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The action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures

Egor Shelukhin

Abstract. We extend the definition of Weinstein's action homomorphism to Hamiltonian actions with equivariant moment maps of (possibly infinite-dimensional) Lie groups on symplectic manifolds, and show that under conditions including a uniform bound on the symplectic areas of geodesic triangles the resulting homomorphism extends to a quasimorphism on the universal cover of the group. We apply these principles to finite-dimensional Hermitian Lie groups like the linear symplectic group, reinterpreting the Guichardet–Wigner quasimorphisms, and to the infinite-dimensional groups of Hamiltonian diffeomorphisms of closed symplectic manifolds that act on the space of compatible almost complex structures with an equivariant moment map given by the theory of Donaldson and Fujiki. We show that the quasimorphism on the universal cover of the Hamiltonian group obtained in the second case is symplectically conjugation-invariant and compute its restrictions to the fundamental group via a homomorphism introduced by Lalonde–McDuff–Polterovich, answering a question of Polterovich; to the subgroup of Hamiltonian biholomorphisms via the Futaki invariant; and to subgroups of diffeomorphisms supported in an embedded ball via the Barge–Ghys average Maslov quasimorphism, the Calabi homomorphism and the average Hermitian scalar curvature. We show that when the first Chern class vanishes this quasimorphism is proportional to a quasimorphism of Entov and when the symplectic manifold is monotone, it is proportional to a quasimorphism due to Py. As an application we show that a Sobolev distance on the universal cover of the Hamiltonian group is unbounded, similarly to the results of Eliashberg–Ratiu.

Mathematics Subject Classification (2010). 53D35, 53D20, 58B25.

Keywords. Quasimorphism, moment map, Hamiltonian diffeomorphism, Hermitian scalar curvature, Hermitian Lie group, compatible almost complex structures, action homomorphism.

Contents

1. Introduction and main results

1.1. Introduction. In [6] Barge and Ghys have introduced a quasimorphism on the fundamental groups Γ of surfaces of genus $g \geq 2$ (cf. [75]). Their construction uses in a fundamental way the discrete action of Γ by isometries on the hyperbolic upper half-plane $\mathbb{H}$. Indeed, choosing a Γ-invariant one-form α on $\mathbb{H}$ whose differential is bounded in the way $|d\alpha| \leq C_\alpha |\sigma_{\mathbb{H}}|$ for a constant C_α with respect to the hyperbolic Kähler form $\sigma_{\mathbb{H}}$ on $\mathbb{H}$, the quasimorphism is given by integrating α over the geodesic $l(x, \gamma \cdot x)$ between a fixed base-point x and its image $\gamma \cdot x$ under the action of an element $\gamma \in \Gamma$. Using these quasimorphisms Barge and Ghys have obtained results on the second bounded cohomology $H_b^2(\Gamma)$ of such groups Γ. Further results on the second bounded cohomology of discrete groups following from their actions upon certain spaces with "negative enough" curvature – e.g. Gromov-hyperbolic groups – were studied extensively in [40], [45], [56], [57], [70] to name a few works in such a direction. The second bounded cohomology of finite-dimensional Lie groups was also studied extensively. For example, in the works [55], [34] and others, the action of simple Hermitian symmetric Lie groups G upon their symmetric space $X = G/K$ of non-compact type was utilized to construct bounded 2-cocycles on G. The basic construction of such cocycles similarly uses the integration of the natural Kähler form σ_X on X on simplices with geodesic boundaries.

We shall first formulate a general setting in terms of the action of a group $\mathcal{G}$ on a space $\mathcal{X}$ for constructions related to integration on geodesic simplices to yield bounded 2-cocycles. Then we formulate a general principle, again in terms of such actions, for the construction of primitives to such cocycles in the (unbounded) group cohomology, to wit – quasimorphisms – functions that satisfy the homomorphism property up to a uniformly bounded error. For one, our construction gives a symplectic formula for the quasimorphisms on the universal covers $\widetilde{G}$ of simple Hermitian symmetric Lie groups whose differentials equal the Guichardet–Wigner cocycles (cf. [55], [34], [25], [85], [17]). A key notion in our construction is the use of *equivariant moment maps* for the Hamiltonian action of a group $\mathcal{G}$ on a space $\mathcal{X}$ with a symplectic form Ω. Another key notion is that of the action homomorphism of A. Weinstein [92] that generalizes to general Hamiltonian actions with equivariant moment maps. As our construction is rather formal, or "soft" in the terminology of Gromov [54] in that it does not require the solution of partial differential equations or the convergence of certain series, it readily applies to the infinite-dimensional case.

Indeed there have been many constructions of equivariant moment maps for actions of infinite-dimensional Lie groups on infinite-dimensional symplectic spaces $(\mathcal{X}, \Omega)$. Starting with the work of Atiyah and Bott [4], [3] – for the action of gauge groups of principal bundles over Riemann surfaces on the corresponding spaces of connections, with numerous later developments including an extension to higher dimensions – a general framework for the Hitchin–Kobayashi correspondence [28], [88], [29], the works of Donaldson [32], [31], [30] and Fujiki [44] for actions of diffeomorphism groups upon spaces of mappings (submanifolds or sections of bundles), and more recent advances e.g. [47], [41] this has been an active and fruitful area of research for over three decades, with many applications – for example to Kähler geometry. Of these the Donaldson–Fujiki [30], [44] framework of the scalar curvature as a moment map for the action of the Hamiltonian group on the space of compatible almost complex structures fits the setting of our construction. We shall, therefore, apply this framework to build new quasimorphisms on the Hamiltonian group, or its universal cover, of an arbitrary symplectic manifold of finite volume (and of an arbitrary closed symplectic manifold in particular). Similarly to the finite-dimensional case, our quasimorphism provides a group-cohomological primitive for the restriction to the Hamiltonian group of a certain 2-cocycle that was constructed using the natural notion of geodesic simplices in spaces of almost complex structures by Reznikov [81], [80], [82] in his studies of the cohomology of the group of symplectomorphisms.

The intriguing topic of the study of quasimorphisms on groups of (Hamiltonian) symplectomorphisms has a long history. A very early work of Eugenio Calabi [20] constructs a homomorphism on the group of compactly supported symplectomorphisms of the symplectic ball of arbitrary dimension $2n$. An early example of a quasimorphism on a symplectomorphism group that is not a homomorphism was constructed by Ruelle [83] on the group of compactly supported volume preserving diffeomorphisms of the two-dimensional disk, as a certain average asymptotic rotation number. This result was generalized using the Maslov quasimorphism on the universal cover $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ of the linear symplectic group by Barge and Ghys [7] to the group of compactly supported symplectomorphisms of the symplectic ball of arbitrary dimension $2n$. A quasimorphism on the universal cover $\widetilde{\mathrm{Symp}}(M,\omega)$ of closed symplectic manifolds (M,ω) with $c_1(TM,\omega) = 0$ was rather recently constructed by Entov [36], generalizing the previous quasimorphism in the sense that it equals the Barge–Ghys average Maslov quasimorphism when restricted to each subgroup of diffeomorphisms supported in an embedded ball – we shall say that it has the Maslov local type. In a recent work of Py [78], [79] a quasimorphism on $\widetilde{\mathrm{Ham}}(M,\omega)$ for closed symplectic manifolds (M,ω) with $c_1(TM,\omega) = \kappa[\omega]$ for $\kappa \neq 0$ was constructed as a rotation number using the notion of a prequantization of an integral symplectic manifold. The local type of the Py quasimorphism is Calabi–Maslov – it equals a certain linear combination of the Calabi homomorphism and the Barge–Ghys average Maslov quasimorphism when evaluated on diffeomorphisms supported in a given embedded ball. A compelling discovery of quasimorphisms of Calabi local type was made by Entov and Polterovich in [37] – one distinctive feature of which is that the embedded balls should be small enough – using "hard" methods of Hamiltonian Floer homology and the algebraic properties of quantum homology. These methods were since generalized and extended to a large class of manifolds [72], [38], [73], [91], [89], a very recent result due to Usher [90] showing e.g. the existence of Calabi quasimorphisms on $\widetilde{\mathrm{Ham}}$ of every one-point blowup of a closed symplectic manifold. The sequent question of constructing a "soft" quasimorphism of Calabi local type on the Hamiltonian group of a closed symplectic manifold was recently solved for the two-torus and for surfaces of genus $g \geq 2$ by Py [78]. The first case builds upon the works of Ghys and Gambaudo [48], [49] in dimension 2 that describe the Calabi homomorphism and a large number of quasimorphisms, using such methods as the action of diffeomorphism groups upon the configuration spaces of distinct points in a surface (these works have been since developed in many other papers – cf. [14]). The second case uses prequantizations and the notion of the bounded Euler class (which is again related to the boundedness of the symplectic area of geodesic triangles), and can be extended to compact quotients of simple Hermitian symmetric spaces X of non-compact type by discrete groups of isometries [79]. Another quasimorphism on $\widetilde{\mathrm{Ham}}(M,\omega)$ for (M,ω) the complex projective space $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$ with the natural Fubini–Study Kähler form can be derived from the work of Givental [52] that uses methods of generating functions, which also has the Calabi property by the work of Ben Simon [9] and can easily be shown to descend to $\mathrm{Ham}(M,\omega)$ itself by results from [84]. In fact necessary and sufficient conditions for the above quasimorphisms on a group $\widetilde{\mathcal{G}}$ to descend to $\mathcal{G}$ are given by the vanishing of certain homomorphisms $\pi_1(\mathcal{G}) \to \mathbb{R}$. This happens automatically for surfaces where the fundamental group of $\mathcal{G} = \mathrm{Ham}(M,\omega)$ is finite, which is also known to be the case for certain four-dimensional symplectic manifolds – e.g. $(\mathbb{C}P^2, \omega_{\mathrm{FS}})$, $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{FS}})$ [53] (cf. [67]). Remarkably, for all monotone examples – (M,ω) such that $c_1(TM,\omega) = \kappa[\omega]$ for $\kappa \neq 0$ – the homomorphism is the same one [39] – the action-Maslov homomorphism of Polterovich [76] (cf. [84]).

