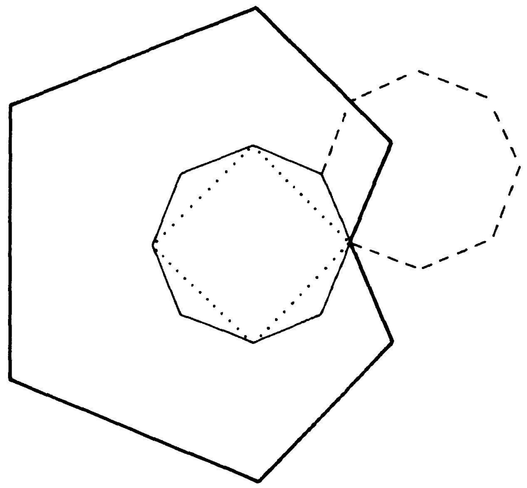

Figure 3. The polygonal cardioid generated by an octagon.

For polygonal nephroids we have

$$A_e(2,n)=4\left[\frac{1}{2}(2n)R^2\sin(2\pi/2n)\right](2+\cos(\pi/2n)).$$

In view of (7) we see that the length and area of the polygonal nephroid are each $4(2+\cos(\pi/2n))$ times the respective diameter and area of the rolling polygon. On the other hand

$$A_e(2,n)=8\left[\frac{1}{2}(2n)R^2\sin(2\pi/2n)\right]+4\left[\frac{1}{2}nR^2\sin(2\pi/n)\right];$$

thus the area of a polygonal nephroid generated by rolling a regular $2n$-gon is eight times the area of the rolling polygon plus four times the area of the regular n-gon of the same radius.

2. Polygonal Hypocycloids

Now let $q \geq 2$ and $n \geq 1$ with $qn \geq 3$, and roll a regular qn-gon of circumradius R along the interior of a regular qn-gon of circumradius qR. In the same manner as before we generate a polygonal curve we call the *(q, n)-polygonal hypocycloid* (Fig. 4). The number q still represents the number of cusps, although these now have the form of radial spikes at every qth vertex of the fixed polygon. The number n remains the number of sides of the fixed polygon between successive cusps. The rolling polygon turns through $2\pi/qn$ radians when pivoting about a point which is between vertices of the fixed polygon. Every qth pivot point coincides with a vertex of the fixed polygon, and here the pivot angle is 0.

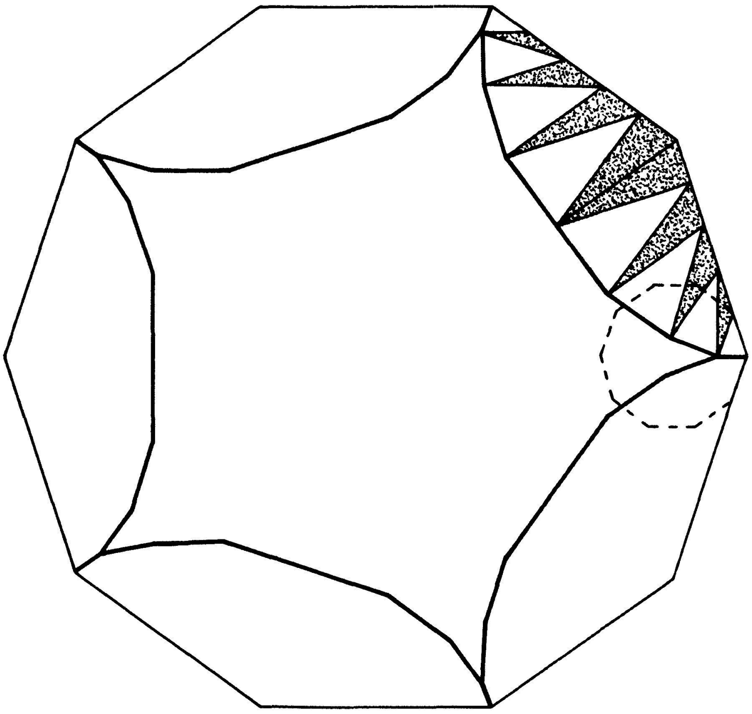

Figure 4. A polygonal hypocycloid.

a. Length

Let $L_h(q, n)$ denote the length of the (q, n)-polygonal hypocycloid. Referring to Fig. 4, we see that

$$L_h(q, n)/q = \sum_{\substack{k=1 \\ q \nmid k}}^{qn-1} 2 r_k \sin(\pi/qn)$$

where $r_k = 2R \sin(k\pi/qn)$. The sum can be put into closed form by following steps analogous to those in the preceding section. Because $\cot(\pi/2) = 0$ it turns out that the formula derived for the case $n \geq 2$ is valid even when $n = 1$. Of course, we also have $L_h(q, 1) = L_e(q, 1)$, with $L_e(q, 1)$ given by (6).

