The knot Floer complex and the smooth concordance group

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Jennifer Hom

Abstract. We define a new smooth concordance homomorphism based on the knot Floer complex and an associated concordance invariant ε. As an application, we show that an infinite family of topologically slice knots are independent in the smooth concordance group.

1. Introduction

The set of isotopy classes of knots in S^3, under the operation of connected sum, forms a monoid. Two knots are *concordant* if they cobound a smooth, properly embedded cylinder in $S^3 \times [0, 1]$. The monoid of knots, modulo concordance, forms the *concordance group*, denoted $\mathcal{C}$. If we loosen the conditions and only require that the cylinder be locally flat, rather than smooth, we obtain the *topological concordance group*. Understanding the difference between these two groups sheds some light on the distinction between the smooth and topological categories.

Ozsváth and Szabó [OS04], and independently Rasmussen [Ras03], defined an invariant, knot Floer homology, associated to a knot in S^3. This invariant comes in many different flavors, the most robust being $CFK^\infty(K)$, a $\mathbb{Z}$-filtered chain complex over the ring $\mathbb{F}[U, U^{-1}]$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and U is a formal variable. There is a second filtration induced by $-(U\text{-exponent})$ allowing us to view $CFK^\infty(K)$ as a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex. The filtered chain homotopy type of this complex is an invariant of the knot K. The weaker invariant, $\widehat{CFK}(K)$, takes the form of a $\mathbb{Z}$-filtered chain complex over $\mathbb{F}$, and is obtained by taking the degree zero part of the associated graded object with respect to one of the filtrations.

Within the complex $\widehat{CFK}(K)$ lives a $\mathbb{Z}$-valued concordance invariant, $\tau(K)$, defined by Ozsváth and Szabó in [OS03b]. The total homology of $\widehat{CFK}(K)$ has rank one, and τ measures the minimum filtration level where this homology is supported. The invariant τ gives a surjective homomorphism from the smooth concordance group

$\mathcal{C}$ to the integers:

$$\tau\colon \mathcal{C} \to \mathbb{Z},$$

which gives a new proof of the Milnor conjecture [OS03b] and is strong enough to obstruct topologically slice knots from being smoothly slice (for example, [Liv04]).

Often, we would like to be able to show that a collection of n knots is linearly independent, that is, that they freely generate a subgroup of rank n in $\mathcal{C}$. One way to accomplish this is to define a concordance homomorphism whose domain has rank at least n, and to show that the image of this collection of knots has span equal to n. Thus, the $\mathbb{Z}$-valued concordance homomorphism τ is not sufficient for this type of result.

We turn to the more robust invariant $CFK^\infty(K)$. In [Hom14], we defined a $\{-1, 0, 1\}$-valued concordance invariant, $\varepsilon(K)$. The invariant ε is associated to the $\mathbb{Z} \oplus \mathbb{Z}$ filtered chain complex CFK^∞ in a manner similar to how τ is associated to the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}$; that is, we ask when certain natural maps vanish on homology. We will sometimes write $\varepsilon(CFK^\infty(K))$, rather than $\varepsilon(K)$, to emphasize that ε is an invariant associated to the knot Floer complex of K.

The goal of this paper is to use ε to define a new concordance homomorphism that is strong enough to detect linear independence in $\mathcal{C}$. The main idea is to turn the monoid of chain complexes $CFK^\infty(K)$ (under tensor product) into a group, which we will denote $\mathcal{F}$, in much the same way that the monoid of knots (under connected sum) can be made into the group $\mathcal{C}$ by quotienting by slice knots.

Let $CFK^\infty(K)^*$ denote the dual of $CFK^\infty(K)$; see Section 2 for the precise definition of the dual complex.

Definition 1. Define the group $\mathcal{F}$ to be

$$\mathcal{F} = \big(\{CFK^\infty(K) \mid K \subset S^3\}, \otimes\big)/\sim,$$

where

$$CFK^\infty(K_1) \sim CFK^\infty(K_2) \iff \varepsilon\big(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*\big) = 0.$$

Theorem 2. *The map*

$$\mathcal{C} \to \mathcal{F},$$

sending a class in $\mathcal{C}$ represented by K to the class in $\mathcal{F}$ represented by $CFK^\infty(K)$ is a group homomorphism.

This group $\mathcal{F}$ has the advantage that it can be studied from an algebraic perspective, much like the algebraic concordance group defined by Levine [Lev69a], [Lev69b] in terms of the Seifert form. However, Levine's homomorphism factors through the topological concordance group, while ours does not.

One algebraic feature of $\mathcal{F}$ is that it is totally ordered, with an additional well-defined notion of domination, "$\ll$"; see Definition 4.1 and Proposition 4.2. Moreover, we can use the relation $\ll$ to define a filtration on $\mathcal{F}$ that can be used to show linear independence of certain classes. Let 0 denote the identify of the group $\mathcal{F}$, i.e., $[CFK^\infty(\text{unknot})]$. Given a chain

$$0 < [CFK^\infty(K_1)] \ll [CFK^\infty(K_2)] \ll \cdots \ll [CFK^\infty(K_n)],$$

it follows that the collection

$$\left\{[CFK^\infty(K_i)]\right\}_{i=1}^n$$

is linearly independent in $\mathcal{F}$, and hence

$$\left\{[K_i]\right\}_{i=1}^n$$

is independent in $\mathcal{C}$. (It is also possible to use spectral sequences to define a second, independent filtration on the group $\mathcal{F}$.) One consequence of this filtration is that $\mathcal{F}$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$; see Theorem 3 below. We will use this rich structure on $\mathcal{F}$ to better understand $\mathcal{C}$.

Let $T_{p,q}$ denote the (p,q)-torus knot, $K_{p,q}$ the (p,q)-cable of K (where p denotes the longitudinal winding and q denotes the meridional winding), and D the (positive, untwisted) Whitehead double of the right-handed trefoil. We write $T_{m,n;p,q}$ to denote the (p,q)-cable of the (m,n)-torus knot. Let $-K$ denote the reverse of the mirror image of K, that is, the inverse of K in $\mathcal{C}$.

Theorem 3. *The topologically slice knots*

$$D_{p,p+1}\# - T_{p,p+1}, \quad p \geq 1,$$

are independent in the smooth concordance group; that is, they freely generate a subgroup of infinite rank.

The first example of an infinite family of smoothly independent, topologically slice knots was given by Endo [End95]. His examples consist of certain pretzel knots. More recently, Hedden and Kirk [HK12] showed that an infinite family of (untwisted) Whitehead doubles of certain torus knots are smoothly independent. The structure of $\mathcal{F}$ shows that our examples (when $p > 1$) are smoothly independent from both of these earlier families.

Let $P(K)$ denote the satellite of K with pattern P; that is, P is a knot in $S^1 \times D^2$, which we then glue into the (zero framed) knot complement $S^3 - \text{nbd } K$ to obtain the knot $P(K) \subset S^3$. Recall that the map $P(-) \colon \mathcal{C} \to \mathcal{C}$ given by

$$[K] \mapsto [P(K)]$$

is well defined, by "following" the concordance along the satellite.

We obtain a similar well-defined map on $\mathcal{F}$:

Proposition 4. *The map* $P(-)\colon \mathcal{F} \to \mathcal{F}$ *given by*

$$[CFK^{\infty}(K)] \mapsto [CFK^{\infty}(P(K))]$$

is well defined.

By composing P with τ, we obtain a new concordance invariant

$$\tau_P(K) = \tau(P(K)),$$

since K_1 being concordant to K_2 implies that $P(K_1)$ is concordant to $P(K_2)$. In the following theorem, we relate this to $[CFK^{\infty}(K)]$.

Theorem 5. $[CFK^{\infty}(K_1)] = [CFK^{\infty}(K_2)]$ *if and only if* $\tau_P(K_1) = \tau_P(K_2)$ *for all patterns* $P \subset S^1 \times D^2$.

Recall that τ is associated to the weaker, $\mathbb{Z}$-filtered chain complex $\widehat{CFK}$. The above theorem says that knowing information about a weaker invariant, namely τ, of satellites of K tells us information about the stronger invariant, CFK^{∞}, of the knot itself.

Does the map $P(-)\colon \mathcal{C} \to \mathcal{C}$ always take linearly independent collections of knots to linearly independent collections of knots? We address this question for cables in the following theorem:

Theorem 6. *For each* $n \in \mathbb{N}$, *there exists a collection of linearly independent knots*

$$\{K^i\}_{i=1}^n$$

such that for $m \geq n^2 - n - 1$,

$$\{K^i_{2,2m+1}\}_{i=1}^n$$

is a collection of linearly independent knots in $\mathcal{C}$.

This result should be compared to the work of Hedden and Kirk [HK12], where they use instantons to prove that the Whitehead doubles of $(2, 2^n + 1)$-torus knots are linearly independent.

Central to the definition of $\mathcal{F}$ is the concordance invariant ε, which exhibits the following properties:

- If K is smoothly slice, then $\varepsilon(K) = 0$.

- If $\varepsilon(K) = 0$, then $\tau(K) = 0$.

- There exist knots K with $\tau(K) = 0$ but $\varepsilon(K) \neq 0$; that is, ε is *strictly stronger* than τ at obstructing sliceness.

- $\varepsilon(-K) = -\varepsilon(K)$.

- If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K\#K') = \varepsilon(K)$. If $\varepsilon(K) = 0$, then $\varepsilon(K\#K') = \varepsilon(K')$.

These facts are proved in [Hom14]; we give sketches of their proofs in Section 3. Notice that since $\varepsilon(K) = 0$ implies that $\tau(K) = 0$, the map

$$\tau \colon \mathcal{C} \to \mathbb{Z}$$

factors through $\mathcal{F}$.

Organization. We begin by recounting the necessary definitions and properties of the complex CFK^{∞} (Section 2) and the concordance invariant ε (Section 3). With these definitions in place, we proceed to define the group $\mathcal{F}$, describe its various algebraic properties, and give examples (Section 4). We study satellites in Section 5. We conclude with the algebraic details in Section 6.

We work with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ throughout.

