Kleine Mitteilungen

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Functionals Close to Each Other

Bishop and Phelps [1] proved the fundamental theorem that every real or complex Banach space is subreflexive, that is the functionals (real or complex) attaining their suprema on the unit sphere of the space are dense in the dual space. (For an extension see [2].) The proof is based on a very useful lemma of Phelps [3, 4], stating an intuitively obvious fact. We shall give a completely elementary proof of this lemma.

Let E be a real normed space with unit ball B and dual E'. Suppose $f, g \in E'$, $\|f\| = \|g\| = 1$ and $f^{-1}(0) \cap B \subset g^{-1}[-\varepsilon, \varepsilon]$ $(\varepsilon < 1/2)$. Then $\|f - g\| \leqslant 2\varepsilon$ or $\|f + g\| \leqslant 2\varepsilon$.

Proof. If $f = \pm g$ there is nothing to prove. If not, choose $a, b \in E$ such that

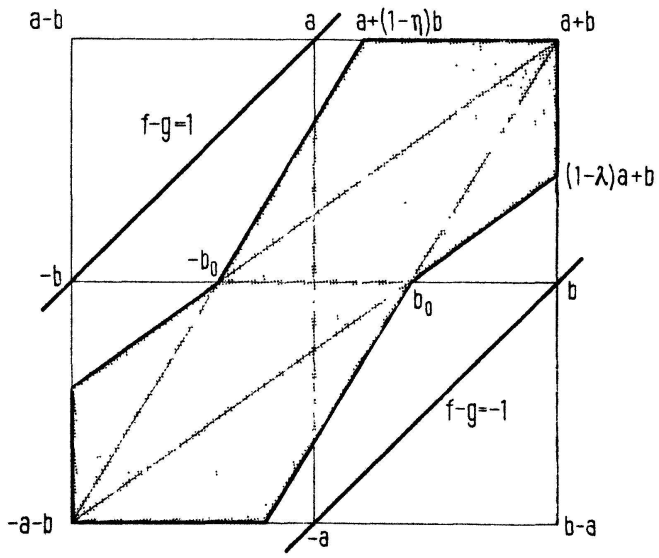
$$f(a) = g(b) = 1 \quad \text{and} \quad f(b) = g(a) = 0 .$$

Denote by M the plane spanned by a and b. Put $N = f^{-1}(0) \cap g^{-1}(0)$. Evidently E is the direct sum of N and M. Denote by D the projection of B into M parallel to N.

Choosing D as the unit ball, define a norm $\|\cdot\|'$ in M. Putting $f_0 = f_{|M}$ and $g_0 = g_{|M}$, by construction we have $\|\alpha f_0 + \beta g_0\|' = \|\alpha f + \beta g\|$ $(\alpha, \beta \in \mathbf{R})$, so we can suppose that E is two-dimensional, $D = B$.

$f^{-1}(0) \cap B \subset g^{-1}[-\varepsilon, \varepsilon]$ implies $\|b\| \geqslant \varepsilon^{-1}$, i.e. B intersects the line determined by b between $-\varepsilon b$ and εb, say ∂B intersects the line at $-b_0$ and b_0.

Take two parallel lines through b_0 and $-b_0$ whose strip contains B. As $\|f\| = \|g\| = 1$, the strip must contain either $a + b$ or $a - b$. By symmetry one can suppose that it contains $a + b$. But then B is contained in the striped region in the Figure.



Comparing this region with the lines $f - g = \pm 1$, one sees immediately that $\|f - g\| \leqslant \max(\eta, \lambda)$. As $\eta \leqslant 2\varepsilon$ and $\lambda \leqslant 2\varepsilon/(1 + \varepsilon)$, we have $\|f - g\| \leqslant 2\varepsilon$.

Remarks. 1. It is evident from the Figure that the result is best possible.

2. The figure also shows that if $f^{-1}(0) \cap B \subset g^{-1}[-\varepsilon, \varepsilon]$ and $g^{-1}(0) \cap B \subset f^{-1}[-\varepsilon, \varepsilon]$ then $\|f - g\| \leqslant 2\,\varepsilon/(1+\varepsilon)$ or $\|f + g\| \leqslant 2\,\varepsilon/(1+\varepsilon)$.

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A Distribution Property of the Sequence of Lucas Numbers

Let $\{L_n\}$ $(n = 1, 2, \ldots)$ be the Lucas sequence: 1, 3, 4, 7, 11, ... Then we have the following lemmas:

Lemma 1. L_n divides L_m if and only if $m = (2\,k - 1)\,n$; $n > 1$; $k = 1, 2, \ldots$ [1].

Lemma 2. If L_m is divided by L_n, $m > n$, then the remainder R is zero, or R is a Lucas number, or $L_n - R$ is a Lucas number [2].

