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On the geometry of horospheres

Cicero Pedro Aquino and Henrique Fernandes de Lima

Abstract. The aim of this paper is to investigate Bernstein-type properties of horospheres of the hyperbolic space $\mathbb{H}^{n+1}$. Our approach is based on the use of appropriate generalized maximum principles in order to obtain new characterization results of such horospheres. Furthermore, by supposing a linear dependence between support functions naturally attached to a hypersurface, we also establish a classification theorem concerning horospheres and hyperbolic cylinders of $\mathbb{H}^{n+1}$.

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1. Introduction

In the theory of isometric immersions, the study of Bernstein-type properties concerning complete hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$ constitutes an important theme. In this research branch, do Carmo and Lawson [9] have used the well know Alexandrov's reflexion method to show that any complete hypersurface properly embedded with constant mean curvature in $\mathbb{H}^{n+1}$ with a single point at the asymptotic boundary is a horosphere. Moreover, they also observed that the statement is no longer true if we replace embedded by immersed. Later on, Alías and Dajczer [2] have proved that a surface properly immersed in $\mathbb{H}^3$ with constant mean curvature $-1 \leq H \leq 1$ and contained in a *slab* (that is, the region between two horospheres that share the same point in the asymptotic boundary) must be, in fact, a horosphere.

In [8], the second author and Caminha have studied complete vertical graphs of constant mean curvature in $\mathbb{H}^{n+1}$. Under appropriate restrictions on the values of the mean curvature and the growth of the height function, they used a generalized maximum principle due to Akutagawa [1] to establish necessary conditions for the existence of such a graph. Moreover, in $\mathbb{H}^3$, they proved that such a graph must be a horosphere. In [7], by extending a technique of Yau [21], the second author jointly with Camargo and Caminha obtained rigidity results concerning to the horospheres of $\mathbb{H}^{n+1}$, without the assumption of the constancy of the mean curvature. Proceeding, they also treated the case of the higher order mean curvatures. More recently, the authors [3] generalized the results of [8] to the context of warped products obeying an appropriate convergence condition. Moreover, in [5], they obtained characterizations theorems of the totally umbilical hypersurfaces of $\mathbb{H}^{n+1}$ under natural restrictions on their Lorentz Gauss mapping.

Here, motivated by these works described above, we treat the following question: *under what reasonable geometric restrictions must a complete hypersurface immersed in the hyperbolic space be a horosphere*?

In order to obtain satisfactory answers for such question, in Section 3 of this paper we apply some appropriate generalized maximum principles which enable us to establish suitable rigidity theorems related to the horospheres of $\mathbb{H}^{n+1}$. In our approach, an important point is the understanding of the geometry of support functions naturally attached to a hypersurface of $\mathbb{H}^{n+1}$, as well as, the study of the behavior of the corresponding Lorentz Gauss mapping.

Finally, in Section 4, we characterize horospheres and hyperbolic cylinders as the only complete hypersurfaces with constant mean curvature of $\mathbb{H}^{n+1}$ whose support functions determined by a nonzero null vector are linearly related. We point out that such characterization result deals with the case that was not contemplated in Theorem 4.1 of [5].

2. Preliminaries

In order to obtain our first results, it will be convenient to consider the hyperbolic space as a hyperquadric of the Minkowski space $\mathbb{L}^{n+2}$. So, we will represent by $\mathbb{L}^{n+2}$ the vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2}.$$

and the hyperbolic space will be identified with

$$\mathbb{H}^{n+1} = \{p \in \mathbb{L}^{n+2}\,;\, \langle p, p\rangle = -1,\ p_{n+2} \geq 1\}$$

equipped with the Riemannian induced metric from $\mathbb{L}^{n+2}$. In this setting horospheres, hyperspheres and spheres can be obtained intersecting $\mathbb{H}^{n+1}$ with affine hyperplanes of $\mathbb{L}^{n+2}$. For example, as it has been observed by López and Montiel in [13], any horosphere of $\mathbb{H}^{n+1}$ is given by

$$L_\tau = \{p \in \mathbb{H}^{n+1}\,;\, \langle p, a\rangle = \tau\}, \tag{2.1}$$

where $a \in \mathbb{L}^{n+2}$ is a nonzero null vector, that is, $\langle a, a\rangle = 0$, and τ is a positive number. When one fixes that vector a and moves $\tau \in \mathbb{R}^+$ one obtains a foliation of