The quasimorphism we construct has Calabi–Maslov local type – it restricts to the difference of suitable multiples of the Calabi homomorphism [20], [66] and of the Barge–Ghys average Maslov quasimorphism on the subgroup of Hamiltonian diffeomorphisms supported in a small ball. Its restriction to the fundamental group of $\mathcal{G}$ is equal by construction to the generalized action homomorphism, involving in this case the Hermitian scalar curvature, and is also computed via a homomorphism earlier introduced in [63] using a Hamiltonian fiber bundle obtained by the clutching construction. A previous work that applies the theory of the Hermitian scalar curvature as a moment map to the study of the topology of the Hamiltonian group is [1], [2].

Furthermore, our quasimorphism agrees with the quasimorphisms of Py and Entov whenever these quasimorphisms are defined. While, having a Maslov component in the local type, our quasimorphism can at best be continuous in the C^1-topology, it is rather easily seen to be coarse-Lipschitz in the Sobolev L^2_2-metric, using the isoperimetric property of Kähler manifolds with a bounded primitive of the Kähler form. This allows us to prove that the Sobolev L^2_2-metric is unbounded on $\widetilde{\mathcal{G}}$ of every symplectic manifold of finite volume, extending a consequence from previous works of Eliashberg–Ratiu [35] on the L^2_1-metric in the case when the symplectic manifold is exact. Moreover, we show that on manifolds like the blowup $\mathrm{Bl}_1(\mathbb{C}P^2)$, where the restriction of the quasimorphism to $\pi_1\mathcal{G}$ does not vanish, the metric is not bounded on $\pi_1\mathcal{G}$ either. We conclude with some questions and discussion related to the topics presented in the paper.

As an aside, it is curious to note that this paper touches upon two directions that both have their origins with Eugenio Calabi – the study of canonical metrics on Kähler manifolds (e.g. [19], [22], [21]) and the theory of the Calabi homomorphism ([20]).

1.2. Moment maps. Assume that a Lie group $\mathcal{G}$ acts $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$, $(g, x) \mapsto g \cdot x$, on a symplectic manifold $(\mathcal{X}, \Omega)$ in a Hamiltonian fashion. Here both the group and the manifold can be infinite-dimensional. The action gives a homomorphism $\mathcal{G} \to \mathrm{Diff}(\mathcal{X})$, $\phi \mapsto \bar{\phi}$, with the property that to each element $X \in \mathrm{Lie}(\mathcal{G})$ there corresponds an element $\mu(X) \in C^\infty(\mathcal{X}, \mathbb{R})$, such that

(1) the equation $\iota_\Xi \Omega = -d\mu(X)$ holds for $\Xi \in$ V.F.$(\mathcal{X})$ – the vector field on $\mathcal{X}$ corresponding to X

(2) the resulting map $\mathrm{Lie}(\mathcal{G}) \to C^\infty(\mathcal{X}, \mathbb{R})$ is a homomorphism of Lie algebras (the Lie structure on the latter is given by the Poisson bracket of the symplectic form Ω).

The second condition is equivalent to the linearity and *equivariance* of the map $X \mapsto \mu(X)$ – for all $X \in \mathrm{Lie}(\mathcal{G})$ and $\phi \in \mathcal{G}$ we have

$$\mu(\mathrm{Ad}_\phi X) = \mu(X) \circ \bar{\phi}^{-1}.$$

In one direction one differentiates this equality and the other can be found in [66], Lemma 5.16.

Note that the map $X \mapsto \mu(X)$ gives us a pairing $\mu\colon \mathrm{Lie}(\mathcal{G}) \times \mathcal{X} \to \mathbb{R}$ that is linear in the first variable, and therefore a map $x \mapsto \mu(-)(x)\colon \mathcal{X} \to (\mathrm{Lie}(\mathcal{G}))^*$. The equivariance condition corresponds to the invariance of the pairing with respect to the diagonal action of $\mathcal{G}$ – for all $X \in \mathrm{Lie}(\mathcal{G})$, $x \in \mathcal{X}$ and $\phi \in \mathcal{G}$ we have

$$\mu(\mathrm{Ad}_\phi X)(\phi \cdot x) = \mu(X)(x).$$

We call μ in any one of these three equivalent formulations a *moment map* for the Hamiltonian action of $\mathcal{G}$ on $\mathcal{X}$.

Remark 1.2.1. For infinite-dimensional Lie groups we use the approach of regular Fréchet Lie groups (cf. [68] and references therein), while one could also use the inverse limit (ILH or ILB) approach of Omori [71]. In any case, as we are interested only in the soft features of the theory of Lie groups and our infinite-dimensional example is a diffeomorphism group where all computations can be carried out as explicit differential-geometric formulae, the foundational theory of infinite-dimensional Lie groups can for the most part be ignored. The same remark applies to infinite-dimensional symplectic manifolds.

1.3. The action homomorphism. Assume that $\pi_1(\mathcal{X}) = 0$. Denote by $\mathcal{P}_\Omega \subset \mathbb{R}$ the spherical period group $\langle \Omega, \pi_2(\mathcal{X})\rangle$ of Ω. Following Weinstein [92], we define the action homomorphism $\pi_1(\mathcal{G}) \to \mathbb{R}/\mathcal{P}_\Omega$ as follows.

Suppose a class $a \in \pi_1(\mathcal{G})$ is represented by a path $\{\phi_t\}$ based at the identity element Id. Pick a point $x \in \mathcal{X}$. Consider its trace $\phi_x = \{\phi_t \cdot x\}_{t=0}^1$ under the action of the loop. Pick a disk D that spans ϕ_x, that is, $D\colon \mathbb{D} \to \mathcal{X}$ is a smooth map from $\mathbb{D} = \{|z| \le 1\} \subset \mathbb{C}$ to $\mathcal{X}$ that satisfies $D(e^{2\pi i t}) = \phi_t \cdot x$ for all $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Then the action homomorphism is defined as

$$\mathcal{A}_\mu(a) = \int_D \Omega - \int_0^1 \mu(X_t)(\phi_t \cdot x)\, dt.$$

It is independent of $x \in \mathcal{X}$ by the first property of μ and of $\{\phi_t\}$ in the homotopy class $a \in \pi_1(\mathcal{G}, \mathrm{Id})$ by the second property of μ. It does depend on the spanning disk D, however the ambiguity lies in $\mathcal{P}_\Omega$. At last, the homomorphism property follows by a short concatenation argument. Detailed proofs can be found in Section 2.

Remark 1.3.1. Note that when $\pi_2(X) = 0$, the action homomorphism takes values in $\mathbb{R}$, since $\mathcal{P}_\Omega = 0$.

Remark 1.3.2. This definition extends the original definition because given a closed symplectic manifold (M, ω), the group $\mathcal{G} = \mathrm{Ham}(M, \omega)$ acts on (M, ω) in a Hamiltonian fashion with the equivariant moment map $\mu(X) = H_X$ where $H_X \in C^\infty(M, \mathbb{R})$

is the zero-mean normalized Hamiltonian function of X. On an open symplectic manifold (M, ω) the group $\mathcal{G} = \mathrm{Ham}_c(M, \omega)$ of compactly supported Hamiltonian diffeomorphisms acts in a Hamiltonian fashion with the equivariant moment map $\mu(X) = H_X$ where H_X is the compact-support normalized Hamiltonian function of X. To ensure the existence of a contracting disk, we assume that the manifold is simply connected in the open case. In the closed case the contracting disk always exists by Floer theory, by the existence of the Seidel element or by a direct geometric degeneration argument [65].

1.4. Preliminaries on quasimorphisms. A *quasimorphism* ν on a group $\mathcal{G}$ is a function $\nu\colon \mathcal{G} \to \mathbb{R}$ that satisfies the additivity property up to a uniformly bounded error. That is for all $x \in \mathcal{G}$ and $y \in \mathcal{G}$ we have

$$\nu(xy) = \nu(x) + \nu(y) + b(x, y),$$

where

$$|b(x, y)| \leq C_\nu$$

for a constant C_ν depending only on ν (and not on x, y). In such cases the limit

$$\overline{\nu}(x) := \lim_{k\to\infty} \frac{1}{k} \nu(x^k)$$

exists by Fekete's lemma on subadditive sequences and is also a quasimorphism. Moreover, it is *homogenous* that is

$$\overline{\nu}(x^k) = k\ \overline{\nu}(x)$$

for all $x \in \mathcal{G}$ and $k \in \mathbb{Z}$ and satisfies

$$\overline{\nu} \simeq \nu,$$

where for any two functions $a, b\colon \mathcal{G}^m \to \mathbb{R}$ we write

$$a \simeq b \tag{1}$$

if they differ by a uniformly bounded function $d\colon \mathcal{G}^m \to \mathbb{R}$ – that is $|d(x_1, \dots, x_m)| \leq C_d$ for a constant C_d independent of $x_1, \dots, x_m$. We refer to the book [23] by Calegari for these statements and for additional information about quasimorphisms.

We will use the following simple fact.