Theorem 3. Let $L_h(q,n)$ denote the length of the (q,n)-polygonal hypocycloid generated by rolling a regular qn-gon of circumradius R along the interior of a fixed regular qn-gon of circumradius qR. Then

$$L_h(q,n) = 4\,qR\,\left(1 + \cos(\pi/qn) - \sin(\pi/qn)\cot(\pi/2n)\right). \tag{12}$$

Since $\lim_{n\to\infty} L_h(q,n) = 8R(q-1)$ we see that the length of the hypocycloid of q cusps is $8(q-1)$ times the radius of the rolling circle. We also see that the polygonal hypocycloid of $q=2$ cusps is just a twice-covered diameter of the fixed polygon. In fact, $L_h(2,n)=8R$. It is interesting to note that there is no dependence here on n.

b. Area

To determine $A_h(q,n)$, the area of a (q,n)-polygonal hypocycloid, we see from Fig. 4 that the area of the unshaded triangles between two successive cusps is

$$\sum_{k=1}^{qn-1} \frac{1}{2} r_k^2 \sin(2\pi/qn),\; r_k = 2R\sin(k\pi/qn). \tag{13}$$

In the case $n=1$, we can apply identity (2) directly to (13) to get $2a$, where $a=(1/2)\,qnR^2\sin(2\pi/qn)$ is the area of the rolling polygon. In the case $n\geq 2$ we first put (13) into the form

$$\frac{4a}{qn}\left[\sum_{k=1}^{qn-1}\sin^2(k\pi/qn) - \sum_{k=1}^{n-1}\sin^2(k\pi/n)\right]. \tag{14}$$

Identity (2) now applies, showing that the quantity (14) is $2a\,(1-q^{-1})$.

The area total of all the triangles, shaded and unshaded, between cusps is thus

$$3a, \quad n=1, \qquad (3-2q^{-1})\,a, \quad n\geq 2. \tag{15}$$

Since $A_h(q,n)$ is the area q^2a of the fixed polygon less q times the area given by (15), we easily arrive at the following result.

Theorem 4. The area of the (q,n)-polygonal hypocycloid is

$$A_h(q,n) = (q-1)(q-2)a, \qquad n\geq 2, \tag{16}$$

$$A_h(q,1) = q(q-3)a, \tag{17}$$

where a is the area of the rolling qn-gon.

Formula (16) is quite remarkable, since it is in direct analogy to the result that hypocycloids of q cusps have area $(q-1)(q-2)$ times that of the generating circle [4, p. 146]. From our remarks just following Theorem 3 we could have anticipated that $A_h(2,n)=0$ for all $n\geq 2$.

The case $n=1$ corresponds to that where a polygonal cycloid is erected, toward the interior, on each side of the fixed q-gon. The (3,1)-polygonal hypocycloid consists of just three (doubly-traced) radial segments from the center of an equilateral triangle to its vertices; of course from (17) $A_h(3,1)=0$. The (4,1)-polygonal hypocycloid is a "spiked" square; the square's side length is twice that of the rolling square. The (5,1)-polygonal hypocycloid is a star-shaped figure with radial spikes; a diagram and other discussion can be found in DeTemple [2].

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Konvexe Körper approximierende Polytopklassen

Es bezeichne $\mathfrak{P}(m,n)$ die Klasse konvexer Polytope des dreidimensionalen euklidischen Raumes E^3, deren Ecken höchstens m-valent sind und deren Seiten höchstens n Ecken besitzen, $m\geq 3, n\geq 3$.

$\mathfrak{P}(m,n)$ heisse *approximierende Klasse der Klasse* $\mathfrak{K}$ aller konvexen Körper des E^3, wenn es zu jedem $K\in\mathfrak{K}$ eine Folge $\{P_i\}_{i\in\mathrm{N}}$ konvexer Polytope $P_i\in\mathfrak{P}(m,n)$ gibt, die im Sinne der Hausdorffmetrik gegen K konvergiert.

G. Ewald hat die Frage gestellt [3], welche der Klassen $\mathfrak{P}(m,n)$ $\mathfrak{K}$ approximieren. Wir beantworten diese Frage vollständig mit Hilfe des folgenden Satzes:

Satz. Die Klassen $\mathfrak{P}(4,4)$, $\mathfrak{P}(3,6)$, $\mathfrak{P}(6,3)$ sind approximierende Klassen von $\mathfrak{K}$.

Bemerkung 1. Damit ist die Frage nach den approximierenden Klassen von $\mathfrak{K}$ geklärt, denn die Klassen $\mathfrak{P}(3,3)$, $\mathfrak{P}(3,4)$, $\mathfrak{P}(4,3)$, $\mathfrak{P}(3,5)$, $\mathfrak{P}(5,3)$ enthalten nur endlich viele kombinatorische Klassen konvexer Polytope und können daher nicht approximierend sein. Alle übrigen Klassen enthalten eine der im Satz aufgeführten Klassen als Teilklasse und sind daher approximierend.

Bemerkung 2. Eine weitergehende Frage für geschlossene Flächen beliebigen Geschlechts im E^3 wird mit graphentheoretischen Methoden, die bei obigem Satz zu versagen scheinen, in einer Arbeit von G. Ewald [2] behandelt.

Bemerkung 3. Die Frage von G. Ewald [3], ob es endliche approximierende Klassen der Klasse $\mathfrak{K}^n$ aller n-dimensionalen konvexen Körper, $n\geq 4$, gibt, bleibt offen.