$\mathbb{H}^{n+1}$ by means of horospheres having the same point at the infinity. It is easy to see that

$$\xi(p) = -p - \frac{1}{\tau}a \tag{2.2}$$

is a unit normal field on L_τ with respect to which the horosphere has constant mean curvature 1.

Now, let $\psi\colon \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be an orientable hypersurface immersed into the hyperbolic space. We will denote by A the shape operator of Σ^n with respect to a globally defined unit normal vector field N. In order to set up the notation, let us represent by ∇^0,$\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{L}^{n+2}$, $\mathbb{H}^{n+1}$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for Σ^n in $\mathbb{H}^{n+1}$ are given, respectively, by

$$\nabla^0{}_X Y = \nabla_X Y + \langle AX, Y\rangle N + \langle X, Y\rangle \psi$$

and

$$AX = -\overline{\nabla}_X N = -\nabla^0{}_X N,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

By fixing an arbitrary vector $a \in \mathbb{L}^{n+2}$, we will consider two support functions, $f_a = \langle N, a\rangle$ and $l_a = \langle \psi, a\rangle$, naturally attached to the immersion $\psi\colon \Sigma^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$. It is immediate to verify that

$$\nabla l_a = a^\top \quad \text{and} \quad \nabla f_a = -A(a^\top),$$

where $a^\top \in \mathfrak{X}(\Sigma)$ denotes the tangential component of a along the immersion ψ, that is,

$$a^\top = a - f_a N + l_a \psi. \tag{2.3}$$

For $0 \le r \le n$ and $p \in \Sigma^n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p; in this way one gets n smooth functions $S_r\colon \Sigma^n \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by convention. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

Also, we define the r-th mean curvature H_r of Σ^n, $0 \le r \le n$, by $\binom{n}{r} H_r = S_r$. We observe that $H_0 = 1$, while H_1 is the usual mean curvature H of Σ^n. For

$0 \le r \le n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = S_r I - AP_{r-1}.$$

On the other hand, given $f \in C^\infty(\Sigma)$, for each $0 \le r \le n$, the second order differential operator L_r is defined as follows:

$$L_r f = \mathrm{tr}(P_r \mathrm{Hess} f).$$

When $r = 0$, L_0 is nothing but the Laplacian operator Δ. Moreover, for a smooth function $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $f \in C^\infty(\Sigma)$, it follows from the properties of the Hessian that

$$L_r(\varphi \circ f) = \varphi'(f) L_r(f) + \varphi''(f)\langle P_r \nabla f, \nabla f\rangle.$$

Based on the ideas of Reilly [18], Rosenberg in [19] showed the following

Lemma 2.1. *Let $x\colon \Sigma^n \to \mathbb{H}^{n+1}$ be a orientable hypersurface immersed in the hyperbolic space $\mathbb{H}^{n+1}$. Then, for the support functions f_a and l_a previously defined, we have*

$$L_r l_a = (r+1) S_{r+1} f_a + (n-r) S_r l_a \tag{2.4}$$

and

$$L_r f_a = -\big(S_1 S_{r+1} - (r+2) S_{r+2}\big) f_a - (r+1) S_{r+1} l_a - \langle \nabla S_{r+1}, a^\top\rangle. \tag{2.5}$$

In order to obtain some of our results, we will also need the well known generalized maximum principle of Omori–Yau [17], [20].