Lemma 3. Let p be a prime number $\geqslant 2$ and with the property that it divides some Lucas number. Suppose that L_t is the smallest Lucas number with $L_t \equiv 0 \pmod p$, where t is an integer > 1. Then there does not exist an L_q with $L_{(2k-1)t} < L_q < L_{(2k+1)t}$ such that $L_q \equiv 0 \pmod p$.

Proof. By lemma 1 we have $L_{(2k-1)t} \equiv 0 \pmod p$. Now suppose that there exists an L_q with $L_{(2k-1t)} < L_q < L_{(2k+1)t}$ such that $L_q \equiv 0 \pmod p$. Let L_q be divided by L_t, then, by lemma 2, the remainder $R = 0$ or R is a Lucas number or $L_t - R$ is a Lucas number. But $R = 0$ is impossible according to lemma 1, for $R = 0$ implies that L_t divides L_q, which in its turn implies that q must be an odd multiple of t. If R would be a Lucas number, then $R \equiv 0 \pmod p$, for by the division algorithm $R = -a\,L_t + L_q$, where a is an integer, and L_t, $L_q \equiv 0 \pmod p$. Now $R < L_t$ and $R \equiv 0 \pmod p$ yields a contradiction. Similarly, if $L_t - R$ would be a Lucas number, then by the same argument we see that $L_t - R \equiv 0 \pmod p$ and $L_t - R < L_t$ which again yields a contradiction.

Definition. The sequence $\{x_n\}$ $(n = 1, 2, \ldots)$ of integers is said to be uniformly distributed mod m where $m \geqslant 2$ is an integer, provided that

$$\lim_{N\to\infty} \frac{1}{N} \cdot A(N, j, m) = \frac{1}{m}, \qquad (*)$$

for each $j = 0, 1, \ldots, m-1$, where $A(N, j, m)$ is the number of x_n, $n = 1, 2, \ldots, N$, that are congruent to j mod m.

Theorem 1. Let $\{L_n\}$ $(n = 1, 2, \ldots)$ be the Lucas sequence. Then $\{L_n\}$ is not uniformly distributed mod p for any prime $p \geqslant 2$.

Proof. Let p be a prime $\geqslant 2$. We distinguish two cases.

(1) If p does not divide any Lucas number, then $A(N, 0, p) = 0$ which violates (*) for $m = p$ and $j = 0$.

(2) Let p divide the Lucas number L_t, where the index t is the least possible. Then by virtue of lemma 1 and lemma 3 p divides all $L_{(2k-1)t}$ $(k = 1, 2, \ldots)$ and no other Lucas numbers. Consequently

$$A(N, 0, p) = \frac{N-r}{2t} \quad \text{or} \quad \frac{N-r}{2t} + 1 \quad (0 \leqslant r < 2t)$$

and therefore

$$\lim_{N\to\infty} \frac{1}{N} A(N, 0, p) = \lim_{N\to\infty} \left(\frac{1}{2t} - \frac{r}{2Nt}\right) = \frac{1}{2t} \quad \text{as} \quad N \to \infty .$$

But $2t$ is even and thus $\neq p$ if $p > 2$. Moreover, the Lucas numbers are not uniformly distributed mod 2. This completes the proof.

Theorem 2. Let $\{L_n\}$ $(n = 1, 2, \ldots)$ be the Lucas sequence. Then $\{L_n\}$ is not uniformly distributed mod m for any composite integer $m \geqslant 2$.

Proof. If we assume that $\{L_n\}$ is uniformly distributed mod m for some composite integer $m \geqslant 2$, then $\{L_n\}$ would be uniformly distributed mod p, where p is a prime factor of m, according to theorem 5.1 of [3], which says that a sequence of integers which is uniformly distributed mod m, where m is composite, is also uniformly distributed with respect to any positive divisor of m. This contradicts theorem 1.

Consequently, from theorem 1 and theorem 2 we have:

Theorem 3. The sequence of Lucas numbers is not uniformly distributed mod m for any integer $m \geqslant 2$.

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Note on Arithmetical Progressions with Equal Products of Five Terms

It is not known if there exist two arithmetical progressions each formed by five positive integers and such that the products of their terms are equal (cf. [2]). This problem leads to the equation

$$(a-2r)(a-r)\,a\,(a+r)(a+2r) = (b-2R)(b-R)\,b\,(b+R)(b+2R)$$

or

$$a\,(a^2 - r^2)\,(a^2 - 4\,r^2) = b\,(b^2 - R^2)\,(b^2 - 4\,R^2)\,,$$

where we look for solutions in positive integers a, b, r, R such that $a - 2\,r > 0$, $b - 2\,R > 0$.

We are giving here all solutions of this equation in integers in the case when $a = b$, i.e. when the middle terms of our progressions are equal, and the progressions consist of integers.