2. The knot Floer complex CFK^{∞}

To a knot $K \subset S^3$, Ozsváth and Szabó [OS04], and independently Rasmussen [Ras03], associate $CFK^{\infty}(K)$, a $\mathbb{Z}$-filtered chain complex over $\mathbb{F}[U, U^{-1}]$, whose filtered chain homotopy type is an invariant of K. The complex CFK^{∞} can be considered as a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex, with the second filtration induced by $-(U$-exponent). The ordering on $\mathbb{Z} \oplus \mathbb{Z}$ is given by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$. We assume the reader is familiar with this invariant, and the various related flavors, $CFK^{-}(K)$ and $\widehat{CFK}(K)$; for an expository introduction to these invariants, see [OS06]. The knot K is specified by a doubly pointed Heegaard diagram, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, and the generators (over $\mathbb{F}[U, U^{-1}]$) of $CFK^{\infty}(K)$ are the usual g-tuples of intersection points between the α- and β-circles, where g is the genus of Σ and each α-circle and each β-circle is used exactly once. The differential is defined as

$$\partial x = \sum_{y \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mathrm{ind}(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\, U^{n_w(\phi)} \cdot y.$$

This complex is endowed with a homological $\mathbb{Z}$-grading, called the *Maslov grading M*, as well as a $\mathbb{Z}$-filtration, called the *Alexander filtration A*. The relative Maslov

and Alexander gradings are defined as

$$M(x) - M(y) = \text{ind}(\phi) - 2n_w(\phi) \quad \text{and} \quad A(x) - A(y) = n_z(\phi) - n_w(\phi),$$

for $\phi \in \pi_2(x, y)$. The differential, ∂, decreases the Maslov grading by one, and respects the Alexander filtration; that is,

$$M(\partial x) = M(x) - 1 \quad \text{and} \quad A(\partial x) \leq A(x).$$

Multiplication by U shifts the Maslov grading and the Alexander filtration as follows:

$$M(U \cdot x) = M(x) - 2 \quad \text{and} \quad A(U \cdot x) = A(x) - 1.$$

It is often convenient to view this complex in the (i, j)-plane, where the i-axis represents $-(U$-exponent$)$ and the j-axis represents the Alexander filtration. The Maslov grading is suppressed from this picture. We place a generator x at position $(0, A(x))$; more generally, an element of the form $U^i \cdot x$ will have coordinates $(-i, A(x) - i)$.

Given $S \subset \mathbb{Z} \oplus \mathbb{Z}$, let $C\{S\}$ denote the set of elements in the plane whose (i, j)-coordinates are in S together with the arrows between them. The complex $CFK^-(K)$ is the subcomplex $C\{i \leq 0\}$, that is, the left half-plane. The complex $\widehat{CFK}(K)$ is the subquotient complex $C\{i = 0\}$.

A basis $\{x_i\}$ for a filtered chain complex (C, ∂) is called a *filtered basis* if the set $\{x_i \mid x_i \in C\{S\}\}$ is a basis for $C\{S\}$ for all filtered subcomplexes $C\{S\} \subset C$. Given a filtered basis for CFK^∞, we may visualize the differential by placing an arrow from a generator x to a generator y if y appears in ∂x. The differential points non-strictly to the left and down. Often, it will be convenient to consider only the part of the differential that preserves the Alexander grading, i.e., the horizontal arrows. We will denote this by ∂^{horz}. Similarly, we will use ∂^{vert} to denote the part of the differential that preserves the filtration by powers of U, i.e., the vertical arrows.

The integer-valued smooth concordance invariant $\tau(K)$ is defined in [OS03b] to be

$$\tau(K) = \min\{s \mid \iota \colon C\{i = 0, j \leq s\} \to C\{i = 0\}$$
$$\text{induces a non-trivial map on homology}\},$$

where ι is the natural inclusion of chain complexes. Alternatively, $\tau(K)$ may be defined in terms of the U-action on $HFK^-(K)$, as in Appendix A of [OST08]:

$$\tau(K) = -\max\{s \mid \text{there is } [x] \in HFK^-(K, s) \text{ such that } U^d[x] \neq 0 \text{ for all } d \geq 0\},$$

where $HFK^-(K, s) = H_*(C\{i \leq 0, j = s\})$.

The complex $CFK^\infty(K)$ satisfies certain symmetry and rank properties [OS04], Section 3. The complex obtained by interchanging the roles of i and j is filtered

chain homotopic to the original. Also, the rank of the homology of any column or row is one; more generally, modulo grading shifts, any column or row is filtered chain homotopic to $\widehat{CFK}(K)$.

By Theorem 7.1 of [OS04], we have the filtered chain homotopy equivalence

$$CFK^\infty(K_1 \# K_2) \simeq CFK^\infty(K_1) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(K_2).$$

Let $-K$ denote the reverse of the mirror image of K. The knot Floer complex is not sensitive to changes in orientation of the knot, but it is sensitive to changes in the orientation of the ambient manifold [OS04], Section 3.5. In particular,

$$CFK^\infty(-K) \simeq CFK^\infty(K)^*,$$

where $CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$, i.e.,

$$\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(CFK^\infty(K), \mathbb{F}[U, U^{-1}]).$$

In order to depict the complex $CFK^\infty(K)^*$ in the (i, j)-plane, we take the complex $CFK^\infty(K)$ and reverse the direction of all of the arrows as well as the direction of both of the filtrations. (In practice, we can accomplish this by reversing the direction of all of the arrows and then turning our heads upside down.)

We point out that when we write $CFK^\infty(K)$, we are really denoting an equivalence class of filtered chain complexes. We may always choose as our representative the E_1 page of the spectral sequence associated to one of these complexes, that is, the homology of the associated graded object together with the induced differentials. In other words, we may choose our representative to be reduced, in the sense that any differential strictly lowers the filtration (in at least one direction).

3. The invariant ε

The invariant ε can be defined in terms of the (non-) vanishing of certain cobordism maps, which, using the relation between large surgery and knot Floer homology ([OS04], Theorems 4.1 and 4.4) has an algebraic interpretation in terms of the filtered chain complex $CFK^\infty(K)$.

Let N be a sufficiently large integer. (It turns out that $N > 2g(K)$ will suffice; see [OS08], Theorem 1.1, and [OS04], Theorem 5.1.) We consider the map

$$F_s \colon \widehat{HF}(S^3) \to \widehat{HF}(S^3_{-N}(K), [s]),$$

induced by the 2-handle cobordism, W^4_{-N}. As usual, $[s]$ denotes the restriction to $S^3_{-N}(K)$ of the Spin^c structure $\mathfrak{s}_s$ over W^4_{-N} with the property that

$$\langle c_1(\mathfrak{s}_s), [\widehat{F}] \rangle + N = 2s,$$

where $|s| \leq \frac{N}{2}$ and $\widehat{F}$ denotes the capped off Seifert surface in the four manifold. We also consider the map

$$G_s \colon \widehat{HF}(S_N^3(K), [s]) \to \widehat{HF}(S^3),$$

induced by the 2-handle cobordism, $-W_N^4$, that is, the cobordism W_N^4 turned "upside down" and with its orientation reversed.

The maps F_s and G_s can be defined algebraically by studying certain natural maps on subquotient complexes of $CFK^\infty(K)$, as in [OS04]. The map F_s is induced by the chain map

$$C\{i = 0\} \to C\{\min(i, j - s) = 0\}$$

consisting of quotienting by $C\{i = 0, j < s\}$, followed by inclusion. Similarly, the map G_s is induced by the chain map

$$C\{\max(i, j - s) = 0\} \to C\{i = 0\}$$

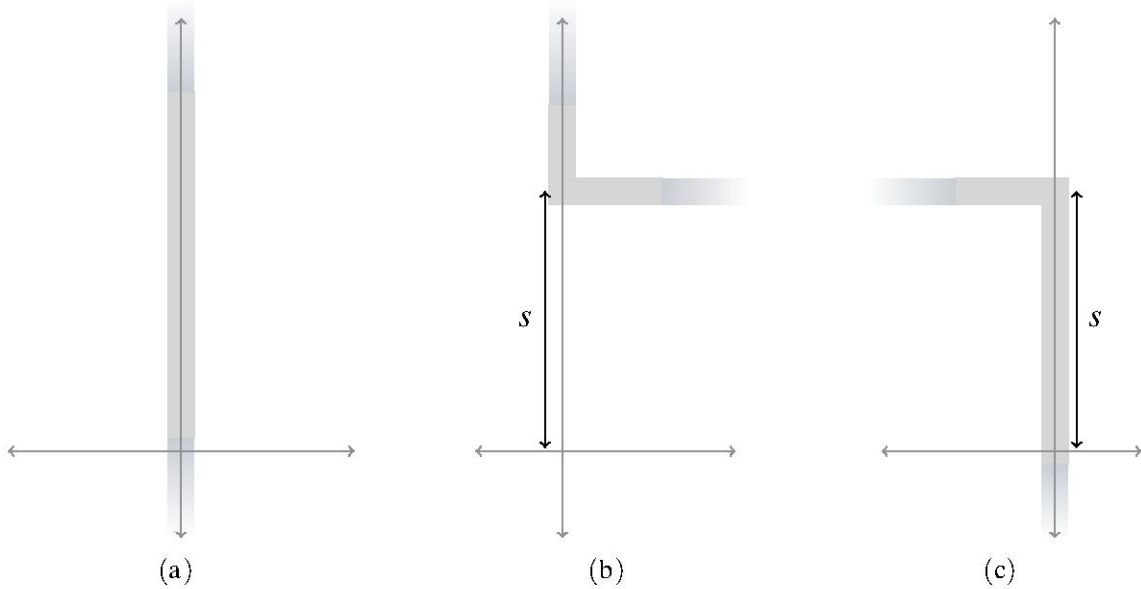consisting of quotienting by $C\{i < 0, j = s\}$, followed by inclusion.

Figure 1. Left, the subquotient complex $C\{i = 0\}$. Center, the subquotient complex $C\{i = 0, j < s\}$. Right, the subquotient complex $C\{\max(i, j - s) = 0\}$.

For ease of notation, we will often write simply τ for $\tau(K)$ when the meaning is clear from context. Notice that for $s > \tau$, F_s is trivial, since quotienting $C\{i = 0\}$ by $C\{i = 0, j < s\}$ will induce the trivial map, as the homology of $C\{i = 0\}$ is supported in filtration level τ.

For $s < \tau$, F_s is non-trivial, since any generator of $H_*(C\{i = 0\})$ will still be in the kernel, but not the image, of the differential on $C\{\min(i, j - s) = 0\}$.

The map F_τ may be trivial or non-trivial, depending on whether the class representing a generator of $H_*(C\{i = 0\})$ lies in the image of the differential on $C\{\min(i, j - \tau) = 0\}$ or not.

The maps G_τ behaves similarly. For $s > \tau$, the map G_s is non-trivial, and for $s < \tau$, G_s is trivial. The map G_τ will be non-trivial if the class representing a generator of $H_*(C\{i = 0\})$ lies in the kernel of the differential on $C\{\max (i, j - s) = 0\}$, and trivial otherwise.

Because $C\{j = \tau\}$ is a chain complex, and so $\partial^2 = 0$, it follows that F_s and G_s cannot both be trivial; that is, a class cannot lie in the image but not in the kernel of the differential. (This is made precise in [Hom14].) Therefore, there are three possibilities for F_τ and G_τ: either exactly one vanishes, or neither vanishes.

Definition 3.1. The invariant ε is defined in terms of F_τ and G_τ as follows:

- $\varepsilon(K) = 1$ if and only if F_τ is trivial (in which case G_τ is necessarily non-trivial).