Lemma 2.2. *Let Σ^n denote an n-dimensional complete Riemannian manifold having Ricci curvature bounded from below. Then, for any C^2 function $u\colon \Sigma^n \to \mathbb{R}$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence of points $\{p_k\}_{k\ge 1}$ in Σ^n satisfying the following properties:*

(i) $u(p_k) > u^* - \frac{1}{k}$,

(ii) $|\nabla u|(p_k) < \frac{1}{k}$, *and*

(iii) $\Delta u(p_k) < \frac{1}{k}$,

for all $k \ge 1$.

3. Uniqueness results in the hyperbolic space

Along this work, we will always suppose that all considered hypersurfaces are orientable and connect. The next lemma plays an essential role along this work.

Lemma 3.1. *Let $\psi \colon \Sigma^n \to \mathbb{H}^{n+1}$ be a hypersurface immersed in $\mathbb{H}^{n+1}$ with mean curvature $-1 \leq H \leq 1$. Then the function l_a^2 is subharmonic, for all nonzero null vector $a \in \mathbb{L}^{n+2}$.*

Proof. From Lemma 2.1 we obtain that

$$\Delta l_a^2 = 2nHf_a l_a + 2nl_a^2 + 2|\nabla l_a|^2.$$

Now, by a direct computation, it is easy to verify that

$$\Delta l_a^2 = n\big(f_a + Hl_a\big)^2 + n\big(l_a^2 - f_a^2\big) + n\big(1 - H^2\big)l_a^2 + 2|\nabla l_a|^2. \tag{3.1}$$

Finally, using that a is a nonzero null vector, we have from (2.3) that

$$l_a^2 - f_a^2 = |\nabla l_a|^2.$$

Thus, from (3.1), we get

$$\Delta l_a^2 = n\big(f_a + Hl_a\big)^2 + n\big(1 - H^2\big)l_a^2 + (n+2)\,|\nabla l_a|^2. \tag{3.2}$$

Therefore, from our restriction on H, we conclude that l_a^2 is a subharmonic function on Σ^n. □

We recall that the Gauss mapping N of a hypersurface Σ^n of $\mathbb{H}^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ can be regarded as a map $N \colon \Sigma^n \to \mathbb{S}_1^{n+1}$, where $\mathbb{S}_1^{n+1}$ denotes the $(n+1)$-dimensional unitary de Sitter space, that is,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = 1\}.$$

In this setting, N is called the *Lorentz Gauss mapping* of Σ^n. In a dual context, given a spacelike hypersurface Σ^n of $\mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ (that is, a hypersurface of $\mathbb{S}_1^{n+1}$ whose induced metric is a Riemannian metric), its Gauss mapping N can be thought of as a map $N \colon \Sigma^n \to \mathbb{H}^{n+1}$; so, N is said the *hyperbolic Gauss mapping* of Σ^n.

In [16], Montiel have proved that if a complete spacelike hypersurface Σ^n in the de Sitter space $\mathbb{S}_1^{n+1}$ with constant mean curvature $H \geq 1$ is such that the image of its hyperbolic Gauss mapping is contained in the closure of the interior domain enclosed by a horosphere, then its mean curvature is, in fact, equal to 1. When $n = 2$, this implies that Σ^2 is also an umbilical surface and the image of its hyperbolic Gauss mapping is exactly a horosphere. On the other hand, from (2.2) we have that the image of the Lorentz Gauss mapping of the horospheres of $\mathbb{H}^{n+1}$ are the following hypersurfaces of $\mathbb{S}_1^{n+1}$:

$$\mathscr{L}_\tau = \{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = \tau\},$$

for some nonzero null vector $a \in \mathbb{L}^{n+2}$, which are totally umbilical hypersurfaces of $\mathbb{S}_1^{n+1}$, isometric to the Euclidean space $\mathbb{R}^n$ and with mean curvature $H^2 = 1$ (cf. [14]). In this sense, we will call $\mathscr{L}_\tau$ a hyperplane of $\mathbb{S}_1^{n+1}$.