Lemma 1.4.1. *For every quasimorphism $\nu\colon \mathcal{G} \to \mathbb{R}$ we have $\nu(x) \simeq -\nu(x^{-1})$ as functions $\mathcal{G} \to \mathbb{R}$.*

Proof. Indeed $\nu(x) \simeq \overline{\nu}(x) = -\overline{\nu}(x^{-1}) \simeq -\nu(x^{-1})$. □

Explicit constructions of quasimorphisms on Lie groups often use rotation numbers. For this purpose we require the notion of the variation of angle of a continuous path $\delta\colon [0,1] \to S^1$.

Definition 1.4.1. We define the full *variation of angle* of $\delta\colon [0,1] \to S^1$ as

$$\mathrm{varangle}(\delta) = \tilde{\delta}(1) - \tilde{\delta}(0)$$

for any continuous lift $\tilde{\delta}\colon [0,1] \to \mathbb{R}$ of δ to the universal cover $\mathbb{R} \xrightarrow{\mathbb{Z}} S^1$.

1.5. A general principle for constructing quasimorphisms. The general principle says that when groups act well enough on spaces of negative enough curvature, then they have quasimorphisms and non-trivial bounded (or bounded-continuous) cohomology. While usually this principle is applied to proper discontinuous actions of discrete groups, we propose a version of this principle for smooth actions of (possibly infinite-dimensional) Lie groups. Firstly, we propose a version of "negative enough curvature" – (possibly infinite-dimensional) symplectic manifolds $(\mathcal{X}, \Omega)$ with bounded Gromov norm of Ω. We make, more specifically, the following definition.

Definition 1.5.1 (Domic–Toledo space $(\mathcal{X}, \Omega, \mathcal{K})$). Assume that $\mathcal{X}$ has $\pi_1(\mathcal{X}) = 0$ (as before) and $\pi_2(\mathcal{X}) = 0$ also. Moreover assume that there is a system $\mathcal{K}$ of paths $[x, y] := \gamma(x, y)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{X}$, such that for all $x, y, z \in \mathcal{X}$

$$\left| \int_{\Delta(x,y,z)} \Omega \right| < C_{\mathcal{X}},$$

for a constant $C_{\mathcal{X}}$ that does not depend on x, y, z. Here $\Delta = \Delta(x, y, z)$, which we will call a *geodesic triangle* is any disk with boundary $\partial\Delta = [x, y] \cup [y, z] \cup [z, x]$. We call the triple $(\mathcal{X}, \Omega, \mathcal{K})$ a *Domic–Toledo* space.

Next we propose a version for "act well enough" – by "isometries" with an equivariant moment map. More exactly, we make the following definition.

Definition 1.5.2 (Hamiltonian–Hermitian group $\mathcal{G}$). We call a (possibly infinite-dimensional) Lie group $\mathcal{G}$ *Hamiltonian–Hermitian* if it acts on a Domic–Toledo space $(\mathcal{X}, \Omega, \mathcal{K})$ – preserving $\mathcal{K}$ and Ω – with an *equivariant moment map*

$$\mu\colon \mathcal{X} \times \mathrm{Lie}(\mathcal{G}) \to \mathbb{R}.$$

We say that the action of $\mathcal{G}$ on $(\mathcal{X}, \Omega, \mathcal{K})$ preserves $\mathcal{K}$ if for every two points $x \in \mathcal{X}$ and $y \in \mathcal{X}$ and every $g \in \mathcal{G}$ we have

$$g \cdot [x, y] = [g \cdot x, g \cdot y].$$

Remark 1.5.1. All examples of Domic–Toledo spaces known to the author are (possibly infinite-dimensional) Kähler manifolds $(\mathcal{X}, \Omega, J)$ with $[x, y]$ being the geodesic segment between $x \in \mathcal{X}$ and $y \in \mathcal{X}$. A first set of examples is given by Hermitian symmetric spaces $\mathcal{D}$ of non-compact type (bounded Hermitian domains) [27], [26]. The second one (trivially containing the first) is given by spaces of global sections of bundles with fiber $\mathcal{D}$ over a manifold (M, ϕ) with a volume form ϕ of finite volume.

Remark 1.5.2. Examples of finite-dimensional Hamiltonian–Hermitian groups are given by Hermitian symmetric Lie groups – like $\mathrm{Sp}(2n, \mathbb{R})$ – since they act by Hamiltonian biholomorphisms on the corresponding symmetric spaces of non compact type equipped with the Bergman Kähler structure, which is Kähler–Einstein. Therefore, the natural lift (by use of the differential) of these diffeomorphisms to the top exterior power of the tangent bundle furnishes the action with an equivariant moment map (note that the Kähler–Einstein condition implies that $(-i)$ times the curvature of the Chern connection on these bundles, given by the Hermitian metric, equals to the Kähler form on one hand, and on the other hand the corresponding connection form is surely preserved by the lifts). Details are presented in Section 1.9.

Infinite-dimensional examples are given by groups $\mathrm{Ham}(M, \omega)$ of closed symplectic manifolds (M, ω) since these act on the spaces $\mathcal{J}$ of compatible almost complex structures, which is a Domic–Toledo space – since it is the space of global sections of a bundle over (M, ω) with fiber the Siegel upper half-space. This class of examples can be extended to arbitrary symplectic manifolds of finite volume. Details are presented in Section 1.7.

We now construct a quasimorphism on the universal cover of a Hamiltonian–Hermitian group $\mathcal{G}$ with an equivariant moment map μ and Domic–Toledo space $(\mathcal{X}, \Omega, \mathcal{K})$. Given a path $\{g_t\}_{t=0}^1$ in $\mathcal{G}$ with $g_0 = \mathrm{Id}$, $g_1 = g$ representing a class $\tilde{g}$ in $\widetilde{\mathcal{G}}$, consider the loop $\{g_t \cdot x\}_{t=0}^1 \# [g \cdot x, x]$ for a fixed basepoint $x \in \mathcal{X}$. Fill it by any disk $D = D_{\{g_t\}_{t=0}^1}$. Then define

$$\nu_x(\tilde{g}) = \int_D \Omega - \int_0^1 \mu(X_t)(g_t \cdot x)dt, \tag{2}$$

where $\{X_t\}_{t=0}^1$ is the path in $\mathrm{Lie}(\mathcal{G})$ corresponding to the path $\{g_t\}_{t=0}^1$. In Section 2.2 we show that this value is well-defined and gives a real-valued quasimorphism $\nu_x \colon \widetilde{\mathcal{G}} \to \mathbb{R}$ on the universal cover of $\mathcal{G}$.

Theorem 1. *Any Hamiltonian–Hermitian group $\mathcal{G}$ acting with an equivariant moment map μ on the corresponding Domic–Toledo space $(\mathcal{X}, \Omega, \mathcal{K})$ admits a real-valued quasimorphism $\nu_x \colon \widetilde{\mathcal{G}} \to \mathbb{R}$ on its universal cover for each point $x \in \mathcal{X}$, given by Equation (2). Moreover, the homogeneization ν of ν_x does not depend on the basepoint x. By construction, the quasimorphism ν restricts to the homomorphism $\mathcal{A}_\mu$ on $\pi_1(\mathcal{G})$.*

Remark 1.5.3. If we assume additionally that the loop $[x, x] \in K$ is the constant path at x, then the quasimorphism ν_x also restricts to $\mathcal{A}_\mu$ on $\pi_1(\mathcal{G})$.

Note that this theorem does not state that the homogenous quasimorphism ν is necessarily not a homomorphism, or even not trivial. It can in principle be identically equal to zero. However, in all the known examples it turns out to be non-trivial and not a homomorphism.

The key feature of the proof which we defer to Section 2.2 is that the differential of ν_x in group cohomology satisfies

$$b(g,h) = \nu_x(\tilde{g}\tilde{h}) - \nu_x(\tilde{g}) - \nu_x(\tilde{h}) = \int_{\Delta(x, g\cdot x, gh\cdot x)} \Omega \tag{3}$$

for $\tilde{g}, \tilde{h} \in \tilde{\mathcal{G}}$ with endpoints $g, h \in \mathcal{G}$. The latter is a bounded cocycle by the properties of Domic–Toledo spaces and "isometric" actions upon them.

Remark 1.5.4. From Equation (3), given that for all $x \in \mathcal{X}$, $[x, x]$ is the constant path at x, it follows that for all $\tilde{\phi} \in \tilde{G}$ we have

$$\nu_x(\tilde{\phi}^{-1}) = -\nu_x(\tilde{\phi}).$$

Indeed the difference equals $\int_{\Delta(x,\phi\cdot x,x)} \Omega = 0$, since we can choose a degenerate filling disk.

Furthermore, we would like to explore the invariance of the quasimorphism with respect to larger groups extending a given action of a Hamiltonian–Hermitian group $\mathcal{G}$ on a Domic–Toledo space. For this we have the following proposition, which we prove in Section 2.2.

Proposition 1.5.1. *Assume $\mathcal{G} \subset \mathcal{H}$ is a normal subgroup, $\mathcal{G}$ is a Hamiltonian–Hermitian group acting with an equivariant moment map μ on the Domic–Toledo space $(\mathcal{X}, \Omega, \mathcal{K})$, and $\mathcal{H}$ is a (possibly infinite-dimensional) Lie group that acts on $(\mathcal{X}, \Omega, \mathcal{K})$ preserving Ω and $\mathcal{K}$ and extending the action of $\mathcal{G}$ (however not necessarily with a moment map). Assume moreover that the moment map $\mu\colon \mathrm{Lie}(\mathcal{G}) \times \mathcal{X} \to \mathbb{R}$ is equivariant with respect to the action of $\mathcal{H}$ (note that as $\mathcal{G} \subset \mathcal{H}$ is normal, $\mathcal{H}$ acts on $\mathrm{Lie}(\mathcal{G})$ by the adjoint representation). Then $\nu_x(h\tilde{g}h^{-1}) = \nu_{h^{-1}x}(\tilde{g})$ for all $\tilde{g} \in \tilde{\mathcal{G}}$ and $h \in \mathcal{H}$. Consequently, by the independence of the homogeneization upon the basepoint, we have*

$$\nu(h\tilde{g}h^{-1}) = \nu(\tilde{g}),$$

for all $\tilde{g} \in \tilde{\mathcal{G}}$ and $h \in \mathcal{H}$. Equivalently $\nu(\tilde{h}\tilde{g}\tilde{h}^{-1}) = \nu(\tilde{g})$, for all $\tilde{g} \in \tilde{\mathcal{G}}$ and $\tilde{h} \in \tilde{\mathcal{H}}$.