- $\varepsilon(K) = -1$ if and only if G_τ is trivial (in which case F_τ is necessarily non-trivial).

- $\varepsilon(K) = 0$ if and only if both F_τ and G_τ are non-trivial.

Let $[x]$ be a generator of $H_*(C\{i = 0\})$, the so-called "vertical" homology. In light of the preceding discussion, the definition of ε corresponds to viewing $[x]$ as a class in the "horizontal" complex $C\{j = \tau\}$ as follows:

- $\varepsilon(K) = 1$ if and only if $[x]$ is in the image of horizontal differential.

- $\varepsilon(K) = -1$ if and only if $[x]$ is not in the kernel of the horizontal differential.

- $\varepsilon(K) = 0$ if and only if $[x]$ is in the kernel but not the image of the horizontal differential.

Notice that ε is an invariant of the filtered chain homotopy type of CFK^∞; at times, to emphasize this point, we will write $\varepsilon(CFK^\infty(K))$ rather than simply $\varepsilon(K)$.

This idea of associating numerical invariants to filtered chain complexes is common; for example, to any $\mathbb{Z}$-filtered chain complex whose total homology has rank one, we can define an integer-valued invariant that measures the minimum filtration level at which this homology is supported, e.g., $\tau(K)$, which is an invariant of the $\mathbb{Z}$-filtered chain homotopy type of $\widehat{CFK}(K)$.

Similarly, to any $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex whose "vertical" homology has rank one, we can define a $\{-1, 0, 1\}$-valued invariant that measures how this class appears in the "horizontal" complex, i.e., in the image of the horizontal differential, in the kernel but not the image, or not in the kernel, respectively. In particular, when $\varepsilon(K) = 0$, then $CFK^\infty(K)$ is filtered chain homotopic to a complex with a distinguished generator that is non-trivial in both the vertical and the horizontal homology.

Proposition 3.2 ([Hom14]). *The following are properties of $\varepsilon(K)$:*

(1) *If K is smoothly slice, then $\varepsilon(K) = 0$.*

(2) *If $\varepsilon(K) = 0$, then $\tau(K) = 0$.*

(3) $\varepsilon(-K) = -\varepsilon(K)$.

(4)   (a) *If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K \# K') = \varepsilon(K) = \varepsilon(K')$.*

     (b) *If $\varepsilon(K) = 0$, then $\varepsilon(K \# K') = \varepsilon(K')$.*

For completeness, we sketch the proof below.

Sketch of proof. To prove (1), we consider the d-invariants of large surgery along K. If K is slice, then the surgery correction terms defined in [OS03a] vanish, i.e., agree with the surgery correction terms of the unknot, and the maps

$$\widehat{HF}(S_N^3(K), [0]) \to \widehat{HF}(S^3) \quad \text{and} \quad \widehat{HF}(S^3) \to \widehat{HF}(S_{-N}^3(K), [0])$$

are non-trivial. Indeed, the surgery corrections terms can be defined in terms of the maps

$$HF^+(S^3) \to HF^+(S_{-N}^3(K), [s])$$

and we have the commutative diagram

$$
\begin{array}{ccc}
\widehat{HF}(S^3) & \xrightarrow{F_s} & \widehat{HF}(S_{-N}^3(K), [s]) \\
\iota \downarrow & & \iota_s \downarrow \\
HF^+(S^3) & \xrightarrow{F_s^+} & HF^+(S_{-N}^3(K), [s]).
\end{array}
$$

Let $N \gg 0$. If the surgery corrections terms vanish (that is, agree with those of the unknot), then F_τ^+ is an injection [Ras04], Section 2.2, and so the composition $\iota \circ F_\tau^+$ is non-trivial. By commutativity of the diagram, it follows that F_τ must be non-trivial. A similar diagram in the case of large positive surgery shows that G_τ must be non-trivial as well. Hence $\varepsilon(K) = 0$.

The proof of (2) follows from the fact that if $\varepsilon(K) = 0$, then there is a class x in $CFK^\infty(K)$ which generates both $H_*(C\{i = 0\})$ and $H_*(C\{j = 0\})$. In the former complex, x has Alexander grading $A(x)$, and in the latter, viewed as a $\mathbb{Z}$-filtered complex, x has filtration level $-A(x)$. Hence $\tau(K) = -\tau(K) = 0$.

The proof of (3) follows from the symmetry properties of the knot Floer complex [OS04], Section 3.5; in particular, we have the filtered chain homotopy equivalence $CFK^\infty(-K) \simeq CFK^\infty(K)^*$.

To prove the first part of (4): if $[x]$ and $[x']$ are generators of $H_*(\widehat{CFK}(K))$ and $H_*(\widehat{CFK}(K'))$, respectively, then $[x \otimes x']$ is a generator of $H_*(\widehat{CFK}(K \# K'))$.

(Here, we are identifying $\widehat{CFK}$ with $C\{i = 0\}$.) Suppose $\varepsilon(K) = \varepsilon(K') = 1$. Then both $[x]$ and $[x']$ are both in the image of the horizontal differential, and hence $[x \otimes x']$ is also. The other cases follow similarly. $\square$

Notice that Proposition 3.2 implies that ε is a concordance invariant. If K and K' are concordant, then $\varepsilon(K\# - K') = 0$, in which case $\varepsilon(K) = -\varepsilon(-K')$ by (4), or $\varepsilon(K) = \varepsilon(K')$.

Note that we have the following subgroup of $\mathcal{C}$:

$$\{[K] \mid \varepsilon(K) = 0\} \subset \mathcal{C}.$$

This subgroup will appear again in the next section.

4. The group $\mathcal{F}$

In this section, we define the group $\mathcal{F}$ as well as some of its algebraic structure. We will give examples of knots that demonstrate the richness of this structure. In particular, we give an infinite family of topologically slice knots that are linearly independent in $\mathcal{F}$, and hence also in the smooth concordance group $\mathcal{C}$, as needed for the proof of Theorem 3.

4.1. Definition of the group $\mathcal{F}$. We define the group $\mathcal{F}$ as

$$\mathcal{F} = \left(\{CFK^\infty(K) \mid K \subset S^3\}, \otimes\right) / \sim,$$

where

$$CFK^\infty(K_1) \sim CFK^\infty(K_2) \iff \varepsilon\left(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*\right) = 0,$$

$CFK^\infty(K)^*$ denotes the dual of $CFK^\infty(K)$, and the tensor product is over $\mathbb{F}[U, U^{-1}]$. We have the well-defined group homomorphism

$$\mathcal{C} \to \mathcal{F},$$

given by

$$[K] \mapsto [CFK^\infty(K)].$$

Theorem 2, including well-definedness, follows from the following facts (the first two from Section 3.5 of [OS04] and the last from Proposition 3.2):

- $CFK^\infty(-K) \simeq CFK^\infty(K)^*$.
- $CFK^\infty(K_1 \# K_2) \simeq CFK^\infty(K_1) \otimes CFK^\infty(K_2)$.
- If K is smoothly slice, then $\varepsilon(CFK^\infty(K)) = 0$.

Notice that $\mathcal{F}$ is isomorphic to the quotient

$$\mathcal{F} \cong \mathcal{C}/\{[K] \mid \varepsilon(K) = 0\}.$$

For ease of notation, from now on, we will write

$$[\![K]\!]$$

to denote $[CFK^\infty(K)]$, and, when convenient, we will write

$$[\![K_1]\!] + [\![K_2]\!]$$

to denote the operation on the group, which can be thought of as either $[CFK^\infty(K_1) \otimes CFK^\infty(K_2)]$ or $[CFK^\infty(K_1 \# K_2)]$. Note that $-[\![K]\!] = [\![-K]\!]$. We denote the identity of $\mathcal{F}$, $[\![\text{unknot}]\!]$, by 0.

The group $\mathcal{F}$ has a rich algebraic structure: it has a total ordering, and a "$\ll$" relation that satisfies the certain properties (e.g., Lemma 4.3) and induces a filtration on the group. This algebraic structure on $\mathcal{F}$ will in turn be useful in understanding the structure of the smooth concordance group $\mathcal{C}$.

Proposition 4.1. *The group $\mathcal{F}$ is totally ordered, with the ordering given by*

$$[\![K_1]\!] > [\![K_2]\!] \iff \varepsilon(K_1 \# - K_2) = 1.$$

Proof. We may think of $\varepsilon(K)$ as the "sign" of $[\![K]\!]$, and then the order relation between any two classes is determined by the sign of their difference.

This relation is clearly transitive, since given

$$[\![K_1]\!] > [\![K_2]\!] \quad \text{and} \quad [\![K_2]\!] > [\![K_3]\!],$$

it follows that

$$[\![K_1]\!] > [\![K_3]\!].$$

Indeed,

$$\varepsilon(K_1 \# - K_3) = \varepsilon(K_1 \# - K_2 \# K_2 \# - K_3)$$
$$= 1,$$

by (4) of Proposition 3.2 since $\varepsilon(K_1 \# - K_2) = 1$ and $\varepsilon(K_2 \# - K_3) = 1$.

This relation is also translation invariant. Given

$$[\![K_1]\!] > [\![K_2]\!],$$

it follows that

$$[\![K_1]\!] + [\![K_3]\!] > [\![K_2]\!] + [\![K_3]\!],$$

since

$$\varepsilon(K_1 \# K_3 \# - K_3 \# - K_2) = \varepsilon(K_1 \# - K_2)$$
$$= 1. \qquad \square$$

Totally ordered groups give rise to many natural algebraic constructions, which we will utilize below. For example, we have a notion of absolute value; that is, given an element $[\![K]\!]$, either $[\![K]\!]$ or $-[\![K]\!]$ is greater than the identity, so we define the absolute value as

$$\big|[\![K]\!]\big| = \begin{cases} [\![K]\!] & \text{if } \varepsilon(K) \geq 0, \\ -[\![K]\!] & \text{otherwise.} \end{cases}$$

A natural question to ask is: Do there exist knots K_1 and K_2 with $\varepsilon(K_1) = \varepsilon(K_2) = 1$ (i.e., they are both "positive" with respect to the ordering), and

$$[\![K_1]\!] > n[\![K_2]\!] \quad \text{for all } n \in \mathbb{N}?$$

The answer, it turns out, is yes, motivating the following definition:

Definition 4.2. The class $[\![K_1]\!]$ *dominates* $[\![K_2]\!]$, denoted

$$[\![K_1]\!] \gg [\![K_2]\!],$$

if $[\![K_1]\!] > n[\![K_2]\!] > 0$ for all $n \in \mathbb{N}$.