Motivated by this previous discussion, we will state our first result. In what follows, we say that a hypersurface Σ^n is *under a horosphere* L_τ of $\mathbb{H}^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ determinated by a nonzero null vector $a \in \mathbb{L}^{n+2}$, when its corresponding support function l_a satisfies $l_a \le \tau$. Moreover, in this same context, we say that the image of the Lorentz Gauss mapping of Σ^n is *under a plane* $\mathcal{L}_{\tilde{\tau}}$ of $\mathbb{S}^{n+1}_1 \hookrightarrow \mathbb{L}^{n+2}$, when its corresponding support function f_a satisfies $f_a \le \tilde{\tau}$.

Theorem 3.2. *Let $\psi\colon \Sigma^2 \to \mathbb{H}^3$ be a complete surface with non-negative Gaussian curvature and under a horosphere L_τ of $\mathbb{H}^3$, with (not necessarily constant) mean curvature H satisfying $-1 \le H \le 1$. If the image of its Lorentz Gauss mapping is under a plane $\mathcal{L}_{|\tau|-\varepsilon}$ of $\mathbb{S}^3_1$, for some $\varepsilon \in (0, |\tau|)$, and contained in the closure of the interior domain enclosed by a plane $\mathcal{L}_\beta$, for some $\beta > 0$, then Σ^2 is a horosphere.*

Proof. Consider $a \in \mathbb{L}^{n+2}$ the nonzero null vector that determines the horosphere $L_{|\tau|}$ in $\mathbb{H}^3$ and the planes $\mathcal{L}_\beta$ and $\mathcal{L}_{|\tau|-\varepsilon}$ in $\mathbb{S}^3_1$. The hypothesis on the Lorentz Gauss mapping of Σ^2 assure us that $\beta \le f_a \le |\tau| - \varepsilon$. Using equation (2.3), we also have

$$|\nabla l_a|^2 + f_a^2 = l_a^2, \tag{3.3}$$

from which we conclude that $l_a^2 \ge \beta^2$ and thus either $l_a \ge \beta$ or $l_a \le -\beta$ on Σ^2.

By the hypothesis on the mean curvature H, we have from Lemma 3.1 that l_a^2 is a subharmonic function. Now, suppose that $l_a \ge \beta$ on Σ^2. Since Σ^2 is under a horosphere L_τ, we obtain that $\beta \le l_a \le \tau$ and thus l_a^2 is a bounded subharmonic function. However, a classical result due to A. Huber [11] assures that complete surfaces of nonnegative Gaussian curvature must be parabolic. Therefore, l_a is constant on Σ^2, that is, Σ^2 is a horosphere of $\mathbb{H}^3$. Let us consider the case that $l_a \le -\beta$, and suppose that $\tau > 0$. Thus, $l_a < 0 < \tau$ and, since Σ^2 is under the horosphere L_τ, we must have by continuity that either $l_a^2 < \tau^2$ or $l_a^2 > \tau^2$ on Σ^2. Suppose that $l_a^2 > \tau^2$. Since l_a is bounded from above, we have from Omori–Yau maximum principle (cf. Lemma 2.2) that there exists a sequence of points $\{p_k\}_{k\ge1}$ in Σ^2 such that $\lim l_a(p_k) = \sup l_a$ and $|\nabla l_a|(p_k) < 1/k$, for all $k \ge 1$. Therefore, from equation (3.3) we conclude that

$$\lim_{k\to\infty} f_a^2(p_k) \ge \tau^2,$$

and this gives us a contradiction, since $f_a^2 \le (|\tau| - \varepsilon)^2 < \tau^2$ on Σ^2. The previous argument guarantees that $l_a^2 < \tau^2$ and from this we can apply once more the result of A. Huber to conclude that Σ^2 is a horosphere of $\mathbb{H}^3$. Finally, if $\tau < 0$, since by hypothesis $l_a \le \tau < 0$, we get that $l_a^2 > \tau^2$ and, hence, we can reason as before to assure again that Σ^2 is a horosphere of $\mathbb{H}^3$. □