1.6. The scalar curvature as a moment map. Given a compact symplectic manifold (M,ω) consider the space $\mathcal{J}$ of ω-compatible almost complex structures. This space can be given the structure of an infinite-dimensional Kähler manifold $(\mathcal{J},\Omega,\mathbb{J})$ as follows. Consider the bundle $\mathcal{S}\to M$, the general fibre of which over $x\in M$ is the space $\mathrm{J}_c(T_xM,\omega_x)\cong \mathrm{Sp}(2n)/\mathrm{U}(n)$ of ω_x-compatible complex structures on T_xM. As J_c posses a canonical $\mathrm{Sp}(2n)$-invariant Kähler form $\sigma=\sigma_{\mathrm{trace}}$ we have a fiberwise-Kähler form σ on $\mathcal{S}$. Note now that $\mathcal{J}=\Gamma(M,\mathcal{S})$ – the space of global sections of the bundle $\mathcal{S}\to M$. Now define $\Omega(A,B):=\int_M\sigma_x(A_x,B_x)\omega^n(x)$. The complex structure $\mathbb{J}$ on $\mathcal{J}$ is defined as $\mathbb{J}_JA=JA$ for $A\in T_J\mathcal{J}$. Surely Ω and $\mathbb{J}$ are compatible.

Note that the group $\mathcal{G}=\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms acts on $\mathcal{J}$ by $\phi\cdot J:=\phi_*J$. This action can be shown to be Hamiltonian [30], [44] with respect to the form Ω. The moment map is given as follows.

First note that the Lie algebra of $\mathcal{G}$ is isomorphic to the space $C^\infty(M,\mathbb{R})/\mathbb{R}\cong C_0^\infty(M,\mathbb{R})$. The latter space consists of smooth functions F on M with integral zero: $\int_M F\omega^n=0$. For an element $\phi\in\mathcal{G}$, the adjoint action is given in these conventions by

$$\mathrm{Ad}_\phi H=(\phi^{-1})^*H. \tag{4}$$

To a function $H\in\mathrm{Lie}(\mathcal{G})\cong C_0^\infty(M,\mathbb{R})$ there corresponds the function $\mu(H)$ on $\mathcal{J}$ given in [30], [44] by the formula

$$\mu(H)(J)=\int_M S(J)H\omega^n, \tag{5}$$

where $S(J)\in C^\infty(M,\mathbb{R})$ is the *Hermitian scalar curvature* of the Hermitian metric $h(J)=g(J)-i\omega$ defined as follows. Consider the Hermitian line bundle $L=\Lambda^n_{\mathbb{C}}(TM,J,h(J))$. It has a natural connection ∇^n induced from the canonical connection ∇ on $(TM,J,h(J))$ (cf. [51], Section 2.6, [61], and [87], Section 2, and references therein) defined by the properties

$$\nabla J=0,\quad \nabla h=0,\quad T_\nabla^{(1,1)}=0.$$

This connection can also be equivalently (by [51], Section 2) defined by use of $\bar\partial$-operators, as in [30]. The connection ∇^n has curvature $i\tilde\rho$ for the lift $\tilde\rho$ of a real valued closed two form $\rho\in\Omega^2(M,\mathbb{R})$ on M by the natural projection $L\to M$. We define $S(J)\in C^\infty(M,\mathbb{R})$ by

$$S(J)\omega^n=n\rho\wedge\omega^{n-1}. \tag{6}$$

Whenever J is integrable $S(J)$ coincides with the scalar curvature of the Riemannian metric $g(J)$. In the above $g(J)$ is the Riemannian metric corresponding to J given by $g(J)(\xi,\eta)=\omega(\xi,J\eta)$. Note that $g(\phi_*J)=(\phi^{-1})^*g(J)$ and consequently the same is true for $h(J)$. Hence, for all $\phi\in\mathcal{G}$,

$$S(\phi_*J)=(\phi^{-1})^*S(J). \tag{7}$$

From Equalities (4), (5) and (6) we obtain $\mu(H)(\phi_*^{-1}J) = \int_M S(\phi_*^{-1}J)H\omega^n = \int_M \phi^* S(J)H\omega^n = \int_M S(J)(\phi^{-1})^* H\omega^n = \mu((\phi^{-1})^* H)(J) = \mu(\mathrm{Ad}_\phi H)(J)$. Therefore the moment map is equivariant.

We remark that the action of $\mathcal{G}$ on $\mathcal{J}$ can be extended to the action of $\mathcal{H} = \mathrm{Symp}(M,\omega)$ that preserves Ω and $\mathcal{J}$. Moreover $\mathcal{G} \subset \mathcal{H}$ is a normal subgroup and (by the same computation as above) the moment for map $\mu\colon \mathrm{Lie}(\mathcal{G}) \times \mathcal{J} \to \mathbb{R}$ for the action of $\mathcal{G}$ on $\mathcal{J}$ is equivariant with respect to the action of $\mathcal{H}$ (which acts on $\mathrm{Lie}(\mathcal{G})$ by the adjoint action $\mathrm{Ad}_\psi H = (\psi^{-1})^* H$, $\psi \in \mathcal{H}$).

1.7. Quasimorphisms on the Hamiltonian groups of symplectic manifolds. Here we apply the general principle for constructing quasimorphisms to the group $\mathcal{G} = \mathrm{Ham}(M,\omega)$ acting on $(\mathcal{J},\Omega,\mathbb{J})$ and study the resulting object to obtain the main results of this paper. Corollary 1 and Theorem 3 are of special note.

First, it is rather easy to prove that the space $(\mathcal{J},\Omega,\mathcal{K})$ for the system $\mathcal{K}$ of paths consisting of the fiberwise geodesics is a Domic–Toledo space. In more detail for every two almost complex structures $J_0, J_1 \in \mathcal{J} = \Gamma(\mathcal{S}; M)$ we define $[J_0, J_1]$ to be the fiberwise geodesic path $[J_0, J_1](t)$ that restricts in each fiber $\mathcal{S}_x$ over a point $x \in M$ to the unique geodesic $[(J_0)_x,(J_1)_x](t)$ in $(\mathcal{S}_x,\sigma_x, j_x)$ joining $(J_0)_x$ and $(J_1)_x$. Moreover, for any three elements $J_0, J_1, J_2 \in \mathcal{J}$ we choose $\Delta = \Delta(J_0, J_1, J_2)$ to be the fiberwise geodesic convex hull of J_0, J_1, J_2 so that in each fiber $\mathcal{S}_x$ over $x \in M$, Δ restricts to a geodesic 2-simplex Δ_x with respect to σ_x with vertices $(J_0)_x$, $(J_1)_x$, $(J_2)_x$. Then since

$$\int_{\Delta(J_0,J_1,J_2)} \Omega = \int_M \left(\int_{\Delta_x} \sigma_x \right) \omega^n(x),$$

we estimate

$$\left| \int_{\Delta(J_0,J_1,J_2)} \Omega \right| \le \int_M \left| \int_{\Delta_x} \sigma_x \right| \omega^n(x) \le \mathrm{Vol}(M,\omega^n) C_{\mathcal{S}_n},$$

as $(\mathcal{S}_x,\sigma_x, j_x)$ is a Domic–Toledo space (with geodesics for the system of paths) with the constant $C_{\mathcal{S}_n}$. And surely, $\mathcal{J}$ is contractible so the conditions $\pi_1(J) = 0$ and $\pi_2(J) = 0$ are satisfied.

Second, we show that $\mathcal{G} = \mathrm{Ham}(M,\omega)$ is Hamiltonian–Hermitian with its action on $(\mathcal{J},\Omega,\mathcal{K})$. First, as explained above it acts on $\mathcal{J}$ preserving Ω with an equivariant moment map. It is also easy to deduce from the fact that the action preserves $\mathbb{J}$ that it also preserves $\mathcal{K}$ – though we give a direct proof. Indeed this follows immediately from the fact that for every diffeomorphism $f \in \mathcal{G}$ and for all $x \in M$ the map $\mathcal{S}_{f^{-1}x} \to \mathcal{S}_x$ given by $J_{f^{-1}x} \mapsto (f_{*x})J_{f^{-1}x}(f_{*x})^{-1}$ is an isometry of the Siegel upper half-spaces. The canonical metric ρ_y on $\mathcal{S}_y$ for $y \in M$ is given by $(\rho_y)_{J_y}(A_y, B_y) = \mathrm{const} \cdot \mathrm{trace}(A_y B_y)$ for $A_y, B_y \in T_{J_y}\mathcal{S}_y$ ($J_y \in \mathcal{S}_y$), and surely, trace is preserved by conjugation with a linear isomorphism.

Therefore by Theorem 1 the group $\widetilde{\mathcal{G}}$ admits a homogenous quasimorphism, which we show to be non-trivial by computing its local type in Theorem 3.