Theorem. All integer solutions of the equation

$$(a^2 - r^2)\,(a^2 - 4\,r^2) = (a^2 - R^2)\,(a^2 - 4\,R^2) \qquad (1)$$

are of the form :

a) a, r arbitrary integers, $R = \pm r$,

b) $r = \pm(u^2 + 4\,u\,v - v^2)\,\varrho$, $R = (2\,u^2 - 2\,u\,v - 2\,v^2)\,\varrho$, $a = 2\,(u^2 + v^2)\,\varrho$,

c) $r = \pm(2\,u^2 - 2\,u\,v - 2\,v^2)\,\varrho$, $R = (u^2 + 4\,u\,v - v^2)\,\varrho$, $a = 2\,(u^2 + v^2)\,\varrho$,

where u, v, ϱ are integers.

Proof. The equation (1) can be written as

$$4\,(r^4 - R^4) = 5\,a^2\,(r^2 - R^2)\,.$$

On excluding the case a), we get

$$4\,(r^2 + R^2) = 5\,a^2\,.$$

Now $a = 2\,A$, and

$$r^2 + R^2 = 5\,A^2\,. \qquad (2)$$

All integer solutions of this equation are well-known (cf. [1], p. 48, Ex. 1) :

$$r = \pm p\,\varrho,\ R = q\,\varrho,\ A = s\,\varrho \ \text{ and } \ r = \pm q\,\varrho,\ R = p\,\varrho,\ A = s\,\varrho$$

where $p = u^2 + 4\,u\,v - v^2$, $q = 2\,u^2 - 2\,u\,v - 2\,v^2$, $s = u^2 + v^2$ and u, v, ϱ are integers. Putting here $a = 2\,A$ we obtain solutions b) and c) and the Theorem is proved.

Remark. It may be easily seen that the equation (1) has no solutions giving arithmetical progressions with positive terms. In fact, assume $R > r > 0$ and $a - 2\,R > 0$. Then $2\,A - 2\,R = a - 2\,R > 0$, $A > R$ and $r^2 + R^2 < 2\,R^2 < 5\,R^2$, contrary to (2). Thus if $R > r > 0$, then $a - 2\,R \leqslant 0$.

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Nichtnegative Matrizen mit konstanter Zeilensumme

Mit $M_n(R)$ bezeichnen wir die Menge der quadratischen Matrizen der Ordnung n über dem Körper R der reellen Zahlen.

Eine Matrix $A = (a_{kl}) \in M_n(R)$ heisst *nichtnegativ*, wenn jedes ihrer Elemente nichtnegativ ist. Bekannt ist der folgende

Satz: *Die Matrix A sei nichtnegativ und*

$$0 < \sum_{l=1}^{n} a_{kl} = r \text{ für } k = 1, 2, \dots, n\,. \tag{1}$$

So gilt $a(r)$: A hat den Eigenwert r, $b(r)$: alle Eigenwerte sind dem Betrage nach nicht grösser als r, und $c(r)$: Eigenwerten mit dem Betrage r entsprechen lineare Elementarteiler.

Diesen Satz erhält man z.B. aus einer entsprechenden Aussage über stochastische Matrizen (s. etwa GANTMACHER [1], S. 72ff.). Der Beweis kann aber besonders einfach geführt werden, wenn man den folgenden Satz aus der Theorie der Differenzengleichungen als bekannt voraussetzt (s. HAHN [2], S. 52):

Die Nullösung des Systems linearer Differenzengleichungen mit konstanten Koeffizienten

$$x\,(t+1) = B\,x(t) \tag{2}$$

ist genau dann stabil, wenn für B die Aussagen $b(1)$ und $c(1)$ gelten.

Dabei heisst die Nullösung des Systems (2) *stabil*, wenn sich zu jedem positiven ε ein positives δ finden lässt, so dass $|x(0)| < \delta$

$$|x(t)| < \varepsilon\,, \quad t = 1, 2, \dots,$$

nach sich zieht. $|\cdot|$ bedeutet hier eine beliebige Vektornorm. Wir zeigen, dass die Nullösung von (2) stabil ist, wenn $A = r\,B$ nichtnegativ ist und (1) erfüllt.

Setzen wir $|x| = \max_i |x_i|$ und bezeichnet $Z(A)$ die zu dieser Vektornorm passende Zeilennorm von A (s. etwa ZURMÜHL [3], S. 202ff.), so ist

$$|x\,(t+1)| \leqslant Z(B)\,|x(t)| = \frac{1}{r}\,r\,|x(t)| = |x(t)|\,,$$

und daher

$$|x(t)| \leqslant |x(0)|\,, \quad t = 1, 2, \dots,$$

wie behauptet.

Also gelten die Aussagen $b(1)$ und $c(1)$ für B, und daher $b(r)$ und $c(r)$ für A. $a(r)$ folgt daraus, dass A den Eigenvektor $\operatorname{col}(1, 1, \dots, 1)$ hat.

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