Now, we apply once more Huber's result [11], which was quoted along the proof of Theorem 3.2, to obtain the following

Theorem 3.3. *Let $\psi : \Sigma^2 \to \mathbb{H}^3$ be a complete surface immersed in a slab of $\mathbb{H}^3$ with nonnegative Gaussian curvature. If the (not necessarily constant) mean curvature H of Σ^2 satisfies $-1 \leq H \leq 1$, then Σ^2 is a horosphere.*

Proof. Let us assume that the Σ^2 is contained in a slab of $\mathbb{H}^3$. From this, we have that there exists a nonzero null vector $a \in \mathbb{L}^3$ and positive constants τ_1 and τ_2 such that the support function $l_a = \langle \psi, a \rangle$ satisfies $\tau_1 \leq l_a(p) \leq \tau_2$, for all $p \in \Sigma^2$. Now, from the Lemma 3.1 we have that l_a^2 is a bounded subharmonic function on Σ^2. Thus, we are in position to use again Huber's result [11] to conclude that l_a is a constant function on Σ^2, that is, Σ^2 is a horosphere of $\mathbb{H}^3$. □

In the paper [21], Yau obtained the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold Σ^n: *if $\omega \in \Omega^{n-1}(\Sigma)$ is an $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that $B_i \subset B_{i+1}$, $\Sigma^n = \bigcup_{i \geq 1} B_i$ and*

$$\lim_{i \to +\infty} \int_{B_i} d\omega = 0.$$

By applying this result to $\omega = \iota_{\nabla f}$, where $f : \Sigma^n \to \mathbb{R}$ is a smooth function, ∇f denotes its gradient and $\iota_{\nabla f}$ the contraction in the direction of ∇f, Yau established an extension of H. Hopf's theorem on a complete noncompact Riemannian manifold. In what follows, $\mathcal{L}^1(\Sigma)$ denotes the space of Lebesgue integrable functions on Σ^n.

Lemma 3.4 (Corollary on page 660 of [21]). *Let Σ^n be an n-dimensional, complete Riemannian manifold and let $f : \Sigma^n \to \mathbb{R}$ be a smooth function. If f is a subharmonic (or superharmonic) function with $|\nabla f| \in \mathcal{L}^1(\Sigma)$, then f must actually be harmonic.*

In [2], Alías and Dajczer studied complete surfaces properly immersed in a slab of $\mathbb{H}^3$. Using the warped structure of $\mathbb{H}^3$, they obtained a Bernstein-type result for the case of constant mean curvature $-1 \leq H \leq 1$ (cf. Theorem 1 of [2]). Now, with a new approach, we are able to give an extension of such result.

Theorem 3.5. *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface immersed in a slab of $\mathbb{H}^{n+1}$ determined by the nonzero null vector $a \in \mathbb{L}^{n+2}$ with (not necessarily constant) mean curvature $-1 \leq H \leq 1$. If $|a^\top| \in \mathcal{L}^1(\Sigma)$ then Σ^n is a horosphere.*

Proof. From Lemma 3.1 we conclude that l_a^2 is a subharmonic function on Σ^n. On the other hand, observing that $\nabla l_a = a^\top$ and $|\nabla l_a^2| = 2|l_a||\nabla l_a|$ is integrable on Σ^n, we have from Lemma 3.4 that l_a^2 is a harmonic function. Now, using the equation (3.2) we have that $|\nabla l_a|^2 = 0$ on Σ^n therefore l_a is constant and this shows that Σ^n is a horosphere of $\mathbb{H}^{n+1}$. □

The following lemma is known as the *tangency principle* in the hyperbolic space, which is a celebrated geometric consequence of the classical Hopf's maximum principle (cf. [9] for details and definitions; see also Theorem 3.1 of [6]).