Corollary 1. *The universal cover $\widetilde{\mathcal{G}}$ of the group of Hamiltonian diffeomorphisms $\mathcal{G} = \mathrm{Ham}(M,\omega)$ of an arbitrary closed symplectic manifold (M,ω) admits a non-trivial homogenous quasimorphism $\mathfrak{S}\colon \widetilde{\mathcal{G}} \to \mathbb{R}$.*

By construction the restriction $\mathfrak{S}|_{\pi_1(\mathcal{G})}$ equals $\mathcal{A}_\mu\colon \pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}$. In more detail, for an element $\phi = [\{\phi_t\}] \in \pi_1(\mathrm{Ham}(M,\omega))$ with mean-normalized Hamiltonian $H_t \in C_0^\infty(M,\mathbb{R})$, we have

$$A_\mu(\phi) = \int_D \Omega - \int_0^1 dt \int_M S((\phi_t)_* J) H_t(x)\omega^n,$$

where $J \in \mathcal{J}$ is an arbitrary element and D is a disk in $\mathcal{J}$ spanning the loop $\{(\phi_t)_* J\}_{t\in\mathbb{R}/\mathbb{Z}}$. We now compute the homomorphism $\mathcal{A}_\mu$ in terms of a previously known homomorphism on $\pi_1(\mathrm{Ham}(M,\omega))$ [63].

Definition 1.7.1 (The homomorphism $I_{c_1}: \pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}$). As usual with topological groups, there is a bijective correspondence between $\pi_1(\mathrm{Ham}(M,\omega))$ and the isomorphism classes of bundles $P \xrightarrow{M} S^2$ over the 2-sphere with fiber M, such that their structure group is contained in $\mathrm{Ham}(M,\omega)$ [63]. Such bundles are called *Hamiltonian fiber bundles* (or *fibrations*) over the 2-sphere. Over such a bundle, the vertical tangent bundle $T_V P$ is naturally endowed with the structure of a symplectic vector bundle. Hence it has Chern classes, called the *vertical Chern classes*, of which we shall use the first $c_1^V := c_1(T_V P)$. There is also a natural characteristic class $u \in H^2(P,\mathbb{R})$ of such bundles with the defining properties $u|_{\text{fiber}} = [\omega]$ and $\int_{\text{fiber}} u^{n+1} = 0$ (or in the case when the base is 2-dimensional $u^{n+1} = 0$) – cf. [63], [76] and references therein. It is called the *coupling class* of the Hamiltonian fibration. With these two characteristic classes we compose the monomial $c_1^V u^n$, where $n = \frac{1}{2}\dim M$ and integrate over P. This yields a homomorphism $\pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}$ that we denote I_{c_1}. The formula for $I_{c_1}(\gamma)$ for a loop γ in $\mathrm{Ham}(M,\omega)$ based at Id is therefore

$$I_{c_1}(\gamma) = \int_{P_\gamma} c_1^V u^n,$$

where P_γ is the Hamiltonian fibration corresponding to γ.

Theorem 2. *The two homomorphisms A_μ and I_{c_1} from $\pi_1(\mathcal{G})$ to the reals are equal.*

Remark 1.7.1. Assume now that the almost complex structure J_0 is integrable – that is (M,ω,J_0) is a Kähler manifold. Note that the restriction $\iota^* A_\mu$ of A_μ to the π_1 of the finite-dimensional compact Lie subgroup $K := \mathcal{G}_{J_0}$ of $\mathcal{G}$ consisting of

Hamiltonian biholomorphisms satisfies $\iota^* A_\mu = -F$, for the Futaki invariant F [46] since the filling disk D can be chosen to be trivial. The equality is understood via the isomorphism $\pi_1(K) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Lie}(K)/[\mathrm{Lie}(K), \mathrm{Lie}(K)]$ which holds by a classical result of Chevalley and Eilenberg [24] (a short account can be found in [13]). The consequence of Theorem 2 that I_{c_1} restricts to the (Bando–)Futaki invariant on $\mathcal{G}_{J_0}$ has previously been shown in [84] using methods of equivariant characteristic classes.

As a corollary we answer a question of Polterovich (cf. [84], Discussion and Questions, 2).

Corollary 2. *We have the equality* $\mathfrak{S}|_{\pi_1(\mathcal{G})} \equiv I_{c_1}$ *on* $\pi_1(\mathrm{Ham}(M, \omega))$.

By Proposition 1.5.1 we have that $\mathfrak{S}$ is $\mathrm{Symp}(M, \omega)$-invariant.

Corollary 3. *The quasimorphism* $\mathfrak{S} \colon \widetilde{\mathcal{G}} \to \mathbb{R}$ *is invariant with respect to conjugation by elements of* $\widetilde{\mathrm{Symp}}(M, \omega)$ *or equivalently by elements of* $\mathrm{Symp}(M, \omega)$.

Moreover we compute the local type of the quasimorphism $\mathfrak{S}$. To state the result of our computation we would first like to make two definitions of the more classical invariants in terms of which we express the answer.

Definition 1.7.2 (Calabi homomorphism on $\mathcal{G}_B = \mathrm{Ham}_c(B^{2n}, \omega_B)$ [20], cf. [66], [79]). Given a Hamiltonian isotopy $\{\phi_t\}_{t=0}^1 \subset \mathrm{Ham}_c(B^{2n}, \omega_B)$ starting at $\phi_0 = \mathrm{Id}$ with endpoint $\phi = \phi_1$ with generating path of vector fields $\{X_t\}_{t=0}^1$, define H_t (for each $t \in [0,1]$) to be the function that vanishes near ∂B and satisfies $i_{X_t}\omega = -dH_t$. Then the Calabi homomorphism is defined as

$$\mathrm{Cal}_B(\{\phi_t\}_{t=0}^1) = \int_0^1 \int_B H_t \omega^n dt.$$

It is, as can be verified using the differential homotopy and the cocycle formulas, a well-defined homomorphism $\widetilde{\mathcal{G}}_B \to \mathbb{R}$. Moreover it vanishes on loops in $\mathcal{G}_B$ hence descending from $\widetilde{\mathcal{G}}_B$ to $\mathcal{G}_B$ itself.

Remark 1.7.2. We present a short proof that Cal_B vanishes on loops in $\mathcal{G}_B$ that differs slightly from the one usually found in the literature. It is well known, cf. [66], [79], that the Calabi homomorphism can be reinterpreted as

$$\mathrm{Cal}_B(\{\phi_t\}_{t=0}^1) = -\frac{1}{n} \cdot \int_0^1 \int_B (i_{X_t}\lambda)\, \omega^n dt,$$

for a primitive λ of ω_B in B. Hence

$$\mathrm{Cal}_B(\{\phi_t\}_{t=0}^1) = -\frac{1}{n+1} \cdot \int_0^1 \int_B (i_{X_t}\lambda - H_t)\, \omega^n dt.$$

However for a loop $\{\phi_t\}_{t=0}^1$ this is proportional to

$$\int_B \left(\int_{\{\phi_t x\}_{t=0}^1} \lambda - \int_0^1 H_t(\phi_t x) dt \right) \omega^n(x)$$

wherein the integrand is independent of x, as it is the Hamiltonian action of the periodic orbit $\{\phi_t x\}_{t=0}^1$ of $\{\phi_t\}_{t=0}^1$. Consequently the integral localizes (up to a multiplicative constant) to the value of the integrand at each point $x \in B$ that vanishes

$$\int_{\{\phi_t x\}_{t=0}^1} \lambda - \int_0^1 H_t(\phi_t x) dt = 0$$

for x close enough to ∂B.

Definition 1.7.3 (Cf. [7], the Barge–Ghys average Maslov quasimorphism on $\mathcal{G}_B = \mathrm{Ham}_c(B^{2n}, \omega_B)$). Given a Hamiltonian isotopy $\{\phi_t\}_{t=0}^1 \subset \mathrm{Ham}_c(B^{2n}, \omega_B)$ starting at $\phi_0 = \mathrm{Id}$, choosing a trivialization Θ of the tangent bundle $(TB, \omega_B) \cong B \times (V, \omega_0)$ over B as a symplectic vector bundle (here (V, ω_0) is a certain symplectic vector space e.g. $(T_b B, (\omega_B)_b)$) for some $b \in B$), we obtain from the family of paths of differentials $\{\phi_{t*_x} : T_x B \to T_{\phi_t x} B\}_{t=0}^1$ (as x ranges over B) a family $\{A(x,t) \in \mathrm{Sp}(V, \omega_0)\}_{t=0}^1$ of paths of symplectic linear automorphisms of (V, ω_0). For each $x \in B$ we compute the value $\tau_{\mathrm{Lin}}(\{A(x,t)\}_{t=0}^1)$ on the path $\{A(x,t)\}_{t=0}^1$ of the Maslov quasimorphism on the universal cover of the symplectic linear group. Then the map

$$\tau_{\Theta,B} : \{\phi_t\}_{t=0}^1 \mapsto \int_B \tau_{\mathrm{Lin}}(\{A(x,t)\}_{t=0}^1)(\omega_B)^n(x)$$

does not depend upon homotopies of $\{\phi_t\}_{t=0}^1$ with fixed endpoints and yields a quasimorphism $\tau_{\Theta,B} : \widetilde{\mathcal{G}}_B \to \mathbb{R}$. The Barge–Ghys average Maslov quasimorphism $\tau_B : \widetilde{G} \to \mathbb{R}$ is its homogeneization

$$\tau_B(\tilde{\phi}) = \lim_{k \to \infty} \frac{1}{k} \tau_{\Theta,B}(\tilde{\phi}^k).$$

It does not depend on the choice of the symplectic trivialization Θ. Both $\tau_{\Theta,B}$ and τ_B vanish on loops in $\mathcal{G}_B$ and therefore descend to quasimorphisms $\mathcal{G}_B \to \mathbb{R}$.

