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1. Introduction

In [1] some estimates on minimal areas of polygons circumscribed about a plane convex set were considered. In what follows we shall prove a theorem that leads to very concise proofs of those estimates and some other results concerning circumscribed polygons.

We shall deal mainly with plane convex bodies. If K is a plane convex body, the area of K will usually be denoted by the same symbol K in order to simplify notation. We shall say that two convex n-gons are *parallel* if corresponding sides are parallel. Then we can state the main theorem as follows.

Theorem 1. Suppose K is a plane convex body, p is a polygon inscribed in K, and P is a polygon parallel to p and circumscribed about K. Then

$$K^2 \geq pP. \tag{1}$$

2. Proof of the main theorem

The proof of Theorem 1 depends on Minkowski's concept of the *mixed area*, $A(K, L)$, of two plane convex bodies K and L. In case p and P are parallel n-gons, $A(p, P)$ is easily described as follows. Let 0 be a point fixed interior to P. If l_i is the length of a side of p, let d_i be the distance from 0 to the corresponding parallel side of P. Then

$$A(p, P) = \frac{1}{2}\sum d_i l_i, \tag{2}$$

summed over all sides of p. In [5] one can find a treatment of the properties of mixed areas and a proof of the following fundamental inequality of Minkowski:

$$A(K, L)^2 \geq KL. \tag{3}$$

Now consider a plane convex body K, with inscribed n-gon p and parallel circumscribed n-gon P. Each side of P contains at least one point of K. If we choose one such point on each side of P, then these points, taken together with the vertices of p, are the vertices of a convex $2n$-gon Q inscribed in K. Fix a point 0 inside p. If l_i is the length of a side of p, let d_i be the distance from 0 to the corresponding parallel side of P. Upon making a sketch of the situation, the reader will readily see that the area of Q is given by

$$Q = \frac{1}{2}\sum d_i l_i = A(p, P). \tag{4}$$

Using the fact that $Q \subset K$, and Minkowski's inequality, we then have,

$$K^2 \geq Q^2 = A(p, P)^2 \geq pP , \tag{5}$$

which proves Theorem 1.

3. Applications of the main theorem

We now derive a number of corollaries of Theorem 1, with all proofs following basically the same pattern.

Corollary 1. Any plane convex body K is contained in a triangle T_0 of area not more than twice that of K.

Proof. Let T_0 be a triangle of minimal area containing K. Then the midpoints of the sides of T_0 touch K (see [1] for a proof). Let t be the triangle inscribed in K formed by joining these midpoints, and let T be the triangle parallel to t and circumscribed about K. We have that $t = \frac{1}{4} T_0$ and $T \geq T_0$. Hence

$$K^2 \geq tT \geq \left(\frac{1}{4} T_0\right) (T_0) = \frac{1}{4} T_0^2, \tag{6}$$

so $T_0 \leq 2K$, as we wanted to prove.

Corollary 2. Any plane convex body K is contained in a quadrilateral Q_0 of area not more than $\sqrt{2}$ times that of K.

Proof. Let Q_0 be a quadrilateral of minimal area containing K. Again (see [1]) the midpoints of the sides of Q_0 touch K. Let q be the quadrilateral inscribed in K formed by joining the midpoints of the sides of Q_0. Let Q be the quadrilateral parallel to q circumscribed about K. We have $Q \geq Q_0$, and it is easy to see q is a parallelogram with $q = \frac{1}{2} Q_0$. Hence

$$K^2 \geq qQ \geq \left(\frac{1}{2} Q_0\right) (Q_0) = \frac{1}{2} Q_0^2, \tag{7}$$

so $Q_0 \leq (\sqrt{2})\, K$, as required.

The result given in Corollary 1 is in a sense the best possible, since a parallelogram K is not contained in any triangle of area less than twice the area of K. On the other hand, it is not known if the estimate for minimal circumscribed quadrilaterals in Corollary 2 is best possible, and good estimates for minimal circumscribed n-gons, $n > 4$, are apparently not known. However, the next corollary of Theorem 1 shows how to obtain an inequality by utilizing the maximum inscribed n-gon.

Corollary 3. Any plane convex body K is contained in an n-gon P of area not more than $\frac{2\pi}{n} \csc \frac{2\pi}{n}$ times that of K.