Transitivity of $\gg$ follows exactly as for the total ordering. We have the following lemma, showing a useful property that the $\gg$ relation satisfies:

Lemma 4.3. *If*

$$[\![K_1]\!] \gg [\![K_2]\!] \quad \text{and} \quad [\![K_1]\!] \gg [\![K_3]\!]$$

then

$$[\![K_1]\!] \gg [\![K_2]\!] + [\![K_3]\!].$$

Proof. To see that this is true, we proceed by contradiction. Assume there exists $n \in \mathbb{N}$ such that

$$[\![K_1]\!] \leq n\big([\![K_2]\!] + [\![K_3]\!]\big).$$

Then $2[\![K_1]\!] \leq 2n\big([\![K_2]\!] + [\![K_3]\!]\big)$, i.e.,

$$[\![K_1]\!] - 2n[\![K_2]\!] + [\![K_1]\!] - 2n[\![K_3]\!] \leq 0.$$

But $[\![K_1]\!] - m[\![K_2]\!] > 0$ and $[\![K_1]\!] - m[\![K_3]\!] > 0$ for all $m \in \mathbb{N}$, giving us the desired contradiction. $\square$

Remark 4.4. These ideas could alternatively be phrased in terms of Archimedean equivalence classes. Recall that two elements a and b of a totally ordered group are *Archimedean equivalent* if there exist natural numbers M and N such that $M \cdot |a| > |b|$ and $N \cdot |b| > |a|$. Then we say that $a \gg b$ if $a > b > 0$, and a and b are not Archimedean equivalent. Note that the set of Archimedean equivalence classes is naturally totally ordered, and this ordering corresponds to the $\gg$ relation.

Definition 4.5. Let $\mathcal{F}_K$ denote the collection of elements

$$\mathcal{F}_K = \{[\![J]\!] \mid |[\![J]\!]| \ll |[\![K]\!]|\}.$$

Proposition 4.6. $\mathcal{F}_K$ *is a subgroup of* $\mathcal{F}$.

Proof. If $[\![J]\!]$ is in $\mathcal{F}_K$, then $-[\![J]\!]$ clearly is as well. Given $[\![J_1]\!]$ and $[\![J_2]\!]$ in $\mathcal{F}_K$, is follows immediately that $[\![J_1]\!] + [\![J_2]\!]$ is also in $\mathcal{F}_K$, by Lemma 4.3. $\qquad\square$

Notice that given a sequence of knots $K_1, K_2, \ldots, K_n$ satisfying

$$[\![K_1]\!] \gg [\![K_2]\!] \gg \cdots \gg [\![K_n]\!],$$

we obtain a filtration

$$\mathcal{F}_{K_1} \supset \mathcal{F}_{K_2} \supset \cdots \supset \mathcal{F}_{K_n}.$$

Lemma 4.7. *If* $[\![K_1]\!] \gg [\![K_2]\!] \gg \cdots \gg [\![K_n]\!] > 0$, *then the knots*

$$K_1, \ K_2, \ \ldots, K_n$$

are linearly independent in $\mathcal{F}$ *and hence in* $\mathcal{C}$; *that is, they generate a subgroup of rank* n *in both* $\mathcal{F}$ *and* $\mathcal{C}$.

Proof. By Lemma 4.3, for any positive integer m, $m[\![K_1]\!]$ dominates any linear combination of $[\![K_2]\!], \ldots, [\![K_n]\!]$, and thus cannot be expressed as a linear combination of these classes. Similarly, $m[\![K_i]\!]$ dominates any linear combination of $[\![K_{i+1}]\!], \ldots, [\![K_n]\!]$, for $i < n$. $\qquad\square$

4.2. Examples. We now give examples of families of knots that can be shown to independent in $\mathcal{C}$.

Proposition 4.8. *Let* $0 < p < q$. *Then we have the following relations in the group* $\mathcal{F}$:

  (1) $[\![T_{p,p+1}]\!] \ll [\![T_{q,q+1}]\!]$,
  (2) $[\![D_{p,p+1}]\!] \ll [\![D_{q,q+1}]\!]$,
  (3) $[\![T_{p,p+1}]\!] \ll [\![D_{p,p+1}]\!]$,
  (4) $[\![T_{p,p+1;2,2m+1}]\!] \ll [\![T_{q,q+1;2,2m+1}]\!]$, $m \geq q^2 - q - 1$.

We will prove this proposition at the end of Section 6.

Remark 4.9. A straightforward consequence of (2) and (3) of the preceding proposition is the relation

$$[\![D_{p,p+1}\# - T_{p,p+1}]\!] \ll [\![D_{q,q+1}\# - T_{q,q+1}]\!].$$

We are now ready to prove Theorem 3; that is, we will show that the knots

$$D_{p,p+1} \# - T_{p,p+1}, \quad p \geq 1,$$

are smoothly independent while being topologically slice.

Proof of Theorem 3. Recall that D is the (positive, untwisted) Whitehead double of the right-handed trefoil. The Alexander polynomial of D is equal to one, and so by Freedman [Fre82], it follows that D is topologically slice. Hence, the $(p, p + 1)$-cable of D, $D_{p,p+1}$, is topologically concordant to the $(p, p+1)$-cable of the unknot, i.e., the torus knot $T_{p,p+1}$. Thus, $D_{p,p+1} \# - T_{p,p+1}$ is topologically slice.

It follows from Lemma 4.7 and Remark 4.9 that the knots

$$D_{p,p+1} \# - T_{p,p+1}, \quad p \geq 1,$$

are linearly independent in $\mathcal{F}$, and hence also in $\mathcal{C}$.   □

Proof of Theorem 6. We need to find a collection of linearly independent knots $\{K^i\}_{i=1}^n$ such that the collection $\{K^i_{2,2m+1}\}_{i=1}^n$ is also linearly independent for sufficiently large m.

Let $K^i = T_{i,i+1}$, and consider the $(2, 2m + 1)$-cable of K^i, where $m \geq n^2 - n - 1$. By Lemma 4.7 and Proposition 4.8, it follows that the collection

$$\{K^i\}_{i=1}^n$$

is linearly independent in $\mathcal{F}$, hence also in $\mathcal{C}$. Again, by Lemma 4.7 and Proposition 4.8, the collection

$$\{K^i_{2,2m+1}\}_{i=1}^n,$$

is also linearly independent in $\mathcal{F}$ and thus in $\mathcal{C}$.   □

5. Satellites and $\mathcal{F}$

Recall that $P(K)$ denotes the satellite of K with pattern P; that is, P is a knot in $S^1 \times D^2$, which we then glue into the (zero framed) knot complement $S^3 - \text{nbd } K$. The map $P(-) \colon \mathcal{C} \to \mathcal{C}$ given by

$$[K] \mapsto [P(K)]$$

is well defined, by "following" the concordance along the satellite. We will show that an analogous result holds for the group $\mathcal{F}$; this is a restatement of Proposition 4 from the Introduction.

Proposition 5.1. *The map $P(-)\colon \mathscr{F} \to \mathscr{F}$ given by*

$$[\![K]\!] \mapsto [\![P(K)]\!]$$

is well defined.

The following theorem from [Hom14] gives a formula for $\tau(K_{p,q})$ in terms of $\tau(K)$, $\varepsilon(K)$, p, and q:

Theorem 5.2 ([Hom14]). *Let $K \subset S^3$, and let p, q be relatively prime integers with $p > 0$. Then the behavior of $\tau(K_{p,q})$ is completely determined by p, q, $\tau(K)$, and $\varepsilon(K)$. More precisely:*

(1) *If $\varepsilon(K) = 1$, then $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}$.*

(2) *If $\varepsilon(K) = -1$, then $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}$.*

(3) *If $\varepsilon(K) = 0$, then $\tau(K) = 0$ and $\tau(K_{p,q}) = \tau(T_{p,q}) = \begin{cases} \frac{(p-1)(q+1)}{2} & \text{if } q < 0, \\ \frac{(p-1)(q-1)}{2} & \text{if } q > 0. \end{cases}$*

We see that knowing $\tau(K_{2,1})$ and $\tau(K_{2,-1})$ is sufficient to determine $\varepsilon(K)$. More precisely:

- If $\tau(K_{2,1})$ is odd, then $\varepsilon(K) = -1$.

- If $\tau(K_{2,-1})$ is odd, then $\varepsilon(K) = 1$.

- If $\tau(K_{2,1}) = \tau(K_{2,-1}) = 0$, then $\varepsilon(K) = 0$.

The proof of Proposition 5.1 will rely on this observation.

The proof will also rely on facts from bordered Heegaard Floer homology, as defined by Lipshitz, Ozsváth and Thurston [LOT08]. We will need only a special case of the formal properties of these invariants, which we recount here.

To a framed knot complement Y_K, we associate a left differential graded module $\widehat{CFD}(Y_K)$, whose homotopy equivalence class is an invariant of the framed knot complement [LOT08], Theorem 1.1. Furthermore, the homotopy equivalence class is completely determined by the complex $CFK^\infty(K)$ and the framing n [LOT08], Theorem 11.27 and A.11. For our purposes here, it will be sufficient to let Y_K be the zero framed knot complement. In [Hom14], it is shown that if $\varepsilon(K) = 0$, then

$$\widehat{CFD}(Y_{J\#K}) \simeq \widehat{CFD}(Y_J) \oplus A,$$

for some left differential graded module A which depends on $CFK^\infty(K)$.

To a knot P in $S^1 \times D^2$, we associate a right $\mathcal{A}_\infty$-module $CFA^-(S^1 \times D^2, P)$. Let $gCFK^-(K)$ denote the associated graded complex of $CFK^-(K)$, i.e., $\oplus_s C\{i \leq 0, j = s\}$. Notice that $HFK^-(K) \cong H_*(gCFK^-(K))$. Then the pairing theorem

for bordered Heegaard Floer homology ([LOT08], Theorem 11.21) states that we have the following graded chain homotopy equivalence:

$$gCFK^-(S^3, P(K)) \simeq CFA^-(S^1 \times D^2, P) \widetilde{\otimes} \widehat{CFD}(Y_K),$$

where we choose the zero framing for the knot complement Y_K, and where $\widetilde{\otimes}$ denotes the $\mathcal{A}_\infty$-tensor product, a generalization of the derived tensor product. In particular, $\widetilde{\otimes}$ respects summands.

Proof of Proposition 5.1. Assume $\varepsilon(K\# - J) = 0$. We would like to show that

$$\varepsilon(P(K)\# - P(J)) = 0.$$

Utilizing the observation above, it is sufficient to show that

$$\tau\big((P(K)\# - P(J))_{2,\pm 1}\big) = 0.$$
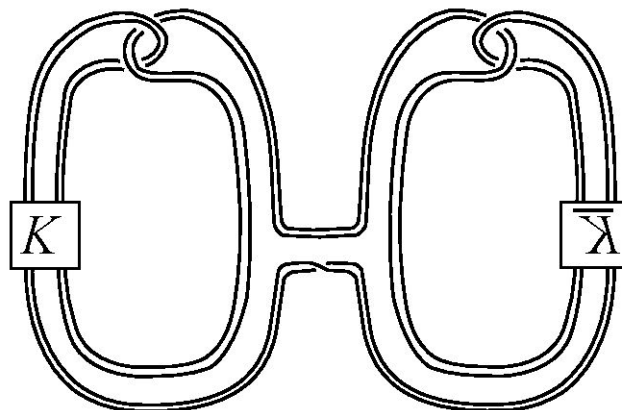

Figure 2. The knot $\big(P(K)\# - P(K)\big)_{2,1}$, in the case where P is the pattern for the Whitehead double.