Remark 1.7.3. The vanishing of $\tau_{\Theta,B}$ on loops can be shown by a similar localization argument as for the Calabi homomorphism. Indeed for a loop $\{\phi_t\}_{t=0}^1$ in $\mathcal{G}_B$ the value $\tau_{\mathrm{Lin}}(\{A(x,t)\}_{t=0}^1)$ equals the Maslov index of the loop $\{A(x,t)\}_{t=0}^1$ which by the homotopy invariance of the Maslov index is independent of x, and for x near ∂B the loop $\{A(x,t)\}_{t=0}^1$ is trivial. Hence the integrand vanishes for all $x \in B$, wherefrom $\tau_{\Theta,B}(\{\phi_t\}_{t=0}^1) = 0$.

Theorem 3. *Let $c = n\int_M c_1\omega^{n-1}/\int_M \omega^n = \int S(J)\omega^n/\mathrm{Vol}(M,\omega^n)$ be the average Hermitian scalar curvature. Then the restriction of $\mathfrak{S}$ to the subgroup $\mathcal{G}_B = \mathrm{Ham}_c(B,\omega|_B) \subset \mathcal{G}$ of Hamiltonian diffeomorphisms supported in an embedded ball B in M satisfies*

$$\mathfrak{S}|_{\mathcal{G}_B} = \frac{1}{2}\tau_B - c\mathrm{Cal}_B,$$

where τ_B is the Barge–Ghys Maslov quasimorphism on $\mathcal{G}_B = \mathrm{Ham}_c(B^{2n},\omega_{\mathrm{std}})$ and Cal_B is the Calabi homomorphism.

We describe the relation of the quasimorphism $\mathfrak{S}$ to the quasimorphisms $\mathfrak{S}_{\mathrm{Py}}$ and $\mathfrak{S}_{\mathrm{En}}$ introduced by Py [78], [79] for closed manifolds (M,ω) with $c_1(TM,\omega) = \kappa[\omega]$ for $\kappa \neq 0$ and Entov [36] for closed manifolds (M,ω) with $c_1(TM,\omega) = 0$. First we state briefly the definitions of the quasimorphisms $\mathfrak{S}_{\mathrm{Py}}$ and $\mathfrak{S}_{\mathrm{En}}$. The detailed definitions appear in the proofs section.

Definition 1.7.4 (A sketch of a definition of $\mathfrak{S}_{\mathrm{Py}}$ [78], [79]). Endow the unit frame bundle $P \xrightarrow{S^1} M$ of $L = \Lambda^n_{\mathbb{C}}(TM,J,\omega)$ for a compatible complex structure $J \in \mathcal{J}$ with the structure α_0 of a prequantization of $(M,-\omega)$. Note that there is a natural map $\det^2\colon \mathcal{L}(TM,\omega) \to P^2$ from the Lagrangian Grassmannian bundle $\mathcal{L}(TM,\omega)$ to the unitary frame bundle P^2 of $L^{\otimes 2}$, since $\mathcal{L}(TM)_x = \mathrm{U}(TM_x,\omega_x,J_x)/\mathrm{O}(n)$. Note that α_0 induces a structure α of prequantization of $(M,-2\omega)$ on P^2. Given a path $\vec{\phi} = \{\phi_t\}_{t=0}^1$ in $\mathcal{G}$ with $\phi_0 = \mathrm{Id}$, choosing a point $L \in \mathcal{L}(TM,\omega)_x$ we have the curve $\{\phi_{t*_x}(L)\}_{0\le t\le 1}$ in $\mathcal{L}(TM,\omega)$ and considering $\vec{\phi}$ as a path of Hamiltonian isotopies of $(M,-2\omega)$ we have the canonical lifting $\{\hat{\phi}_t\}_{0\le t\le 1}$, $\hat{\phi}_0 = \mathrm{Id}$ of $\vec{\phi}$ to the identity component $Q = \mathrm{Quant}(P^2,\alpha)$ of the group of diffeomorphisms of P^2 that preserve α. Consequently, one considers the two curves

$$\{\det^2(\phi_{t*_x}(L))\}_{0\le t\le 1}$$

and

$$\{\hat{\phi}_t(\det^2(L))\}_{0\le t\le 1}$$

in P^2. Both these curves in P^2 start at $\det^2(L)$ and cover the path $\{\phi_t(x)\}_{0\le t\le 1}$ in M and hence differ by an angle:

$$\det^2(\phi_{t*}(L)) = e^{i2\pi\vartheta(t)}\hat{\phi}_t(\overset{2}{\det}(L)),$$

for a continuous function $\vartheta\colon [0,1] \to \mathbb{R}$. Define a continuous function on $\mathcal{L}(TM,\omega)$ by

$$\mathrm{angle}(L,\vec{\phi}) := \vartheta(1) - \vartheta(0).$$

Then the function $\mathrm{angle}(x, \vec{\phi}) = \inf_{L \in \mathcal{L}(TM,\omega)_x} \mathrm{angle}(L, \vec{\phi})$ on M is measurable, bounded and defines the quasimorphism

$$S_2(\vec{\phi}) = -\int_M \mathrm{angle}(x, \vec{\phi})\omega^n(x)$$

that does not depend upon homotopies of $\vec{\phi}$ with fixed endpoints and is thus defined as a real-valued function on $\widetilde{\mathcal{G}}$. Its homogeneization $\mathfrak{S}_{\mathrm{Py}} : \widetilde{\mathcal{G}} \to \mathbb{R}$, defined by $\mathfrak{S}_{\mathrm{Py}}(\tilde{\phi}) := \lim_{k\to\infty} \frac{S_2(\tilde{\phi}^k)}{k}$ is a homogenous quasimorphism on $\widetilde{\mathcal{G}}$ that is independent of the non-canonical structure on P of a prequantization of $(M, -\omega)$, of the prequantization form α on it and of the almost complex structure J.

Definition 1.7.5 (A sketch of a definition of $\mathfrak{S}_{\mathrm{En}}$ [36]). Given a symplectic manifold (M, ω) with $c_1(TM, \omega) = 0$ one first trivializes (TM, ω, J) for $J \in \mathcal{J}$ as a Hermitian vector bundle over the complement $U = M \setminus Z$ of a compact triangulated subset Z of $\mathrm{codim}(Z) \geq 3$, where the differential of the trivialization, appropriately defined, is uniformly bounded. For a path $\vec{\phi} = \{\phi_t\}_{t=0}^1$ in $\mathcal{G}$ by relaxing Z to be a countable union $Z_{\vec{\phi}} = \bigcup_{j\in\mathbb{Z}} Z_j$ depending on $\vec{\phi}$ of sets Z_j of $\mathrm{codim}(Z_j) \geq 2$ one can assume that U is invariant with respect to ϕ_t for all t. Then from the path $\{\phi_{t*_x}\}_{t=0}^1$ for $x \in U$ one obtains a continuous path $\{A(x,t)\}_{t=0}^1$ with $A(x, 0) = \mathrm{Id}$ in $\mathrm{Sp}(2n, \mathbb{R})$ and proceeds to define

$$\mathrm{angle}(x, \vec{\phi}) = \mathrm{varangle}(\{\det^2(A(x,t))\}_{t=0}^1).$$

One then shows that this function extended by 0 on Z is integrable on M and that

$$T_1(\vec{\phi}) = \int_M \mathrm{angle}(x, \vec{\phi})\omega^n(x)$$

does not depend on homotopies of $\vec{\phi}$ with fixed endpoints (by relaxing Z to be a countable union of sets Z_j' of $\mathrm{codim}(Z_j') \geq 1$ depending on a given homotopy), and defines a quasimorphism

$$T_1 : \widetilde{\mathcal{G}} \to \mathbb{R}.$$

Its homogeneization $\mathfrak{S}_{\mathrm{En}} : \widetilde{\mathcal{G}} \to \mathbb{R}$, defined by $\mathfrak{S}_{\mathrm{En}}(\tilde{\phi}) := \lim_{k\to\infty} \frac{T_1(\tilde{\phi}^k)}{k}$ is a homogenous quasimorphism on $\widetilde{\mathcal{G}}$ that is independent of the non-canonical choices of trivialization, of the set $U = M \setminus Z$ and of the almost complex structure J.

We claim that the quasimorphism $\widetilde{\mathcal{G}} \to \mathbb{R}$ obtained from Corollary 1 agrees with these two quasimorphisms in the settings of their definitions.

Theorem 4. 1. *On symplectic manifolds (M, ω) with $c_1(TM, \omega) = \kappa[\omega]$ for $\kappa \neq 0$ we have $2\mathfrak{S} = -\mathfrak{S}_{\mathrm{Py}}$.*

2. *On symplectic manifolds (M, ω) with $c_1(TM, \omega) = 0$ we have $2\mathfrak{S} = \mathfrak{S}_{\mathrm{En}}$.*

Remark 1.7.4. The analogues of Theorem 3 for the cases 1. and 2. above were shown in [36], [78]. The analogue of Corollary 2 was shown in [79]. The agreement of our results with the ones shown in these papers is as follows. For analogues of Theorem 3 note that the average scalar curvature c satisfies $c = n\kappa$ when $c_1(TM,\omega) = \kappa[\omega]$, for every κ. For the analogue of Corollary 2, use the easy Computation 1 from [84], near the end of Section 1.2.