Lemma 3.6. *Let Σ_1^n and Σ_2^n be complete hypersurfaces immersed in $\mathbb{H}^{n+1}$ with mean curvature H_1 and H_2, respectively. In a neighbourhood of a common tangent point, if we have that Σ_1^n lies above Σ_2^n and $H_1 \leq H_2$, then Σ_1^n and Σ_2^n must coincide on such neighbourhood.*

In order to establish our next result, it will be convenient to leave the hyperquadric model of $\mathbb{H}^{n+1}$ that we have utilized before and consider its half-space model, that is, $\mathbb{H}^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} > 0\}$ endowed with the complete metric $\langle\, , \rangle_{\mathbb{H}^{n+1}} = \frac{1}{x_{n+1}^2}(dx_1^2 + \cdots + dx_{n+1}^2)$. In this setting, given a hypersurface $\psi\colon \Sigma^n \to \mathbb{H}^{n+1}$, we define the *normal angle* θ of Σ^n as being the smooth function $\theta\colon \Sigma^n \to [0, \pi]$ given by

$$\cos\theta = \langle N, e_{n+1}\rangle_{\mathbb{H}^{n+1}}.$$

The following result is an extension of Theorem 5.2 of [8] and Theorem 3.3 of [12].

Theorem 3.7. *Let $\psi\colon \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface which lies under a horosphere of $\mathbb{H}^{n+1}$ and with (not necessary constant) mean curvature $-1 \leq H \leq 1$. If the normal angle θ of Σ^n satisfies $|\cos\theta| \geq \sup_\Sigma |H|$, then Σ^n is a horosphere.*

Proof. Let us consider a complete hypersurface Σ^n immersed with mean curvature $|H| \leq 1$ in $\mathbb{H}^{n+1}$, and such that it lies under a horosphere L. This means that Σ^n is included in the open component of the region $\mathbb{H}^{n+1} - L$ where the mean curvature vector of L points. Without lost of generality, we can consider L the hyperplane $\{x \in \mathbb{H}^{n+1}; x_{n+1} = 1\}$. Then, since we are supposing that Σ^n lies under L, we have that $\Sigma^n \subset \{x \in \mathbb{H}^{n+1}; x_{n+1} > 1\}$.

Now, let $H_0 = \sup_\Sigma |H|$. Suppose, by contradiction, that $H_0 < 1$ and consider the family of equidistant hypersurfaces with a given common axis of rotation, having constant mean curvature H_0 and such that their corresponding mean curvature vector is pointing up, coming from the infinity $\{x \in \mathbb{R}^{n+1}; x_{n+1} = 0\}$. By a rigid motion of this family, we arrive until the first contact point of Σ^n with one of such equidistant hypersurfaces, which occurs in some common interior point of both hypersurfaces. Consequently, from Lemma 3.6, we have that Σ^n must be one of these equidistant hypersurfaces. But equidistant hypersurfaces do not lie under a horosphere. So, we arrive at a contradiction and, hence, $H_0 = 1$. Therefore, we use the hypothesis $|\cos\theta| \geq H_0$ to conclude that $\cos\theta = \pm 1$ on Σ^n, that is, Σ^n is a horosphere of $\mathbb{H}^{n+1}$. □

Remark 3.8. As observed in Remark 5.5 of [4], our restriction on the normal angle θ of the hypersurface Σ^n in Theorem 3.7 is motivated by the gradient estimate (19) of [13].

4. Hypersurfaces in $\mathbb{H}^{n+1}$ satisfying $l_a = \lambda f_a$

Our purpose in this last section, is to classify the complete hypersurfaces of $\mathbb{H}^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ whose support functions l_a and f_a, with respect to some fixed nonzero null vector $a \in \mathbb{L}^{n+2}$, are linearly related. In this setting, from (2.2) we observe that the support functions of a horosphere L_τ of $\mathbb{H}^{n+1}$ satisfy $l_a = -f_a$, where a is the nonzero null vector which defines such horosphere. Furthermore, since the horospheres L_τ foliate all hyperbolic space $\mathbb{H}^{n+1}$, for any hypersurface Σ^n immersed in $\mathbb{H}^{n+1}$ we have from (2.1) that its support function l_a has strict sign. Proceeding, we get the following result.

