Kleine Mitteilungen

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Dreiecks der Seitenlänge $2\sqrt{3}\,(1-t)$ der Spanne t. Für Flächeninhalt und Umfang ergeben sich die Formeln

$$f(t) = 3\sqrt{3}\,(1-t^2) + \pi\, t^2; \quad l(t) = 6\sqrt{3}\,(1-t) + 2\pi\, t\,.$$

Offensichtlich sind $f = f(0) = 3\sqrt{3}$ und $l = l(0) = 6\sqrt{3}$ die Masszahlen des Normalrisses P' von P. Mit passender Integration gewinnt man

$$V = [(6\sqrt{3} + \pi)/3]\; h\,,$$

$$F = 3\sqrt{3}\,(1+h) + \pi + 3\int_0^{\pi/3} \sqrt{1 + h^2 - 4\cos w\,(1-\cos w)}\; dw\,.$$

Mit naheliegender einfacher Abschätzung resultiert

$$F < (3\sqrt{3} + \pi)\,(1 + \sqrt{1+h^2})\,,$$

sodass sich gemäss (1)

$$p < \frac{9\sqrt{3} + 3\pi}{12\sqrt{3} + 2\pi}\left(\frac{1 + \sqrt{1+h^2}}{h}\right)$$

ergibt. Mit $h \to \infty$ schliesst man nach (2) auf

$$p_0 \leqslant (9\sqrt{3} + 3\pi)/(12\sqrt{3} + 2\pi) \sim 0{,}924\ldots, \tag{5}$$

sodass also jedenfalls $p_0 < 1$ ausfällt.

Schliesslich wollen wir noch darauf hinweisen, dass sich die Frage nach dem Wert von p_0 nicht etwa dadurch trivialisiert, dass p beliebig kleine Werte annehmen kann, sodass $p_0 = 0$ wäre, sondern dass gezeigt werden kann, dass stets $p > 1/2$ gilt. Der Nachweis kann an dieser Stelle nicht geführt werden. Im Hinblick hierauf wird mit Sicherheit

$$p_0 \geqslant 1/2 \tag{6}$$

gelten. Das hier vorgelegte ungelöste Problem lautet also: Welches ist der Wert des mit (2) angesetzten Infimums p_0 $(0{,}500 \leqslant p_0 \leqslant 0{,}924)$? H. HADWIGER

Kleine Mitteilungen

An Elementary Set Partition Problem

In an earlier note R. SCHNEIDERREIT [2] considers the problem of distributing the numbers 1, 2, ..., n into two boxes so that not more than m consecutive numbers are in the same box; permuting the numbers in a box or interchanging the boxes does not give a new distribution. If $F_m(n)$ denotes the number of such distributions it is shown that

$$F_m(n) = F_m(n-1) + \cdots + F_m(n-m)\,, \quad n > m \tag{1}$$

with $F_m(n) = 2^{n-1}$ for $1 \leqq n \leqq m$. In the particular case $m = 2$ relation (1) with $F_2(n) = f(n)$ becomes $f(n) = f(n-1) + f(n-2)$ with $f(1) = 1$, $f(2) = 2$, $f(n)$ being the well-known Fibonacci number.

By treating the above problem as a partition problem a more general result is herein obtained. Indeed, the method of proof is quite elementary and direct.

Denote by $g_m(n, r)$, m, n, r positive integers, the number of partitions of $\{1, 2, \ldots, n\}$ into r non-empty mutually disjoint subsets or parts such that no part contains more than m consecutive integers. Equivalently $g_m(n, r)$ is the number of ways of distributing n unlike numbered objects $1, 2, \ldots, n$ into r like cells such that no cell is empty and no cell contains more than m consecutively numbered objects. Clearly $g_m(n, r) = 0$ when $n < r$.