Remark 1.7.5. We would also like to note that the general scheme of Theorem 1 applies to the construction of quasimorphisms $\widetilde{\mathcal{G}} \to \mathbb{R}$ for the group $\mathcal{G} = \mathrm{Ham}_c(M,\omega)$ of Hamiltonian diffeomorphisms with compact support of symplectic manifolds (M,ω) of finite volume (without boundary) that are not compact. Indeed, $\mathcal{J}$ here is also a Domic–Toledo space, since $\int_M \omega^n$ is finite, and Donaldson's theory for the scalar curvature as an equivariant moment map [30] applies here nearly verbatim. The only difference is that the symplectic form Ω is not defined on all the tangent space $T_{J_0}\mathcal{J}$ – indeed given $A, B \in T_{J_0}\mathcal{J}$ the function $\sigma_{(J_0)_x}(A_x, B_x)$ may well be non-integrable with respect to ω^n. However, since we compute for diffeomorphisms with compact support, all relevant computations happen in a compact subset of M where all functions that appear are integrable. Moreover, all functions, vector fields, one-forms and sections of endomorphism bundles have compact support, therefore the only non-local part in Donaldson's proof [30] – integration by parts to show the actual integral formulae – goes through (all the other arguments are local). At the same time, when the symplectic volume of M is not finite, $\mathcal{J}$ stops being a Domic–Toledo space (at least with the natural definitions) and hence this approach does not seem to give quasimorphisms. It would be interesting to investigate the restriction to $\pi_1(\mathcal{G})$ of the quasimorphism in the finite volume case. The local type is obtained by nearly the same computation as the one given for the closed case and is given by the Barge–Ghys average Maslov quasimorphism τ. The $\mathcal{H} = \mathrm{Symp}_c(M,\omega)$-invariance holds as before.

A corollary, as obtained in [36] for symplectic manifolds with $c_1(TM,\omega) = 0$, is that the commutator length of $\widetilde{\mathcal{G}}$ is unbounded.

Corollary 4. *The diameter in the commutator length of the group $\widetilde{\mathcal{G}}$ for $\mathcal{G} =$ $\mathrm{Ham}(M,\omega)$ of a closed symplectic manifold (M,ω) and of the perfect* ([5]) *group* $\mathrm{Ker}(\mathrm{Cal}\colon \widetilde{\mathcal{G}} \to \mathbb{R})$ *for $\mathcal{G} = \mathrm{Ham}_c(M,\omega)$ of an open finite volume symplectic manifold (M,ω) is infinite. In the closed case, under the additional assumption $I_{c_1} \equiv 0$, the same conclusion follows for $\mathcal{G}$ itself.*

We also note that for reasons of naturality of the constructions and normalizations of Hamiltonians we have the following proposition.

Proposition 1.7.1 (Embedding functoriality). *Given an open subset $U \subset M$ of a closed symplectic manifold (M,ω), denote by $\mathfrak{S}_M$ the quasimorphism obtained on*

$\mathcal{G}$ for (M,ω) and by $\mathfrak{S}_U$ the quasimorphism obtained on $\mathcal{G}_U$ for $(U,\omega|_U)$. Then

$$\mathfrak{S}_M|_{\mathcal{G}_U} = \mathfrak{S}_U - c \cdot \mathrm{Cal}_U,$$

for the average Hermitian scalar curvature c. Similarly, if M were an open symplectic manifold of finite volume, then

$$\mathfrak{S}_M|_{\mathcal{G}_U} = \mathfrak{S}_U.$$

1.8. Application to the L^2_2-distance on $\widetilde{\mathrm{Ham}}(M,\omega)$. While it is not surprising that our quasimorphism is bounded by a multiple of the Sobolev L^2_2 norm on $\widetilde{\mathrm{Ham}}(M,\omega)$, indeed $\mathfrak{S}_{J_0}$ is surely continuous in the C^1-topology induced to $\mathrm{Ham}(M,\omega)$ from $\mathrm{Diff}(M)$, we present a proof for the sheer simplicity of the argument.

For a Hamiltonian isotopy $\vec{\phi} = \{\phi_t\}_{t=0}^1$ of a symplectic manifold (M,ω) starting at the identity that is generated by the zero-mean-normalized Hamiltonian H_t put

$$\|\vec{\phi}\|_{k,p} = \int_0^1 \|H_t\|_{L^p_k(M,\omega^n)}.$$

Then define the norm of an element $\tilde{\phi} \in \tilde{\mathcal{G}}$ by

$$\|\tilde{\phi}\|_{k,p} = \inf_{[\vec{\phi}]=\tilde{\phi}} \|\vec{\phi}\|_{k,p}.$$

Finally define the norm of $\phi \in \mathcal{G}$ by $\|\phi\|_{k,p} = \inf_{\pi(\tilde{\phi})=\phi} \|\tilde{\phi}\|_{k,p}$ for the natural projection $\pi\colon \tilde{\mathcal{G}} \to \mathcal{G}$. For two elements a, b of the above groups define the distance

$$d_{p,k}(a,b) = \|a^{-1}b\|_{k,p}.$$

The following facts are easy to check.

- For $k \geq 1$ the (p,k)-norms and distances are equivalent to $(p,k-1)$-norms and distances as defined via the vector field X_t generating $\vec{\phi}$.
- For $k \geq 1$ these norms and distances are non-degenerate.

We show in Section 2.9 that $\mathfrak{S}_{J_0}$ calibrates the $(2,2)$-norm as follows:

$$\mathfrak{S}_{J_0}(\tilde{\phi}) \leq C(n,\omega,J_0)\|\tilde{\phi}\|_{2,2}, \tag{8}$$

for a constant $C(n,\omega,J_0)$ that does not depend on $\tilde{\phi}$. As a corollary we obtain that the L^2_2 distance is unbounded on $\tilde{\mathcal{G}}$.

Corollary 5. *The diameter of $\tilde{\mathcal{G}}$ is infinite with respect to the L^2_2-distance for every symplectic manifold (M,ω) of finite volume.*

Remark 1.8.1. Given that $\mathcal{A}_\mu \colon \pi_1 \mathcal{G} \to \mathbb{R}$ vanishes, the same consequence holds for the L^2_2-distance on the group $\mathcal{G}$ itself. For closed manifolds this condition is equivalent to the vanishing of I_{c_1}.

The unboundedness of the L^2_1-metric on compact exact symplectic manifolds was previously proven by Eliashberg and Ratiu [35] (their methods work even for the larger group $\mathcal{H} = \mathrm{Symp}(M, \omega)$ with appropriate definitions), while sharper topological bounds for the 2-disc were obtained by Gambaudo and Lagrange [50] (cf. [8], [15]).

1.9. Finite-dimensional examples: Guichardet–Wigner quasimorphisms. The general principle outlined in Section 1.5 applies also to finite-dimensional Hermitian Lie groups acting on their corresponding Hermitian symmetric spaces of non-compact type. In this section we describe this application, in part for use in the proofs later.

Let G be a simple Hermitian symmetric Lie group. Then the adjoint form of G belongs to those of the following list of Lie groups: $\mathrm{SU}(p,q)$, $\mathrm{SO}_0(2,q)$, $q \neq 2$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}^*(2n)$, $n \geq 2$, and two real forms of the complex simple Lie groups of types E_6 and E_7 respectively. Let us assume that the center of G is finite, so that $\pi_1(G)$ is infinite. Let $K \subset G$ be the analytic subgroup corresponding to the maximal compact Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. In this situation there is a corresponding Hermitian symmetric space $X = G/K$, endowed with a natural complex structure j_X and a Kähler form σ_X that is invariant with respect to the transitive action of the group G (proportional to the Bergman Kähler structure when such a space is realized as a symmetric bounded domain in a complex affine space by the Harish-Chandra embedding) cf. [58], [60], [69]. The works of Domic–Toledo and Ørsted [27], [26] show that when we take the system of paths $\mathcal{K}$ to consist of the geodesics with respect to the invariant Kähler metric, then $(X, \sigma_X, \mathcal{K})$ is a Domic–Toledo space in our terminology.

Moreover, we note that by e.g. [69] these spaces (X, σ_X, j_X) are Kähler–Einstein manifolds (that is, their Ricci forms are proportional to their Kähler forms: $\mathrm{Ric}(\sigma_X) = \lambda \sigma_X$, where for the Bergman metric we have $\lambda = -1$). Note that $\mathrm{Ric}(\sigma_X)$ is equal up to a universal constant to the curvature of the Chern connection on the line bundle $L_X = \Lambda^N_{\mathbb{C}} TX$, with the holomorphic and Hermitian structures induced by j_X and σ_X.