Let U denote the unknot. There exists an embedding Q of $\big(P(U)\# - P(J)\big)_{2,\pm 1}$ into $S^1 \times D^2$ such that

$$Q(K) = \big(P(K)\# - P(J)\big)_{2,\pm 1}.$$

See Figure 3. We consider the bordered invariant

$$CFA^-(S^1 \times D^2, Q)$$

associated to $(S^1 \times D^2, Q)$. Notice that $Q(J) = \big(P(J)\# - P(J)\big)_{2,\pm 1}$ is slice and so $\tau\big((P(J)\# - P(J))_{2,\pm 1}\big) = 0$.
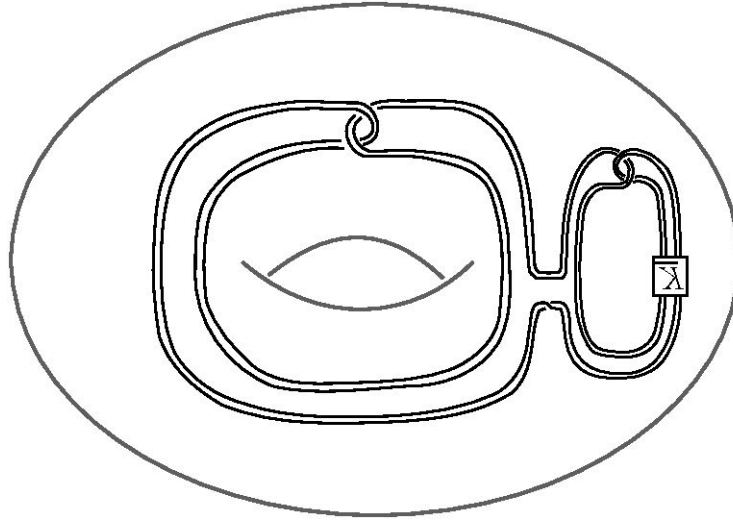
Figure 3. The knot $\left(P(U)\# - P(K)\right)_{2,1}$ embedded in $S^1 \times D^2$ as the pattern knot Q, where again, P is the pattern for the Whitehead double.

The knot K is concordant to the knot $K' = J \# K \# - J$. Since $\varepsilon(K \# - J) = 0$, we have the following chain homotopy equivalence:

$$\widehat{CFD}(Y_{K'}) \simeq \widehat{CFD}(Y_J) \oplus A,$$

for some A.

The knot $Q(K)$ is concordant to $Q(K')$, since K is concordant to K'. The invariant $\tau(Q(K'))$ is determined by

$$gCFK^-(Q(K')) \simeq CFA^-(S^1 \times D^2, Q) \,\widetilde{\otimes}\, \widehat{CFD}(Y_{K'})$$
$$\simeq CFA^-(S^1 \times D^2, Q) \,\widetilde{\otimes}\, \left(\widehat{CFD}(Y_J) \oplus A\right)$$
$$\simeq gCFK^-\left(Q(J)\right) \oplus B$$

where B is the complex $CFA^-(S^1 \times D^2, Q) \,\widetilde{\otimes}\, A$. Notice that $H_*(B)$ is U-torsion, since the ranks of $HFK^-(Q(K'))$ and $HFK^-(Q(J))$ as $\mathbb{F}[U]$-modules are both one. Thus,

$$\tau(Q(K)) = \tau(Q(K')) = \tau(Q(J)) = 0,$$

since $Q(J)$ is slice. Recalling that $Q(K) = \left(P(K)\# - P(J)\right)_{2,\pm 1}$, we have that

$$\tau\left((P(K)\# - P(J))_{2,\pm 1}\right) = 0,$$

implying that

$$\varepsilon(P(K)\# - P(J)) = 0,$$

as desired. $\square$

We now prove Theorem 5, which we restate here:

Theorem 5.3. $[\![K]\!] = [\![J]\!]$ *if and only if* $\tau_P(K) = \tau_P(J)$ *for all patterns* $P \subset S^1 \times D^2$.

Proof. The forward direction is true by Proposition 5.1 and the fact that the map $\tau \colon \mathcal{C} \to \mathbb{Z}$ factors through $\mathcal{F}$. We must now show that if $[\![K]\!] \neq [\![J]\!]$, then there exists some pattern P such that $\tau(P(K)) \neq \tau(P(J))$.

Without loss of generality, we may assume that $\varepsilon(K\# - J) = -1$. Let

$$P(K) = (K\# - J)_{2,1}.$$

Then Theorem 5.2 tells us that

$$\tau(P(K)) = 2(\tau(K) - \tau(J)) + 1 \quad \text{and} \quad \tau(P(J)) = 0,$$

as desired. $\square$

6. Calculations and a refinement of ε

An element of $\mathcal{F}$ is an equivalence class of filtered chain complexes. The goal of this section is to define more tractable invariants associated to such a class, compute these invariants for a few families of knots, and show that these invariants are related to the algebraic structure, namely the $\gg$ relation, on $\mathcal{F}$.

To this end, we will define a refinement of ε. Recall that ε is defined in terms of whether or not certain maps on subquotient complexes of CFK^∞ vanish on homology. Our refinement of ε will be defined in a similar manner.

The invariant $\varepsilon(K)$ is equal to one when the class generating the "vertical" homology of $CFK^\infty(K)$ lies in the image of the horizontal differential. We would like a well-defined way to measure the "length" of the differential that hits that class, that is, how much it decreases the horizontal filtration. We will do this by examining certain natural maps on subquotients of CFK^∞.

The definition of ε involved examining the map F_τ induced by

$$C\{i = 0\} \to C\{\min(i, j - \tau) = 0\}.$$

In particular, if F_τ is trivial, then $\varepsilon(K) = 1$. Consider now the map H_s induced on homology by

$$C\{i = 0\} \to C\{\min(i, j - \tau) = 0, \ i \leq s\},$$

for some non-negative integer s. Notice that H_0 is non-trivial, and for sufficiently large s, H_s agrees with F_τ.

Suppose that $\varepsilon(K) = 1$; that is, F_τ is trivial. Then define $a_1(K)$ to be

$$a_1(K) = \min\{s \mid H_s \text{ is trivial}\}.$$

The idea is that when $\varepsilon(K) = 1$, the class generating the vertical homology lies in the image of the horizontal differential, and a_1 is measuring the "length" of the horizontal differential hitting that class.

Now consider the map $H_{a_1,s}$ induced on homology by

$$C\{i = 0\} \rightarrow C\{\{\min(i, j - \tau) = 0, i \le a_1\} \cup \{i = a_1, \tau - s \le j < \tau\}\},$$

for some non-negative integer s. Clearly, $H_{a_1,0}$ is trivial. Define

$$a_2(K) = \min\{s \mid H_{a_1,s} \text{ is non-trivial}\}.$$

Notice that $a_2(K)$ may be undefined; that is, the map $H_{a_1,s}$ may be trivial for all s. Effectively, a_2 is measuring the "length" of a certain vertical differential, if it exists.

Lemma 6.1. *The invariants a_1 and a_2 are invariants of the class $[\![K]\!]$.*

Proof. Suppose $[\![J]\!] = [\![K]\!]$. Then

$$[\![J]\!] = [\![K]\!] = [\![K\# - J\#J]\!].$$

Since $\varepsilon(K\# - J) = 0$, it follows from Lemma 3.3 of [Hom14] that there exists a basis for $CFK^\infty(K\# - J)$ with a distinguished element, say x_0, with no incoming or outgoing horizontal or vertical arrows. Similarly, there is a basis for $CFK^\infty(J\# - J)$ with a distinguished element y_0. Then we may compute $a_1(K\# - J\#J)$ and $a_2(K\# - J\#J)$ by considering either

$$\{x_0\} \otimes CFK^\infty(J) \quad \text{or} \quad CFK^\infty(K) \otimes \{y_0\},$$

the former giving us $a_1(J)$ and $a_2(J)$, and the latter giving us $a_1(K)$ and $a_2(K)$. $\square$

Lemma 6.2. *Let $a_1 = a_1(K)$. Then there exists a basis $\{x_i\}$ over $\mathbb{F}[U, U^{-1}]$ for CFK^∞ with basis elements x_0 and x_1 with the following properties.*

(1) *There is a horizontal arrow of length a_1 from x_1 to x_0.*

(2) *There are no other horizontal or vertical arrows to or from x_0.*

(3) *There are no other horizontal arrows to or from x_1.*

If we also have that $a_2 = a_2(K)$ is well defined, then there exists a basis $\{x_i\}$ with basis elements x_0, x_1, and x_2 with the following properties, in addition to the ones above:

(4) *There is a vertical arrow of length a_2 from x_1 to x_2.*

(5) *There are no other vertical arrows to or from x_1 or x_2.*

Proof. We will give the proof for the case where a_2 is well defined. The proof in the case where a_2 is not well defined is a straightforward simplification of this proof.