Assume $m < n$. In a partition counted by $g_m(n, r)$, either object n is in a part not containing object $n-1$, or object n is in a part containing object $n-1$ but not object $n-2$, or object n is in a part containing $n-1, n-2$ but not $n-3, \ldots$, or object n is in a part containing $n-1, n-2, \ldots, n-m+1$ but not $n-m$. Now the number of partitions such that $n, n-1, \ldots, n-i+1$ but not $n-i$, $1 \leqq i \leqq m$, are in the same part is

$$g_m(n-i, r-1) + (r-1)\, g_m(n-i, r) \tag{2}$$

since the first number counts those partitions for which $n, n-1, \ldots, n-i+1$ are in the same part with no other objects and $(r-1)\, g_m(n-i, r)$ counts those partitions for which $n, n-1, \ldots, n-i+1$ but not $n-i$ are in the same part with that part containing at least one other object. Summing (2) over all i, $1 \leqq i \leqq m$ we have for $m < n$ the relation

$$\begin{aligned} g_m(n, r) = {} & g_m(n-1, r-1) + g_m(n-2, r-1) + \cdots + g_m(n-m, r-1) + \\ & + (r-1)\,[g_m(n-1, r) + g_m(n-2, r) + \cdots + g_m(n-m, r)]\,. \end{aligned} \tag{3}$$

With $r = 2$ in (3) we have

$$g_m(n, 2) = g_m(n-1, 1) + g_m(n-1, 2) + \cdots + g_m(n-m, 1) + g_m(n-m, 2)\,. \tag{4}$$

Letting $F_m(n) = g_m(n, 1) + g_m(n, 2)$, i.e. the number of partitions of $\{1, 2, \ldots, n\}$ into at most two parts with not more than m consecutive integers in any part, then for $m < n$, $g_m(n, 1) = 0$ and from (4) we obtain relation (1).

Defining $g_m(n, r) = 0$ when $n \leqq 0$, then relation (3) holds for all positive integral values of m. When $m > n - r$ there are no restrictions on the partitions and $g_m(n, r)$ is simply the number of partitions of a set of n elements into r non-empty, mutually disjoint subsets or parts. With $m = n - r + 1$, and $S(n, r) = g_m(n, r)$ we have from (3) the relation

$$\begin{aligned} S(n, r) = {} & S(n-1, r-1) + S(n-2, r-1) + \cdots + S(r-1, r-1) + \\ & + (r-1)\,[S(n-1, r) + S(n-2, r) + \cdots + S(r-1, r)]\,. \end{aligned} \tag{5}$$

Since the number of ways of distributing n unlike objects into r unlike cells with none empty is easily seen to be, by the principle of inclusion and exclusion, $\sum_{j=0}^{r-1} (-1)^j \binom{r}{j} (r-j)^n$, we have the known expression (cf. [1])

$$S(n, r) = \frac{1}{r!} \sum_{j=0}^{r-1} (-1)^j \binom{r}{j} (r-j)^n\,.$$

Therefore for $m \geqq n$, $F_m(n) = S(n, 1) + S(n, 2) = 2^{n-1}$ as noted in [2]. It is easy to see that (5) is equivalent to the relation

$$S(n, r) = S(n-1, r-1) + r\, S(n-1, r) \tag{6}$$

where $S(n-1, r-1)$ counts those partitions with n in a part by itself while $r\, S(n-1, r)$ counts those partitions with n in a part containing at least one other element. Of course the numbers $S(n, r)$ with n, r integers, $S(n, r) = 0$ for $r > n$ and $r < 1$, are the well known Stirling numbers of the second kind (cf. [1]) sometimes defined by

$$\sum_{r=1}^{n} S(n, r)\, x^{(r)} = x^n, \quad n > 0$$

where $x^{(r)} = x\,(x-1)\ldots(x-r+1)$.

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Two More Tetrahedra Equivalent to Cubes by Dissection

Some tetrahedra can be divided by plane cuts into a finite number of pieces which can be assembled to form a cube. A tabulation of the known cases was published by the author [1]. It included the findings of HILL [2] and SYDLER [4]. Since then, a supplementary list of five new tetrahedra was published by LENHARD [5]. The list is not complete. Two more cases, described in the following, were discovered in connection with another problem.

A classic problem in polyhedra is the determination of those polyhedral shapes which can fill space by replication of a single shape. As a special case, SOMMERVILLE derived the space-filling tetrahedra [6]. Another modification is the allowance of the use of isometric tetrahedra; that is, those tetrahedra which are mirror images of each other. These were described recently by DAVIES [7].

The following definitions and symbols will be used in the preparation of a tabulation of the solutions.

DEFINITION: A space-filling tetrahedron (designated by SFT) is one which, together with other congruent tetrahedra, can fill space without overlapping.

DEFINITION: A space-filling twin-tetrahedron (designated by SFTT) is one, which together with congruent tetrahedra and their mirror images, can fill space without overlapping.