Theorem 4.1. *Let $\psi\colon \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface immersed in a slab of $\mathbb{H}^{n+1}$ determined by a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that $l_a = \lambda f_a$, for some smooth function $\lambda\colon \Sigma^n \to \mathbb{R}$, and that the (not necessarily constant) mean curvature H of Σ^n satisfies $\frac{H}{\lambda} \geq -1$. Suppose that one of the following conditions is satisfied:*

(a) *$n = 2$ and the Gaussian curvature of Σ^2 is non-negative.*

(b) $|a^\top| \in \mathcal{L}^1(\Sigma)$.

Then, Σ^n is a horosphere.

Proof. Initially, from the causal character of a, we observe that the function λ has strict sign on Σ^n. From Lemma 2.1 and hypothesis on support functions of Σ^n, we have that

$$\Delta l_a^2 = 2n\Big(\frac{H}{\lambda} + 1\Big) l_a^2 + 2|\nabla l_a|^2. \tag{4.1}$$

Since Σ^n is contained in a slab of $\mathbb{H}^{n+1}$ determined by a, we have that l_a^2 is a bounded subharmonic function on Σ^n.

We observe that if Σ^2 has non-negative Gaussian curvature, by a result due to A. Huber [11], we have that l_a is constant.

Now, suppose that $|a^\top| \in \mathcal{L}^1(\Sigma)$, then ∇l_a^2 has integrable norm on Σ^n. Thus, from equation (4.1), we conclude, from Lemma 3.4, that l_a^2 is harmonic and therefore we have $|\nabla l_a|^2 = 0$, hence we conclude that l_a is constant.

To finish the proof, we note that from the definition of l_a, if $l_a \equiv \tau$ on a complete hypersurface Σ^n, then $\Sigma^n \subset L_\tau$. Therefore, by completeness, we must have $\Sigma^n = L_\tau$. □

Now, we consider an integer k satisfying $0 \leq k < n$. Let us define the smooth function $F\colon \mathbb{H}^{n+1} \to \mathbb{R}$ by

$$F(p) = p_1^2 + \cdots + p_{k+1}^2,$$

where $p = (p_1, \ldots, p_{n+2})$. For $\rho > 0$, let $\Sigma^n = F^{-1}(\rho^2)$. It is not difficult to see that Σ^n is a complete orientable constant mean curvature hypersurface immersed in $\mathbb{H}^{n+1}$. If $p = (p_1, \ldots, p_{n+2})$ is a point of Σ^n then, by considering the standard immersions $\mathbb{S}^k(\rho) \hookrightarrow \mathbb{R}^{k+1}$ and $\mathbb{H}^{n-k}(\sqrt{1+\rho^2}) \hookrightarrow \mathbb{L}^{n-k+1}$, we get

$$\Sigma^n = \mathbb{S}^k(\rho) \times \mathbb{H}^{n-k}(\sqrt{1+\rho^2}) \hookrightarrow \mathbb{H}^{n+1}.$$

Moreover, we have that

$$N(p) = -\frac{\overline{\nabla} F}{|\overline{\nabla} F|}(p) = -\frac{1}{\rho\sqrt{1+\rho^2}}(\nu(p) + \rho^2 p) \tag{4.2}$$

defines a Gauss mapping for Σ^n, where $\nu(p) = (p_1, \ldots, p_{k+1}, 0, \ldots, 0)$, and the Weingarten operator A of Σ^n with respect to N has the following principal curvatures:

$$\lambda_1 = \cdots = \lambda_k = \frac{\sqrt{1+\rho^2}}{\rho} \quad \text{and} \quad \lambda_{k+1} = \cdots = \lambda_n = \frac{\rho}{\sqrt{1+\rho^2}}.$$

Furthermore, from (4.2) we easily verify that

$$l_a = -\frac{\sqrt{1+\rho^2}}{\rho} f_a,$$

where f_a and l_a are the support functions of Σ^n with respect the nonzero null vector $a = (0, \ldots, 1, 1) \in \mathbb{L}^{n+2}$.