For ease of notation, let

$$A = C\{\min(i, j - \tau) = 0, i \le a_1\},$$
$$B = C\{\{\min(i, j - \tau) = 0, i \le a_1\} \cup \{i = a_1, \tau - a_2 \le j < \tau\}\},$$

so that H_{a_1} and H_{a_1,a_2}, respectively, are the maps on homology induced by

$$C\{i = 0\} \to A,$$
$$C\{i = 0\} \to B.$$

See Figure 4. Since H_{a_1} is trivial, it follows that there is a generator, say x_0, of $H_*(C\{i = 0\})$ in position $(0, \tau)$ that is in the image of the differential on A, but not in the image of the differential on B. Since H_{a_1,a_2} is non-trivial, there exists a class x_1 supported in position (a_1, τ) whose boundary, in A, is x_0, and whose boundary, in B, is a class, say $x_0 + x_2$, where x_2 is supported in position $(a_1, \tau - a_2)$. Moreover, we may replace x_0 with $\partial^{\text{horz}} x_1$, since a priori, $\partial^{\text{horz}} x_1$ might include elements with negative i-coordinate. Similarly, we may replace x_2 with $\partial^{\text{vert}} x_1$. We now complete

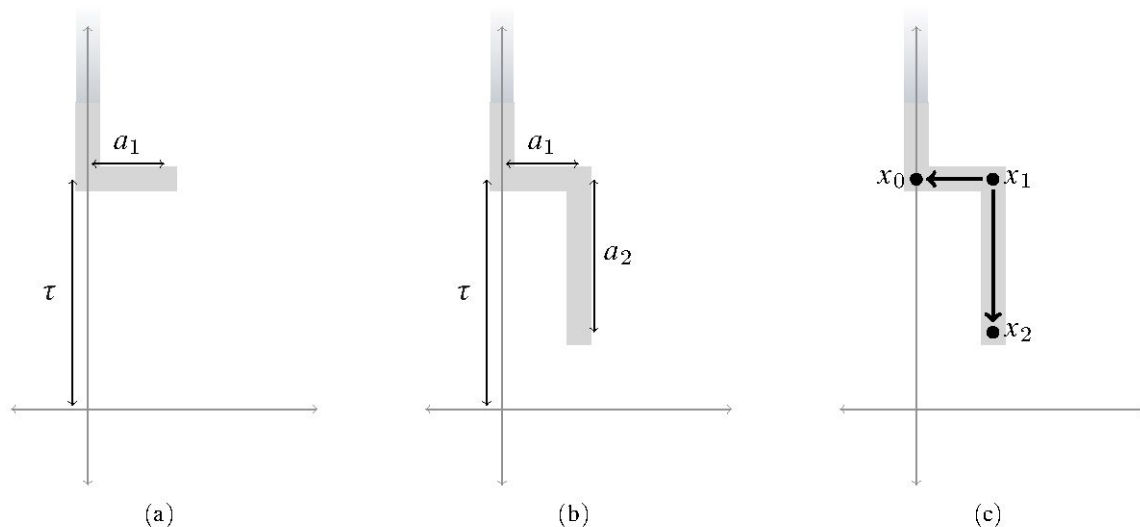
Figure 4. Left, the complex A in the (i, j)-plane. Center, the complex B. Right, part of the basis in Lemma 6.2.

$\{x_0, x_1, x_2\}$ to a basis $\{x_i\}$ for $CFK^\infty(K)$, and conditions (1) and (4) above are satisfied. To satisfy the remaining three conditions, we will use a change of basis in order to remove the unwanted arrows.

There are no vertical arrows leaving x_0, since it is in the kernel of the vertical differential. Since x_0 is not in the image of the vertical differential, if there is an

incoming vertical arrow to x_0 from, say, y, then there is also a vertical arrow from y to, say, z. Changing basis to replace z with $z + x_0$ will remove the vertical arrow to x_0. All of the incoming vertical arrows to x_0 may be removed in this manner, and filtration considerations ensure that we have not changed x_1 or x_2.

Since x_0 is in the image of ∂^{horz}, it follows immediately that there are no horizontal arrows leaving x_0, by the fact that $\partial^{\text{horz}} \circ \partial^{\text{horz}} = 0$. We must now remove any horizontal arrows entering x_0. Suppose there is an arrow of length ℓ from y to x_0. If $\ell < a_1$, we may remove the arrow as in the preceding paragraph. If $\ell \geq a_1$, then we replace y with $y + x_1$. In this manner, we can remove all of other horizontal arrows into x_0.

There are now no horizontal arrows entering x_1, because $\partial^{\text{horz}} x_1 = x_0$, $\partial^{\text{horz}} \circ \partial^{\text{horz}} = 0$, and there are no other horizontal arrows to x_0.

We may remove unwanted vertical arrows involving x_1 and x_2 in the same manner that we removed unwanted horizontal arrows involving x_1 and x_0. $\square$

Note that if we have such a basis $\{x_i\}$ for $CFK^\infty(K)$, then we have a basis $\{x_i^*\}$ for $CFK^\infty(K)^*$ satisfying the following:

- There is a horizontal arrow of length $a_1(K)$ from x_0^* to x_1^*.

- There is a vertical arrow of length $a_2(K)$ from x_2^* to x_1^*.

- There are no other horizontal or vertical arrows to or from x_0^*.

- There are no other horizontal or vertical arrows to or from x_1^*.

- There are no other vertical arrows to or from x_2^*.

If x_k has filtration level (i, j), then x_k^* has filtration level $(-i, -j)$. We will use these types of bases to prove the following lemmas:

Lemma 6.3. *If $a_1(J) > a_1(K)$, then*

$$\llbracket K \rrbracket \gg \llbracket J \rrbracket.$$

Proof. We proceed using induction. We will show that $\varepsilon(K \# - J) = 1$ and that

$$a_1(K \# - J) = a_1(K)$$

from which we can conclude that

$$\varepsilon(K \# - nJ) = 1$$

for all $n \in \mathbb{N}$.

Let $\{x_i\}$ be a basis for $CFK^\infty(K)$ found using the first part of Lemma 6.2. Similarly, let $\{y_i\}$ be such a basis for $CFK^\infty(J)$, and hence $\{y_i^*\}$ is a basis for $CFK^\infty(-J)$. We consider the knot $K \# - J$ and its knot Floer complex. Notice that

$x_0 y_0^*$ generates $H_*(C\{i = 0\})$, the "vertical" homology of $CFK^\infty(K\# - J)$. Let $\tau = \tau(K\# - J)$.

Consider the subquotient complex

$$A = C\{\min(i, j - \tau) = 0\}.$$

There is a direct summand of A consisting of generators $x_0 y_0^*$ and $x_1 y_0^*$, with a horizontal arrow of length $a_1(K)$ from the latter to the former. Hence, $\varepsilon(K\# - J) = 1$ and $a_1(K\# - J) = a_1(K)$, as desired. $\qquad\square$

Lemma 6.4. *If $a_1(J) = a_1(K)$ and $a_2(J) > a_2(K)$, then*

$$[\![J]\!] \gg [\![K]\!].$$

Proof. We again proceed using induction. We will show that $\varepsilon(J\# - K) = 1$ and that

$$a_1(J\# - K) = a_1(J),$$
$$a_2(J\# - K) = a_2(J),$$

from which we can conclude that

$$\varepsilon(J\# - nK) = 1$$

for all $n \in \mathbb{N}$.

Let $\{x_i\}$ be a basis for $CFK^\infty(K)$ found using Lemma 6.2. Similarly, let $\{y_i\}$ be such a basis for $CFK^\infty(J)$. We consider the knot $J\# - K$ and its knot Floer complex. For ease of notation, let $\tau = \tau(J\# - K)$.

Let

$$A = C\{\min(i, j - \tau) = 0, i \le a_1(J)\},$$
$$B = C\{\{\min(i, j - \tau) = 0, i \le a_1(J)\} \cup \{i = a_1(J), \tau - a_2(J) \le j < \tau\}\}.$$

We claim that the element $x_0^* y_0 + x_1^* y_1$ generates $H_*(C\{i = 0\})$, is zero in $H_*(A)$, and is non-zero in $H_*(B)$. Indeed, there is a direct summand of B with the following generators in the following (i, j)-positions:

$$
\begin{array}{ll}
x_0^* y_0, \ x_1^* y_1 & \big(0, \tau(J\# - K)\big), \\
x_0^* y_1 & \big(a_1(J), \tau(J\# - K)\big), \\
x_0^* y_2 & \big(a_1(J), \tau(J\# - K) - a_2(J)\big), \\
x_2^* y_1 & \big(0, \tau(J\# - K) + a_2(K)\big),
\end{array}
$$

and the following differentials:

$$\partial(x_0^* y_1) = x_0^* y_0 + x_1^* y_1 + x_0^* y_2,$$
$$\partial(x_2^* y_1) = x_1^* y_1.$$

See Figure 5 (d). From this observation, the claim readily follows; that is,

$$\varepsilon(J\# - K) = 1,$$
$$a_1(J\# - K) = a_1(J),$$
$$a_2(J\# - K) = a_2(J),$$

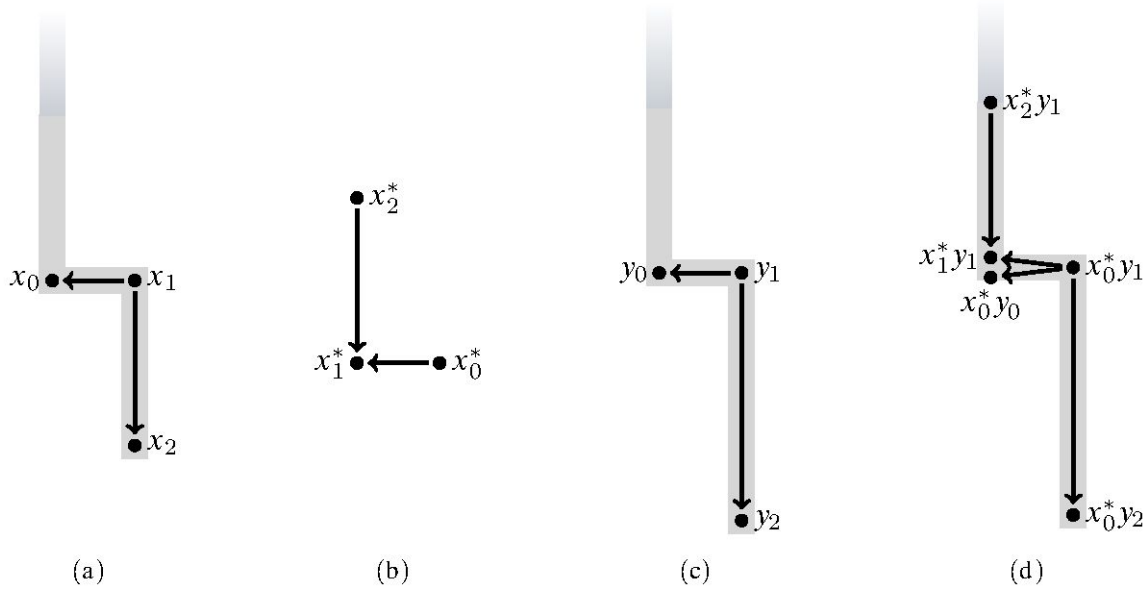as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Figure 5. Far left, a portion of the basis $\{x_i\}$ for $CFK^\infty(K)$, followed by a portion of the basis $\{x_i^*\}$ for $CFK^\infty(K)^*$. Next, a portion of the basis $\{y_i\}$ for $CFK^\infty(J)$. Far right, a direct summand of the subquotient complex B.

Recall that an *L-space* is a rational homology sphere Y for which

$$\operatorname{rk} \widehat{HF}(Y) = |H_1(Y, \mathbb{Z})|.$$

We call a knot $K \subset S^3$ an *L-space knot* if there exists $n \in \mathbb{N}$ such that n-surgery on K yields an L-space. In [OS05], Theorem 1.2, Ozsváth and Szabó prove that if K is an L-space knot, then the complex $CFK^\infty(K)$ has a particularly simple form that can be deduced form the Alexander polynomial of K, $\Delta_K(t)$. (Note that the results in [OS05] are stated in terms of $\widehat{HFK}(K)$, but by considering gradings, they are actually sufficient to determine the full $CFK^\infty(K)$ complex.)