Table 1. Tabulation of space-filling tetrahedra

No.	Description	Type	Illustrated by	
			SOMMERVILLE [6]	DAVIES [7]
1	HILL, first type	SFTT		Top. p. 51
2	HILL, first type, $\alpha = \pi/3$	SFT	Figure 7	Top. p. 51
3	HILL, second type, $\alpha = \pi/4$	SFT	Figure 9	Bot. p. 51
4	HILL, special	SFT	Figure 8	Bot. p. 51
5	SOMMERVILLE	SFT	Figure 10	Bot. p. 51
6	DAVIES (1/2 of SOMMERVILLE)	SFTT		Bot. p. 51

Replications of the foregoing space-filling tetrahedra, Nos. 2 to 5, can be assembled to form a parallelepiped. Therefore, each such tetrahedron is also dissectible to form a cube, as was shown by SYDLER [3, pp. 269–270]. Another exposition of this proof is given by BOLTYANSKII [11, pp. 60–63]. The tetrahedron, designated as No. 6, can be cut into twelve pieces to form its isomer, as was shown by BRICARD [8]. Hence, two tetrahedra of this type can be made into one of No. 5. Therefore, all of the tetrahedra in Table 1 are dissectible to form cubes. However, Numbers 5 and 6, described by SOMMERVILLE and DAVIES, have been overlooked in the previous tabulations of dissectible tetrahedra.

Table 2

Edge	HILL, first type		HILL, second type		HILL, third type		HILL, special	
	Length	Dihedral angle	Length	Dihedral angle	Length	Dihedral angle	Length	Angle
AB	$\sin\alpha$	α	$2\sin\alpha$	α	$2\sin\alpha$	α	$3^{1/2}$	$\pi/3$
AC	$3^{1/2}\cos\alpha$	$\pi/3$	$3^{1/2}\cos\alpha$	$\pi/3$	$12^{1/2}\cos\alpha$	$\pi/6$	$2^{1/2}$	$\pi/2$
AD	1	$\pi/2$	2	$\pi/2$	$\sqrt{2+\sin^2\alpha}$	$\pi-\cos^{-1}(3^{-1/2}\cos\alpha)$	2	$\pi/4$
BC	1	$\pi/2$	$\sqrt{5\sin^2\alpha-1}$	$\pi-\cos^{-1}[(\cot\alpha)/2]$	2	$\pi/2$	1	$\pi/2$
BD	$\sin\alpha$	$\pi-2\alpha$	$2\sin\alpha$	$\pi/2-\alpha$	$\sin\alpha$	$\pi-2\alpha$	$3^{1/2}$	$\pi/3$
CD	$\sin\alpha$	α	$\sqrt{5\sin^2\alpha-1}$	$\cos^{-1}[(\cot\alpha)/2]$	$\sqrt{2+\sin^2\alpha}$	$\cos^{-1}(3^{-1/2}\cos\alpha)$	$2^{1/2}$	$\pi/2$

Edge	SYDLER, T_1		SYDLER, T_2		SYDLER, T_3		SYDLER, T_4		GOLDBERG, T_5		GOLDBERG, T_6	
	Length	Angle	Length	Angle	Length	Angle	Length	Angle	Length	Angle	Length	Angle
AB	τ	$\pi/2$	$3^{1/2}$	$2\pi/3$	$5^{1/4}\tau^{1/2}$	$\pi/5$	$5^{1/4}\tau^{1/2}$	$\pi/5$	$5^{1/4}\tau^{1/2}$	$2\pi/5$	$5^{1/4}\tau^{-1/2}$	$4\pi/5$
AC	τ^{-1}	$\pi/2$	$5^{1/4}\tau^{1/2}$	$\pi/5$	$3^{1/2}$	$\pi/3$	$3^{1/2}$	$\pi/3$	$3^{1/2}$	$\pi/3$	$5^{1/4}\tau^{1/2}$	$\pi/5$
AD	1	$\pi/2$	$5^{1/4}\tau^{1/2}$	$\pi/5$	2	$\pi/2$	$2\tau^{-1}$	$\pi/2$	$3^{1/2}$	$\pi/3$	$5^{1/4}\tau^{1/2}$	$\pi/5$
BC	$3^{1/2}$	$\pi/3$	$5^{1/4}\tau^{-1/2}$	$2\pi/5$	$5^{1/4}\tau^{-1/2}$	$3\pi/5$	$5^{1/4}\tau^{-1/2}$	$\pi/5$	$5^{1/4}\tau^{-1/2}$	$2\pi/5$	$3^{1/2}$	$\pi/3$
BD	$5^{1/4}\tau^{1/2}$	$\pi/5$	$5^{1/4}\tau^{-1/2}$	$2\pi/5$	$3^{1/2}\tau^{-1}$	$\pi/3$	$3^{1/2}\tau^{-1}$	$2\pi/3$	$5^{1/4}\tau^{-1/2}$	$2\pi/5$	$3^{1/2}$	$\pi/3$
CD	$5^{1/4}\tau^{-1/2}$	$2\pi/5$	2	$\pi/2$	$5^{1/4}\tau^{-3/2}$	$2\pi/5$	$5^{1/4}\tau^{-3/2}$	$3\pi/5$	$2\tau^{-1}$	$\pi/2$	2τ	$\pi/2$