In [5], the authors have studied complete constant mean curvature hypersurfaces Σ^n immersed in $\mathbb{H}^{n+1}$ assuming that the support functions of Σ^n satisfy the linear dependence relation $l_a = \lambda f_a$, for some unitary timelike or spacelike vector $a \in \mathbb{L}^{n+2}$ and some real number λ, showing that the Σ^n is either a totally umbilical hypersurface or a hyperbolic cylinder.

Motivated by the previous discussion, now we are able to deal with the case that was not contemplated in Theorem 4.1 of [5]. More precisely, we have the following

Theorem 4.2. *Let $\psi\colon \Sigma^n \to \mathbb{H}^{n+1}$ be a complete hypersurface immersed in $\mathbb{H}^{n+1}$ with constant mean curvature H. If $l_a = \lambda f_a$, for some nonzero null vector $a \in \mathbb{L}^{n+2}$ and some constant $\lambda \in \mathbb{R}$, then Σ^n is either a horosphere or isometric to a hyperbolic cylinder $\mathbb{S}^k(\rho) \times \mathbb{H}^{n-k}(\sqrt{1+\rho^2})$.*

Proof. Suppose that a is a nonzero null vector in $\mathbb{L}^{n+2}$ such that $l_a = \lambda f_a$ for some real number λ. Then, using once more that the support function l_a has strict sign on Σ^n, we have $\lambda \neq 0$. Observe that $\Delta l_a = \lambda \Delta f_a$. Now, by Lemma 2.1 we conclude, from the previous equality, that

$$S_2 = \frac{1}{2}S_1^2 + \Big(\frac{\lambda}{2} + \frac{1}{2\lambda}\Big)S_1 + \frac{n}{2}.$$

on Σ^n. This equality shows that S_2 is also a constant function on Σ^n. Repeating the previous argument for the operator L_1 we have from formulas (2.4) and (2.5) that

$$2S_2 f_a + (n-1)l_a S_1 = -\lambda\big(S_1S_2 - 3S_3\big)f_a - 2\lambda S_2 l_a.$$

Now, using that $\lambda \neq 0$, we obtain from above equality, after a straightforward computation, that

$$S_3 = \frac{2}{3\lambda}S_2 + \frac{(n-1)}{3}S_1 + \frac{1}{3}S_1S_2 + \frac{2\lambda}{3}S_2.$$

on Σ^n. As before, we conclude that S_3 is constant on Σ^n. Iterating this argument we show that S_r is a constant function on Σ^n for all r and from this, by a elementary algebraic argument, we have that all the principal curvatures of Σ^n are constant. Therefore, taking into account the classification of isoparametric hypersurfaces of $\mathbb{H}^{n+1}$ due to É. Cartan [10], we conclude that Σ^n is either a totally umbilical hypersurface or isometric to a hyperbolic cylinder $\mathbb{S}^k(\rho) \times \mathbb{H}^{n-k}(\sqrt{1+\rho^2})$. In the case that Σ^n is a totally umbilical hypersurface, from the description of the foliations of $\mathbb{H}^{n+1}$ due to Montiel in Example 3 of Section 4 of [15] and taking into account the causal character of the vector a, we see that Σ^n must be a horosphere. □

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Cicero Pedro Aquino, Departamento de Matemática, Universidade Federal do Piauí, 64049-550 Teresina, Piauí, Brazil

E-mail: cicero.aquino@ufpi.edu.br

Henrique Fernandes de Lima, Departamento de Matemática e Estatística, Universidade Federal de Campina Grande, 58109-970 Campina Grande, Paraíba, Brazil

E-mail: henrique@dme.ufcg.edu.br