One consequence is that if K is an L-space knot, then the Alexander polynomial of K has the form

$$\Delta_K(t) = \sum_{i=0}^{k} (-1)^i t^{n_i},$$

for some decreasing sequence of non-negative integers $n_0 > n_1 > \cdots > n_k$ with the symmetry condition

$$n_i + n_{k-i} = 2g(K),$$

where we have normalized the Alexander polynomial to have a constant term and no negative exponents. Note that k is always even since there are always an odd number of terms in the Alexander polynomial.

Lemma 6.5. *Let K be an L-space knot with Alexander polynomial*

$$\Delta_K(t) = \sum_{i=0}^{k} (-1)^i t^{n_i},$$

for some decreasing sequence of integers $n_0 > n_1 > \cdots > n_k$. Then

$$a_1(K) = n_0 - n_1,$$
$$a_2(K) = n_1 - n_2.$$

Proof. Theorem 1.2 of [OS05] tells us that for K an L-space knot, $\widehat{HFK}(K)$ is completely determined by $\Delta_K(t)$. Moreover, up to filtered chain homotopy equivalence, $CFK^\infty(K)$ is generated as a $\mathbb{F}[U, U^{-1}]$-module by $\widehat{HFK}(K)$, where $\widehat{HFK}$ is the homology of the associated graded object of $\widehat{CFK}(K) \simeq C\{i = 0\}$. By considering the gradings on the complex $CFK^\infty(K)$, and the fact that the differential decreases the Maslov grading by one, the lemma follows. $\square$

Remark 6.6. More generally, it can be deduced from Theorem 1.2 in [OS05] that there is a basis $\{x_0, \ldots x_k\}$ for $CFK^\infty(K)$ such that

$$\partial x_i = x_{i-1} + x_{i+1} \quad \text{for } i \text{ odd,}$$
$$\partial x_i = 0 \quad\quad\quad\quad\ \text{ otherwise,}$$

where the arrow from x_i to x_{i-1} is horizontal of length $n_i - n_{i-1}$, and the arrow from x_i to x_{i+1} is vertical of length $n_{i+1} - n_i$. The complex looks like a "staircase", where the differences of the n_i give the heights and widths of the steps. See Figure 6.

Recall that positive torus knots are L-space knots since $(pq \pm 1)$-surgery on the torus knot $T_{p,q}$, $p, q > 1$, results in a lens space.

Lemma 6.7. *For $p \geq 3$, the Alexander polynomial of the torus knot $T_{p,p+1}$ is*

$$\Delta_{T_{p,p+1}}(t) = \sum_{i=0}^{k} (-1)^i t^{n_i},$$

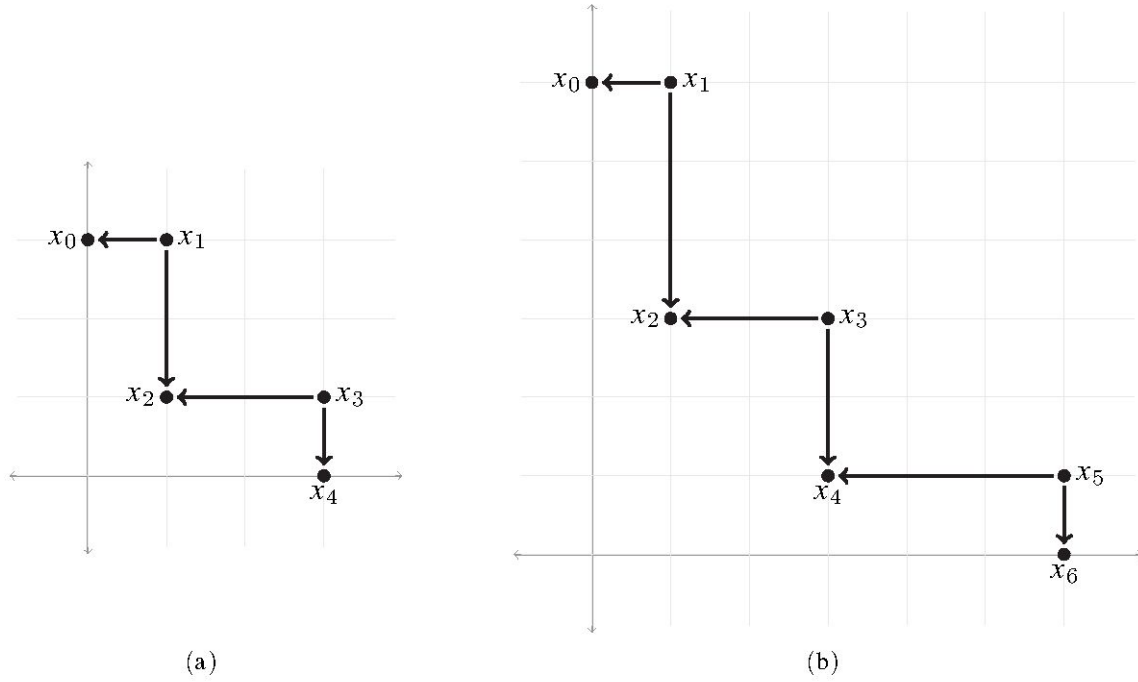(a)   (b)

Figure 6. Left, the basis from Remark 6.6 for CFK^∞ of the torus knot $T_{3,4}$ with Alexander polynomial $\Delta_{T_{3,4}}(t) = t^6 - t^5 + t^3 - t + 1$. Right, the basis for CFK^∞ of the torus knot $T_{4,5}$ with Alexander polynomial $\Delta_{T_{4,5}}(t) = t^{12} - t^{11} + t^8 - t^6 + t^4 - t + 1$. The lengths of the differentials are given by the differences of the exponents of the Alexander polynomial.

for a decreasing sequence of integers $n_0 > n_1 > \cdots > n_k$ *with*

$$n_0 = p^2 - p,$$
$$n_1 = p^2 - p - 1,$$
$$n_2 = p^2 - 2p,$$
$$n_3 = p^2 - 2p - 2.$$

In particular,

$$a_1(T_{p,p+1}) = 1,$$
$$a_2(T_{p,p+1}) = p - 1.$$

Proof. Recall that

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

Following the proof of Proposition 6.1 in [HLR12], we see that

$$\frac{(t^{p(p+1)} - 1)(t - 1)}{(t^p - 1)(t^{p+1} - 1)} = \sum_{i=0}^{p-1} t^{pi} - t \sum_{i=0}^{p-2} t^{(p+1)i}.$$

Indeed, multiplying both sides by $(t^p - 1)(t^{p+1} - 1)$, we obtain two telescoping sums on the right-hand side:

$$(t^p - 1)(t^{p+1} - 1)\left(\sum_{i=0}^{p-1} t^{pi} - t \sum_{i=0}^{p-2} t^{(p+1)i}\right)$$
$$= (t^{p+1} - 1)(t^{p(p-1)+p} - 1) - t(t^p - 1)(t^{(p+1)(p-2)+p+1} - 1)$$
$$= t^{p^2+p+1} - t^{p^2+p} - t + 1$$
$$= (t^{p(p+1)} - 1)(t - 1)$$

as desired.

The last statement now follows from Lemma 6.5.   □

Remark 6.8. For the torus knot $T_{2,3}$, i.e., the case where $p = 2$, we easily see that

$$a_1(T_{2,3}) = 1,$$
$$a_2(T_{2,3}) = 1,$$

since $\Delta_{T_{2,3}}(t) = t^2 - t + 1$.

Remark 6.9. More generally, for the torus knot $T_{p,p+1}$, the horizontal arrows increase in length by one at each "step", from 1 to $p - 1$, and the vertical arrows decrease in length by one at each "step", from $p - 1$ to 1. See Figure 6.

Lemma 6.10. *The iterated torus knot $T_{2,3;p,p+1}$, $p \geq 2$, is an L-space knot with Alexander polynomial*

$$\Delta_{T_{2,3;p,p+1}}(t) = \sum_{i=0}^{k} (-1)^i t^{n_i},$$

for a decreasing sequence of integers $n_0 > n_1 > \cdots > n_k$ with

$$n_0 = p^2 + p,$$
$$n_1 = p^2 + p - 1,$$
$$n_2 = p^2 - 1.$$

In particular,

$$a_1(T_{2,3;p,p+1}) = 1,$$
$$a_2(T_{2,3;p,p+1}) = p.$$

Proof. The fact that $T_{2,3;\ p,p+1}$ is an L-space knot follows from Theorem 1.10 of [Hed09] (cf. [Hom11]), where Hedden gives sufficient conditions for the cable of an L-space knot to again be an L-space knot.

The form of the Alexander polynomial follows from the formula for the Alexander polynomial of the cable of knot, i.e.,

$$\Delta_{T_{2,3;p,p+1}}(t) = \Delta_{T_{2,3}}(t^p) \cdot \Delta_{T_{p,p+1}}(t),$$

and Lemma 6.7. More precisely, for $p \geq 3$,

$$\Delta_{T_{2,3;p,p+1}} = (t^{2p} - t^p + 1)(t^{p^2-p} - t^{p^2-p-1} + t^{p^2-2p} - t^{p^2-2p-2}$$
$$+ \text{lower order terms})$$
$$= t^{p^2+p} - t^{p^2+p-1} + t^{p^2-1} + \text{lower order terms}.$$

The case $p = 2$ follows easily from the fact that

$$\Delta_{T_{2,3;2,3}}(t) = t^6 - t^5 + t^3 - t + 1.$$

$\square$

Lemma 6.11. *For $p \geq 2$, $m \geq p^2 - p - 1$, and $m \neq 1$, the iterated torus knot $T_{p,p+1;2,2m+1}$ is an L-space knot with Alexander polynomial*

$$\Delta_{T_{p,p+1;2,2m+1}}(t) = \sum_{i=0}^{k} (-1)^i t^{n_i},$$

for a decreasing sequence of integers $n_0 > n_1 > \cdots > n_k$ with

$$n_0 = 2p^2 - 2p + 2m,$$
$$n_1 = 2p^2 - 2p + 2m - 1,$$
$$n_2 = 2p^2 - 4p + 2m.$$