Proof. Let p be an n-gon of maximal area inscribed in K, and let P be the circumscribed n-gon parallel to p. By a theorem of Sas (see [4]), we have $p \geq \left(\frac{n}{2\pi} \sin \frac{2\pi}{n}\right) K$. Hence

$$K^2 \geq pP \geq \left(\frac{n}{2\pi} \sin \frac{2\pi}{n}\right) KP, \tag{8}$$

from which the result follows.

Suppose K is a centrally symmetric plane convex body. By a *lattice packing* of K we mean a distribution of translates of K, no pair having interior points in common, with their centers forming a plane lattice. The density of such a packing measures the fraction of the plane covered by these translates of K. The following result, proved in [4] in a different manner, follows readily from Theorem 1.

Corollary 4. Any centrally symmetric plane convex body K can be lattice packed with density at least $\frac{\sqrt{3}}{2}$.

Proof. By a theorem of Dowker [3], there is a centrally symmetric hexagon H_0 of minimum area circumscribed about K. A theorem of Day [2] implies that the midpoints of the sides of H_0 touch K. Let h be the hexagon formed by joining the midpoints of the sides of H_0. Then it is not a difficult exercise to verify that h is the affine image of a regular hexagon, with $h = \frac{3}{4} H_0$. Let H be the centrally symmetric hexagon parallel to h and circumscribed about K. Then $H \geq H_0$, and

$$K^2 \geq hH \geq \left(\frac{3}{4} H_0\right)(H_0) = \frac{3}{4} H_0^2, \tag{9}$$

so $K \geq \left(\frac{\sqrt{3}}{2}\right) H_0$. Since H_0 tiles the plane in a lattice manner, the required result follows.

4. Generalization to higher dimensions

Using mixed volumes in place of mixed areas, the following higher dimensional analogue of Theorem 1 is easily proved.

Theorem 2. Let K be a convex body in Euclidean n-space. Let p be a convex polytope contained in K and let P be a polytope circumscribed about K and parallel to p (that is, the facets of P parallel to corresponding facets of p). Then

$$K^n \geq p^{n-1} P, \tag{10}$$

where we are now using the same notational convention for volumes that we used before for areas.

Corollary 5. Any convex body K in Euclidean n-space is contained in a simplex T_0 of volume not more than n^{n-1} times that of K.

Proof. Let T_0 be a simplex of minimal volume containing K. By the theorem of Day [2], the centroids of the facets of T_0 touch K. Let t be the simplex whose vertices are those centroids, and let T be the simplex parallel to t and circumscribed about K. Then $t = (n^{-n})\, T_0$ and $T \geq T_0$, so

$$K^n \geq t^{n-1}\, T \geq (n^{-n(n-1)}\, T_0^{n-1})\,(T_0)\,, \tag{11}$$

so $T_0 \leq (n^{n-1})\, K$, as we wanted to prove.

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Hypo-Eulerian and Hypo-Traversable Graphs

Introduction

If a graph G does not possess a given property P, and for each vertex v of G the graph $G - v$ enjoys property P, then G is said to be a *hypo-P* graph. Recently, studies have been made where P stands for the graph being *hamiltonian*, *planar*, and *outerplanar* (e.g., see [3]). Here we obtain a characterization of *hypo-eulerian* and *hypo-randomly-eulerian* graphs, and investigate in this respect some of the other concepts arising out of Euler's solution of the classical Königsberg Seven Bridges Problem.

Preliminaries

Following the terminology of [2], a *graph* will be finite, undirected, without loops or multiple edges. A *walk* of a graph G is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ of vertices and edges of G, beginning and ending with vertices and where the edge $e_i = v_{i-1} v_i$ for $i = 1, 2, \ldots, n$. This is a $v_0 - v_n$ walk, and is usually denoted $v_0 v_1 v_2 \ldots v_n$; it is *closed* if $v_0 = v_n$ and *open* otherwise. A walk is a *trail* if all its edges are distinct; it is a *path* if all its vertices are distinct. A closed trail is a circuit and a circuit on distinct vertices is a *cycle*. A cycle on p vertices is denoted C_p, and C_3 is called a *triangle*.

If for every two distinct vertices u and v of a graph G there exists a $u - v$ path, then G is *connected*. A *component* of G is a maximal connected subgraph of G. A vertex