Edge	LENHARD, T_7		LENHARD, T_8		LENHARD, T_9		LENHARD, T_{10}		LENHARD, T_{11}		SOMMERVILLE-GOLDBERG, T_{12}		DAVIES-GOLDBERG, T_{13}	
	Length	Angle	Length	Angle	Length	Angle	Length	Angle	Length	Angle	Length	Angle	Length	Angle
AB	$5^{1/4}\tau^{1/2}$	$3\pi/5$	$5^{1/4}\tau^{1/2}$	$3\pi/5$	$5^{1/4}\tau^{1/2}$	$3\pi/5$	$5^{1/4}\tau^{1/2}$	$3\pi/10$	$5^{1/4}\tau^{1/2}$	$3\pi/10$	$3^{1/2}$	$\pi/6$	$3^{1/2}$	$\pi/6$
AC	$3^{1/2}$	$\pi/3$	$3^{1/2}$	$\pi/6$	$3^{1/2}$	$\pi/3$	$3^{1/2}$	$\pi/3$	$3^{1/2}$	$\pi/6$	$3^{1/2}$	$\pi/6$	$2^{1/2}$	$\pi/2$
AD	$5^{1/4}\tau^{-1/2}$	$\pi/5$	$7^{1/2}/2$	α_1	$5^{1/4}\tau^{-1/2}$	$\pi/5$	$(1+3\tau^{-1})^{1/2}/2$	α_3	1	α_4	$5^{1/2}/2$	$\pi-\alpha_7$	$5^{1/2}/2$	$(\pi-\alpha_7)/2$
BC	$5^{1/4}\tau^{-1/2}$	$\pi/5$	$5^{1/4}\tau^{-1/2}$	$\pi/5$	$5^{1/4}\tau^{-1/2}$	$\pi/10$	$5^{1/4}\tau^{-1/2}$	$\pi/5$	$5^{1/4}\tau^{-1/2}$	$\pi/10$	2	$\pi/4$	1	$\pi/4$
BD	$3^{1/2}$	$\pi/3$	$3^{1/2}/2$	$\pi/3$	$(1+3\tau)^{1/2}/2$	α_2	$(1+3\tau^{-1})^{1/2}/2$	$\pi-\alpha_3$	1	α_5	$5^{1/2}/2$	$(\pi+\alpha_7)/2$	$5^{1/2}/2$	$(\pi-\alpha_7)/2$
CD	$5^{1/4}\tau^{1/2}$	$3\pi/5$	$7^{1/2}/2$	$\pi-\alpha_1$	$(1+3\tau)^{1/2}/2$	$\pi-\alpha_2$	$5^{1/4}\tau^{1/2}$	$3\pi/5$	1	α_6	$5^{1/2}/2$	$(\pi+\alpha_7)/2$	$1/2$	$\pi/2$

The list of all the tetrahedra which can be dissected to form cubes, known at the present time, is given in Table 2. It is not known whether this list is exhaustive. BRICARD [8] and DEHN [9] have shown that a necessary condition on the dihedral angles A, B, C, D, E, F, now known as one of Dehn's conditions, requires that

$$m_1 A + m_2 B + m_3 C + m_4 D + m_5 E + m_6 F = k \pi ,$$

where the m_i and k integers. SYDLER [10] has shown that this condition, plus another DEHN condition on the lengths of the edges, is sufficient. However, it should be noted that, in Table 2, at least two of the dihedral angles of each tetrahedron are rational fractions of π. Furthermore, in all of these cases, $m \Sigma A = k \pi$. This is a stronger condition than the DEHN condition on the angles in which the coefficients may be different integers. It is an open question whether this stronger condition is necessary.