In particular,

$$a_1(T_{2,3;p,p+1}) = 1,$$
$$a_2(T_{2,3;p,p+1}) = 2p - 1.$$

Proof. This iterated torus knot is an L-space knot by [Hed09], Theorem 1.10. The form of the Alexander polynomial follows from the following facts:

$$\Delta_{T_{p,p+1;2,2m+1}}(t) = \Delta_{T_{p,p+1}}(t^2) \cdot \Delta_{T_{2,2m+1}}(t),$$
$$\Delta_{T_{2,2m+1}}(t) = \sum_{i=0}^{2m} (-1)^i t^i,$$

and Lemma 6.7. More precisely,

$$\Delta_{T_{p,p+1;2,2m+1}}(t)$$

$$= \Big(\sum_{i=0}^{p-1} t^{2pi} - t^2 \sum_{i=0}^{p-2} t^{(2p+2)i} \Big) \Big(\sum_{i=0}^{m} t^{2i} - \sum_{i=0}^{m-1} t^{2i+1} \Big)$$

$$= \Big(t^{2p^2-2p} - t^{2p^2-2p-2} + \sum_{i=0}^{p-2} t^{2pi} - t^2 \sum_{i=0}^{p-3} t^{(2p+2)i} \Big) \Big(\sum_{i=0}^{m} t^{2i} - \sum_{i=0}^{m-1} t^{2i+1} \Big)$$

$$= t^{2p^2-2p+2m} - t^{2p^2-2p-2} - t^{2p^2-2p+2m-1} + t^{2p^2-2p-1}$$

$$+ \Big(\sum_{i=0}^{p-2} t^{2pi} - t^2 \sum_{i=0}^{p-3} t^{(2p+2)i} \Big) \Big(\sum_{i=0}^{m} t^{2i} - \sum_{i=0}^{m-1} t^{2i+1} \Big)$$

$$= t^{2p^2-2p+2m} - t^{2p^2-2p-2} - t^{2p^2-2p+2m-1} + t^{2p^2-2p-1}$$

$$+ \big(t^{2p^2-4p+2m} + \text{lower order terms} \big)$$

$$= t^{2p^2-2p+2m} - t^{2p^2-2p+2m-1} + t^{2p^2-4p+2m} + \text{lower order terms},$$

where the last equality follows from the hypothesis that $m > p$.   □

Recall that D denotes the (positive, untwisted) Whitehead double of the right-handed trefoil.

Lemma 6.12. *As elements of the group* $\mathcal{F}$,

$$[\![D]\!] = [\![T_{2,3}]\!].$$

Proof. In [Hed07], Theorem 1.2, Hedden determines the $\mathbb{Z}$-filtered chain homotopy type of $\widehat{CFK}$ of the Whitehead double of K in terms of $\widehat{CFK}(K)$. We can use this result to determine $\widehat{CFK}(D)$, from which we will deduce the class $[\![D]\!]$ using rank and grading considerations.

Using Hedden's result, we see that

$$\widehat{CFK}(D, j) \simeq \begin{cases} \mathbb{F}^2_{(0)} \oplus \mathbb{F}^2_{(-1)}, & j = 1, \\ \mathbb{F}^3_{(-1)} \oplus \mathbb{F}^4_{(-2)}, & j = 0, \\ \mathbb{F}^2_{(-2)} \oplus \mathbb{F}^2_{(-3)}, & j = -1, \end{cases}$$

where the subscript denotes the Maslov, or homological, grading, and j denotes the Alexander grading. Moreover, Hedden proves that every non-trivial differential on this complex lowers the Alexander grading by exactly one, which is sufficient to completely determine the $\mathbb{Z}$-filtered chain homotopy type of $\widehat{CFK}(D)$. Note that $\tau(D) = 1$.

Let x be a generator of $\widehat{HF}(S^3) \cong H_*(C\{i = 0\})$. Note that x necessarily is positioned at $(0, 1)$ in the (i, j)-plane. Then $[x]$ must be zero in $H_*(C\{j = 1\})$ since the homology of $C\{j = 1\}$ is supported in i-coordinate 2. By considering the support of $\widehat{CFK}(D)$, we see that x is in the kernel of ∂^{horz}, so in order to vanish in $H_*(C\{j = 1\})$, it must be in the image of ∂^{horz}, i.e., there exists a class, say y, positioned at $(1, 1)$, such that

$$\partial^{\text{horz}} y = x.$$

The class $[y]$ is equal to zero in $H_*(C\{i = 1\})$ since the homology of $C\{i = 1\}$ is supported in j-coordinate 2. But y cannot be in the image of the differential on $C\{i = 1\}$, since $\partial^2 = 0$, where ∂ is the differential on CFK^∞, and $\partial^{\text{horz}} y \neq 0$. Hence, the boundary of y in $C\{i = 1\}$ must be non-zero; denote this boundary by z. Notice that z has (i, j)-coordinates $(1, 0)$.

Again, for $\partial^2 = 0$ reasons, the boundary of z in $C\{j = 0\}$ must be zero, and by grading considerations, z is not in the image of the differential on $C\{j = 0\}$.

The complex $CFK^\infty(-T_{2,3})$ is generated over $\mathbb{F}[U, U^{-1}]$ by

$$a, \quad b, \quad c,$$

with the differential

$$\partial a = b,$$
$$\partial c = b,$$

where the generators are have the following (i, j)-coordinates:

$$a \quad (0, 1),$$
$$b \quad (0, 0),$$
$$c \quad (1, 0).$$

Then in the tensor product

$$CFK^\infty(-T_{2,3}) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(D)$$

the generator

$$az + by + cx$$

is non-trivial in both vertical and horizontal homology. Indeed, it is clearly in the kernel of the vertical differential, and cannot be in the image of the vertical differential, since cx does not appear in the vertical boundary of any element. Similarly, it is in the kernel but not the image of the horizontal differential.

Thus,

$$\varepsilon\big(CFK^\infty(-T_{2,3}) \otimes_{\mathbb{F}[U,U^{-1}]} CFK^\infty(D)\big) = 0,$$

as desired. $\square$

We are now ready to prove Proposition 4.8, showing that we have the following relations in $\mathscr{F}$, where $0 < p < q$:

- $[\![T_{p,p+1}]\!] \ll [\![T_{q,q+1}]\!]$,
- $[\![D_{p,p+1}]\!] \ll [\![D_{q,q+1}]\!]$,
- $[\![T_{p,p+1}]\!] \ll [\![D_{p,p+1}]\!]$,
- $[\![T_{p,p+1;2,2m+1}]\!] \ll [\![T_{q,q+1;2,2m+1}]\!]$, for $m \geq q^2 - q - 1$.

Proof of Proposition 4.8. The proposition is now an easy consequence of the preceding lemmas. We have from Lemma 6.7 that

$$a_1(T_{p,p+1}) = 1,$$
$$a_2(T_{p,p+1}) = p - 1.$$

Now Lemma 6.4 states that if $a_1(J) = a_1(K)$ and $a_2(J) < a_2(K)$, then $[\![J]\!] \ll [\![K]\!]$, implying that

$$[\![T_{p,p+1}]\!] \ll [\![T_{q,q+1}]\!],$$

which proves the first assertion in the proposition.

From Lemma 6.12, we have that

$$[\![D]\!] = [\![T_{2,3}]\!],$$

and from Proposition 5.1 that

$$[\![D_{p,p+1}]\!] = [\![T_{2,3;p,p+1}]\!].$$

Hence by Lemmas 6.1 and 6.10,

$$a_1(D_{p,p+1}) = 1,$$
$$a_2(D_{p,p+1}) = p,$$

so by Lemma 6.4,

$$[\![D_{p,p+1}]\!] \ll [\![D_{q,q+1}]\!] \quad \text{and} \quad [\![T_{p,p+1}]\!] \ll [\![D_{p,p+1}]\!].$$

Finally, by Lemma 6.11, we have that

$$a_1(T_{p,p+1;2,2m+1}) = 1,$$
$$a_2(T_{p,p+1;2,2m+1}) = 2p - 1,$$

for $p \geq 2$, $m \geq p^2 - p - 1$, $m \neq 1$, and so

$$[\![T_{p,p+1;2,2m+1}]\!] \ll [\![T_{q,q+1;2,2m+1}]\!].$$

This completes the proof of the proposition. $\square$

We conclude this paper by showing that our examples, $\{D_{p,p+1}\# - T_{p,p+1}\}_{p\geq 2}$, of smoothly independent, topologically slice knots are smoothly independent from the examples of Endo [End95] and Hedden–Kirk [HK12]. Recall that Endo's examples are pretzel knots of the form

$$K_t = K(-2t - 1, 4t + 1, 4t + 3), \quad t \geq 1.$$

In particular, they are of genus one. The examples of Hedden–Kirk are (positive, untwisted) Whitehead doubles of certain torus knots.

Proposition 6.13. *If K is a knot of genus one and $\varepsilon(K) = 1$, then either*

$$a_1(K) \neq 1 \quad \text{or} \quad a_1(K) = a_2(K) = 1.$$

Proof. Notice that the assumption that $\varepsilon(K) = 1$ does not cause any loss of generality, since $\varepsilon(-K) = -\varepsilon(K)$.

Assume that $a_1(K) = 1$. We first notice that if K is a knot of genus one and $\varepsilon(K) = 1$, then $\tau(K) \neq -1$. This follows from the adjunction inequality for knot Floer homology [OS04], Theorem 5.1, and the basis from Lemma 6.2

Now, suppose $a_1(K) = 1$ and $\tau(K) = 0$. Using the adjunction inequality [OS04], Theorem 5.1, and a basis found using the first part of Lemma 6.2, we see that the basis element x_1 must be in the kernel of the differential on $C\{i = 1\}$. Moreover, for $\partial^2 = 0$ reasons, it cannot be in the image of the differential on $C\{i = 1\}$. But $[x_1]$ cannot be zero in $H_*(C\{i = 1\}$, because $\tau(K) = 0$ implies that $H_*(C\{i = 1\})$ is supported in (i, j)-coordinate $(1, 1)$.

Hence, we may assume that $a_1(K) = 1$ and $\tau(K) = 1$, in which case the arguments in the proof of Lemma 6.12 lead us to the desired result. $\square$

In the proof of Proposition 4.8, we showed that

$$a_1(D_{p,p+1}\# - T_{p,p+1}) = 1,$$
$$a_2(D_{p,p+1}\# - T_{p,p+1}) = p,$$

Hence, by Proposition 6.13, along with Lemmas 6.3 and 6.4, it follows that when $p > 1$, our examples are independent from those of Endo and Hedden–Kirk.

The following proposition describes the subgroup of $\mathcal{F}$ generated by Whitehead doubles:

Proposition 6.14. *Whitehead doubles are contained in the rank one subgroup of $\mathcal{F}$ generated by the right-handed trefoil.*

Proof. The argument in Lemma 6.12 can be used to show that for a Whitehead double WD with $\varepsilon(WD) = 1$, the class $[\![WD]\!] = [\![T_{2,3}]\!]$ in $\mathcal{F}$. This is sufficient for the result, since $\varepsilon(WD) = -1$ implies that $\varepsilon(-WD) = 1$, and $\varepsilon(WD) = 0$ implies that $[\![WD]\!] = 0$. $\square$

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Jennifer Hom, Department of Mathematics, Columbia University, New York, NY 10027, U.S.A.

E-mail: hom@math.columbia.edu