An excellent summary and exposition of the earlier papers on the subject of the dissection of polygons and polyhedra to form other polygons and polyhedra is given by BOLTYANSKII [11].

In the table, $\tau = (\sqrt{5} + 1)/2$, α is a free variable,

$\alpha_1 \approx 50°$, $\tan\alpha_1 = \sqrt{7/5}$; $\cos 2\alpha_1 = -1/6$;

$\alpha_2 \approx 65°$, $\tan\alpha_2 = \sqrt{9 - 2\sqrt{5}}$; $\cos 4\alpha_2 = -3(\sqrt{5} - 1)/20 = -3\tau^{-1}/10$;

$\alpha_3 \approx 75°$, $\tan\alpha_3 = \sqrt{9 + 2\sqrt{5}}$; $\cos 4\alpha_3 = 3(\sqrt{5} + 1)/20 = 3\tau/10$;

$\alpha_4 \approx 101°$, $\tan\alpha_4 = -3 - \sqrt{5} = -2\tau^2$;

$\alpha_5 \approx 117°$, $\tan\alpha_5 = -2$;

$\alpha_6 \approx 143°$, $\tan\alpha_6 = -3 + \sqrt{5} = -2\tau^{-2}$; $\alpha_4 + \alpha_5 + \alpha_6 = 360°$;

$\alpha_7 \approx 48°$, $\cos\alpha_7 = 2/3$.

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k-Tuples of the First n Natural Numbers

In connection with the problems of infinite sets investigated by ERDÖS and HAJNAL in [1], ERDÖS proposed a problem of the k-tuples of a finite set. This problem in its most general form can be formulated as follows. Take a system of k-tuples of the first n natural numbers. Suppose that the set of the i_1-st, i_2-nd, ..., i_l-th ($l \leqslant k-1$) numbers of an arbitrary k-tuple of the system *does not* coincide with the set of the j_1-st, j_2-nd, ..., j_l-th numbers of another k-tuple of the system. At most how many k-tuples can the system contain? In this paper we solve the problem in a special case.

Denote by $f(n, k)$ the maximal number of k-tuples which can be chosen from the first n numbers ($1 \leqslant k \leqslant n$) such that the last $k-1$ numbers of a k-tuple do not coincide with the first $k-1$ numbers of another selected k-tuple.

It is trivial that $f(n, 1) = n$.

It will be proved that if $k \geqslant 2$ then

$$f(n, k) = \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \binom{n-1-2m}{k-1}.$$

The proof uses induction on n. If $n = k$, $f(k, k) = 1$. If $n = k+1$, only the k-tuple of the first and the k-tuple of the last k numbers cannot be selected at the same time, consequently $f(k+1, k) = k$. Thus the statement holds for $n = k, k+1$, and it may be supposed that $n \geqslant k+2$ and it holds for the numbers less than n.

Put $N = \{1, 2, \ldots, n\}$ and $M = \{2, 3, \ldots, n-1\}$. Consider a maximal system $\mathfrak{N}$ of k-tuples of N, satisfying the condition. Denote by $\varkappa$ a $(k-2)$-tuple. $\varkappa \subset N$ and $\varkappa \subset M$ will mean that the elements of $\varkappa$ are in N and M, respectively. Divide the $(k-1)$-tuples of N into two classes: $\mathfrak{A}$ and $\mathfrak{B}$. If a $(k-1)$-tuple consists of the first $k-1$ elements of a k-tuple of $\mathfrak{N}$, then put the $(k-1)$-tuple into $\mathfrak{A}$, otherwise into $\mathfrak{B}$. Denote by $\alpha(\varkappa)$ the number of $(k-1)$-tuples of $\mathfrak{A}$, the last $k-2$ elements of which form $\varkappa$, and similarly $\beta(\varkappa)$ denotes the number of $(k-1)$-tuples of $\mathfrak{B}$, the first $k-2$ elements of which form $\varkappa$. Because of the maximality of $\mathfrak{N}$ if $\varkappa$ consists of the last $k-2$ elements of a $(k-1)$-tuple of $\mathfrak{A}$ and it is at the same time the set of the first $k-2$ elements of a $(k-1)$-tuple of $\mathfrak{B}$, then $\mathfrak{N}$ contains the k-tuple which is the union of these two $(k-1)$-tuples. It is obvious that this is a unique representation of the elements of $\mathfrak{N}$ as certain unions of elements of $\mathfrak{A}$ and $\mathfrak{B}$. Consequently

$$f(n, k) = \sum_{\varkappa \subset N} \alpha(\varkappa)\, \beta(\varkappa)\,.$$

If $\varkappa$ contains 1, $\alpha(\varkappa) = 0$ and if $\varkappa$ contains n, $\beta(\varkappa) = 0$. Thus it can be supposed that $\varkappa \subset M$ in the sum above. On the other hand if $\varkappa \subset M$, the $(k-1)$-tuple consisting of 1 and of the elements of $\varkappa$ must belong to $\mathfrak{A}$. So if $\alpha'(\varkappa)$ denotes the number of $(k-1)$-tuples of $\mathfrak{A}$ in M, having $\varkappa$ as the 1st $k-2$ elements, then $\alpha(\varkappa) = \alpha'(\varkappa) + 1$. The number $\beta'(\varkappa)$ is defined similarly and $\beta(\varkappa) = \beta'(\varkappa) + 1$. But then

$$f(n, k) = \sum_{\varkappa \subset M} \alpha(\varkappa)\, \beta(\varkappa) = \sum_{\varkappa \subset M} (\alpha'(\varkappa) + 1)\, (\beta'(\varkappa) + 1)$$

$$= \sum_{\varkappa \subset M} \alpha'(\varkappa)\, \beta'(\varkappa) + \sum_{\varkappa \subset M} (\alpha'(\varkappa) + \beta'(\varkappa)) + \sum_{\varkappa \subset M} 1 \leqslant f(n-2, k) + \binom{n-2}{k-1}$$

$$+ \binom{n-2}{k-2} = f(n-2, k) + \binom{n-1}{k-1}.$$

Using the inductional hypothesis this gives

$$f(n, k) \leqslant \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \binom{n-1-2m}{k-1}.$$

To prove the converse inequality take the following system $\mathfrak{N}^*$ of k-tuples of N. If f, l and m are the first, last and an arbitrary middle element of a k-tuple, respectively, then $\mathfrak{N}^*$ contains this k-tuple if and only if $n - l < m \leqslant n - f$. Following the previous proof it is immediate by induction that $\mathfrak{N}^*$ has exactly $\sum_{m=0}^{\left[\frac{n-k}{2}\right]} \binom{n-1-2m}{k-1}$ elements. This proves the required result.

Now consider systems of k-tuples of N satisfying the following condition: there are no two k-tuples in a system, say $\varkappa_1$ and $\varkappa_2$, such that the last element of $\varkappa_1$ is a middle element of $\varkappa_2$ and the first element of $\varkappa_2$ is a middle element of $\varkappa_1$. Denote by $g(n, k)$ the maximal number of elements of these systems.

As this condition is much stronger then the previous one, $f(n, k) \geqslant g(n, k)$ must hold. It is fairly surprising that *equality holds* in this inequality:

$$f(n, k) = g(n, k) .$$

This follows simply from the construction given above, since $\mathfrak{N}^*$ satisfies the stronger condition too. For suppose that f_1, f_2, l_1, l_2 are the first and last elements of two k-tuples of $\mathfrak{N}^*$ for which f_2 is between f_1 and l_1, furthermore l_1 is between f_2 and l_2. But then $n - l_1 < f_2$, $l_1 \leqslant n - f_2$, consequently $n - f_2 < l_1 \leqslant n - f_2$, so this shows there are no such two k-tuples, i.e. $\mathfrak{N}^*$ satisfies the stronger condition.

It is likely that the complete solution of the original problem is very difficult and it needs an entirely different method.

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Sur l'équation diophantienne $(x^2-1)^2 + (y^2-1)^2 = (z^2-1)^2$

W. SIERPIŃSKI a exposé dans [1], p. 55, les quelques résultats connus concernant l'équation:

$$(x^2 - 1)^2 + (y^2 - 1)^2 = (z^2 - 1)^2 \qquad x, y, z \text{ entiers} > 1 , \tag{1}$$

et noté que, pour x, y, z impairs, $x = 2a + 1$, $y = 2b + 1$, $z = 2c + 1$, cette équation pouvait s'écrire:

$$t_a^2 + t_b^2 = t_c^2 \quad a, b, c \text{ entiers} > 0 , \quad t_u = \frac{u(u+1)}{2} . \tag{2}$$

Dans une précédente note [2], nous avons démontré l'impossibilité de (2) si deux au moins des trois nombres a, b, c sont consécutifs.

Supposant $x < y < z$, ce qui est loisible puisque $x \neq y$, nous prouvons cette fois que:

(A) (1) est impossible avec $y - x = 1$.

(B) $(x, y, z) = (10, 13, 14)$ est la seule solution de (1) avec $z - y = 1$.

(C) (1) est impossible avec $y - x$ ou $z - y$ ou $z - x = 2$.

(A) Nous allons montrer que l'équation:

$$(x^2 - 1)^2 + (y^2 - 1)^2 = u^2 \qquad x, y, u \text{ entiers} > 1 , \tag{3}$$

n'a, pour $y - x = 1$, que la solution $(x, y, u) = (3, 4, 17)$. Le résultat s'ensuivra immédiatement puisque $z^2 - 1 \neq 17$.

Avec $y = x + 1$ donc u impair, $u = 2v + 1$, (3) devient $t_v = t_x^2$. Or W. LJUNGGREN [3] puis J. W. S. CASSELS [4] ont montré que t_0, t_1, t_8 $(= t_3^2)$ sont les seuls nombres triangulaires qui sont les carrés de nombres triangulaires. Donc $x = 3$, $v = 8$, $u = 17$.

(B) Si $z = y + 1$, l'équation:

$$u^2 + (y^2 - 1)^2 = (z^2 - 1)^2 \qquad u, y, z \text{ entiers} > 1 , \tag{4}$$

peut s'écrire $(2y + 1)(2y^2 + 2y - 1) = u^2$. Posons $P = 2y + 1$, $Q = 2y^2 + 2y - 1$. On a $PQ = u^2$ avec P, Q entiers > 0; d'où $P = dP_1^2$, $Q = dQ_1^2$ avec $d = (P, Q)$ et P_1, Q_1 entiers > 0. Comme $P^2 - 2Q = 3$ on obtient finalement:

$$d^2 P_1^4 - 2 d Q_1^2 = 3 \qquad \text{avec} \quad d = 1 \text{ ou } 3 .$$

Comme d, P_1, Q_1 sont impairs on a $1 - 2d \equiv 3 \pmod{8}$ donc $d = 3$ et $3 P_1^4 - 2 Q_1^2 = 1$, équation qui vient d'être résolue par R. T. BUMBY [5] et qui donne $(P_1, Q_1) = (1, 1)$ ou $(3, 11)$. La première solution est à écarter car elle conduit à $y = 1$. La seconde fournit $(u, y, z) = (99, 13, 14)$, seule solution de (4) avec $z = y + 1$. En revenant à (1), on a $x^2 - 1 = 99$ soit $x = 10$.

(C) 1°) Si $z - x = 2$, alors $y - x = 1$ et (1) est impossible d'après A.

2°) Si $y - x = 2$, x et y ont même parité et z est impair. x et y ne peuvent être pairs car $(z^2 - 1)^2 \not\equiv 2 \pmod{4}$. Ainsi (1) a lieu avec $x, y = x + 2$, z impairs et implique (2) avec $b = a + 1$, ce qui est impossible d'après [2].

Remarquons qu'en revanche l'équation (3) possède une infinité de solutions avec $y = x + 2$ car alors, x étant impair pour la même raison que ci-dessus, $x = 2p - 1$, p entier > 1, elle s'écrit $32 p^2 (p^2 + 1) = u^2$, d'où on tire successivement $2(p^2 + 1) = v^2$, v pair > 2, $v = 2w$, soit finalement l'équation de PELL $p^2 - 2w^2 = -1$, de solution fondamentale $(p, w) = (1, 1)$.

3°) Si $z - y = 2$, les solutions de l'équation (4) sont $(u, y, z) = (8v^3, 2v^2 - 1, 2v^2 + 1)$, v entier > 1, car cette équation devient $u^2 = [2(y + 1)]^3$. Alors (1) a lieu avec $x, y, z = y + 2$ impairs et implique (2) avec $c = b + 1$, ce qui est impossible d'après [2].

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