We now show that G is Hamiltonian–Hermitian with its action on X. Firstly the group G acts on X by maps preserving j_X and σ_X (symplectic biholomorphisms) and hence preserving the system of geodesics $\mathcal{K}$. We now claim that the group G acts on X with an equivariant moment map $\mu_X \colon \mathrm{Lie}(G) \times X \to \mathbb{R}$. Note that as the Chern connection on TX is given canonically by (σ_X, j_X) and the action preserves these structures, it will also preserve the Chern connection. Consider the natural lift of the action of G on X to an action of G on TX by taking differentials. This induces an action of G on $L_X = \Lambda^N_{\mathbb{C}} TX$. Note that this action preserves the Hermitian structure on L_X, and hence it descends to the circle bundle $P_X \xrightarrow{S^1} X$ of unit vectors in L_X (the unitary frame bundle of the Hermitian vector bundle L_X). The Chern connection on L_X induces a real-valued connection one-form (cf. [84], Appendix A) α_X on the principal S^1-bundle P_X over X that by the Kähler–Einstein property satisfies the relation

$$d\alpha_X = \widetilde{\sigma_X} \tag{9}$$

for the lift $\widetilde{\sigma_X}$ of σ_X to P_X by the natural projection $P_X \to X$, as follows from what is noted above. Now the action of the group G on P_X covering the action of G on X preserves the one-form α_X (by preservation of the Chern connection). This is enough to give an equivariant moment map for the action of G on X. Indeed, it is constructed as follows. A vector $\xi \in \mathrm{Lie}(G)$ induces the vector field $\bar{\xi}$ on X by the action of G on X and a vector field $\hat{\xi}$ on P_X that covers $\bar{\xi}$, by the action of G on P_X. We claim that the equivariant moment map is given by

$$\mu_X(\xi)(x) = (\alpha_X)_y(\hat{\xi}_y) \tag{10}$$

for any $y \in P_X$ over $x \in X$ (indeed $\hat{\xi}$ is equivariant with respect to the natural circle action on P_X as is α_X and hence $(\alpha_X)_y(\hat{\xi}_y)$ does not depend on the choice of y over x). Firstly by relation (9) and the preservation $\mathcal{L}_{\hat{\xi}}\alpha_X = 0$ of the connection by the infinitesimal action we have

$$i_{\bar{\xi}}\sigma_X = -d\mu(\xi)(x).$$

Hence μ_X is a moment map for the action of G on X. For the equivariance we note once again that the action of G on P_X preserves α_X and that the vector field $\hat{\xi}$ has a corresponding equivariance property. Namely for any $g \in G$ and $y \in P$ denoting by $\hat{g} \cdot y$ the action of G on P_X and by $\hat{g}_{*_y}$ the corresponding differential $T_y P_X \to T_{\hat{g}\cdot y} P_X$, we have the very general equivariance property for infinitesimal actions corresponding to Lie group actions on spaces:

$$\hat{\xi}(\hat{g} \cdot y) = \hat{g}_{*_y}(\widehat{\mathrm{Ad}_{g^{-1}}\xi}(y)).$$

Now noting that for $y \in P_X$ over $x \in X$ the point $\hat{g} \cdot y$ is over $g \cdot x$, we obtain

$$\begin{aligned}\mu_X(\mathrm{Ad}_g\xi)(g \cdot x) &= (\alpha_X)_{\hat{g}\cdot y}(\widehat{\mathrm{Ad}_g\xi}(\hat{g} \cdot y)) = (\alpha_X)_{\hat{g}\cdot y}(\hat{g}_{*_y}\hat{\xi}(y)) = (\alpha_X)_y(\hat{\xi}(y)) \\ &= \mu_X(\xi)(x),\end{aligned}$$

showing equivariance.

Hence G is Hamiltonian–Hermitian with Domic–Toledo space (X, σ_X, j_X) and equivariant moment map μ_X, and therefore by Theorem 1 has a homogenous quasimorphism.

Corollary 6. *Theorem 1 gives a homogenous quasimorphism $\nu_G\colon \widetilde{G} \to \mathbb{R}$ for every simple Hermitian symmetric Lie group G.*

It remains to show that it is non-trivial. In fact we show in Section 2.3 that it is equal to the Guichardet–Wigner [55], [34], [25], [85], [17] quasimorphism ϱ_G on $\widetilde{G}$ by comparing them on $\pi_1(G)$ and arguing that a homogenous quasimorphism on $\widetilde{G}$ is determined by its restriction to the fundamental group.

Proposition 1.9.1. *The quasimorphisms ν_G and ϱ_G on $\widetilde{G}$ satisfy the equality*

$$\nu_G = -\varrho_G.$$

We would now like to give a reformulation of the construction of ν_x in the finite-dimensional case as a certain rotation number. Indeed consider once again the principal S^1-bundle $P_X \to X$. Trivialize it by taking parallel transports $\Gamma_{\gamma(y,x)}\colon (P_X)_y \to (P_X)_x$ along geodesics $\gamma(y,x)$ for $y \in X$. Then given a path $\vec{g} = \{g_t\}_{t=0}^1$ with $g_0 = \mathrm{Id}$ in G, the path of differentials $(g_t)_{*x}\colon T_xX \to T_{g_t\cdot x}X$ gives us a path $\Gamma_{\gamma(g_t\cdot x,x)} \circ \hat{g}_t|_{(P_X)_x}\colon (P_X)_x \to (P_X)_x$ which we consider as a path in $\mathrm{U}(1) \cong S^1$. Then

$$\nu_x(\vec{g}) = \mathrm{varangle}(\{\Gamma_{\gamma(g_t\cdot x,x)} \circ \hat{g}_t|_{(P_X)_x}\}_{t=0}^1). \tag{11}$$

Indeed, denoting $\gamma_t := \gamma(g_t \cdot x, x)$ and $\beta_t = \{g_{t'} \cdot x\}_{t'=0}^t$ we have

$$\begin{aligned}
&\mathrm{varangle}(\{\Gamma_{\gamma(g_t\cdot x,x)} \circ \hat{g}_t|_{(P_X)_x}\}_{t=0}^1)\\
&\quad= \mathrm{varangle}(\{\Gamma_{\gamma_t} \circ \Gamma_{\beta_t}\}_{t=0}^1) + \mathrm{varangle}(\{\Gamma_{\bar{\beta}_t} \circ \hat{g}_t|_{(P_X)_x}\}_{t=0}^1)\\
&\quad= \int_{D_{\vec{g}}} \sigma_X - \int_0^1 (\alpha_X)_{\hat{g}_t\cdot y}(\hat{\xi}_t)_{\hat{g}_t\cdot y}\,dt = \int_{D_{\vec{g}}} \sigma_X - \int_0^1 \mu(\xi_t)(g_t\cdot x)\,dt = \nu_x(\vec{g}).
\end{aligned}$$

It is interesting to note that taking this reformulation as a definition for the quasimorphism, its independence upon homotopies with fixed endpoints follows immediately by continuity.

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2. Proofs

2.1. The action homomorphism. We prove that the number $\mathcal{A}_\mu(a)$ defined in Section 1.3 is well defined and determines a homomorphism $\pi_1(\mathcal{G}) \to \mathbb{R}/\mathcal{P}_\Omega$. We refer to Sections 1.2 and 1.3 for the relevant notation and definitions. Let us first prove that it is well defined. First of all, the value $\mathcal{A}_\mu(a) \in \mathbb{R}/\mathcal{P}_\Omega$ obviously does not depend on the spanning disk. Let us prove that it does not depend on the point $x \in \mathcal{X}$. Take another point $x' \in \mathcal{X}$ and choose a path $\beta\colon [0,1] \to \mathcal{X}$ between the two: $\beta(0) = x$, $\beta(1) = x'$. Consider the cylindric cycle $C\colon \mathbb{R}/\mathbb{Z} \times [0,1] \to \mathcal{X}$ defined by $C(t,s) = \phi_t \cdot \beta(s)$. Note that $C(t,0) = \phi_t \cdot x$ and that $C(t,1) = \phi_t \cdot x'$. Define a spanning disk D' for $\phi_{x'}$ by $D' = D \cup_{\phi_x} C$. Then the equality

$$\int_D \Omega - \int_0^1 \mu(X_t)(\phi_t \cdot x)\, dt = \int_{D'} \Omega - \int_0^1 \mu(X_t)(\phi_t \cdot x')\, dt$$

that we are trying to prove reduces to

$$\int_{D'} \Omega - \int_D \Omega = \int_0^1 \mu(X_t)(\phi_t \cdot x')\, dt - \int_0^1 \mu(X_t)(\phi_t \cdot x)\, dt,$$

which is equivalent to

$$\int_C \Omega = \int_0^1 \mu(X_t)(C(t,1))\, dt - \int_0^1 \mu(X_t)(C(t,0))\, dt.$$

This equality is established by direct computation of the left hand side. Indeed

$$\begin{aligned}
\int_C \Omega &= \int_0^1 \int_0^1 \Omega(\partial_s C(s,t), \partial_t C(s,t))\, dsdt = \int_0^1 \int_0^1 \Omega(\partial_s C, \Xi_t(C(t,s)))\, dsdt \\
&= \int_0^1 \int_0^1 d_{C(s,t)}\mu(X_t)(\partial_s C)\, dsdt = \int_0^1 \int_0^1 \partial_s \mu(X_t)(C(s,t))\, dsdt \\
&= \int_0^1 \mu(X_t)(C(1,t)) - \mu(X_t)(C(0,t))\, dt \\
&= \int_0^1 \mu(X_t)(C(t,1))\, dt - \int_0^1 \mu(X_t)(C(t,0))\, dt,
\end{aligned}$$

yielding the desired equality.

Let us now proceed to prove that $\mathcal{A}_\mu(\{\phi_t\}) = \int_D \Omega - \int_0^1 \mu(X_t)(\bar{\phi}_t x)\, dt$ remains invariant when ϕ_t is deformed homotopically with fixed endpoints. Let $\phi_t^s, 0 \le s \le 1$ be such a homotopy. That is $\phi_0^s \equiv \mathrm{Id}$, $\phi_1^s \equiv \mathrm{Id}$ and $(s,t) \to \phi_t^s$ is a smooth map $[0,1] \times [0,1] \to \mathcal{G}$. Surely, it is enough to prove that for all s the derivative $\frac{\partial}{\partial s}|_s \mathcal{A}_\mu(\phi_t^s)$ vanishes. To this end we use the following lemma, which is a direct consequence of the standard differential homotopy formula.

Lemma 2.1.1. *Let X^s_t and Y^s_t be the elements $X^s_t = \frac{\partial}{\partial\tau}|_{\tau=t}\phi^s_\tau \cdot (\phi^s_t)^{-1}$, $Y^s_t = \frac{\partial}{\partial\sigma}|_{\sigma=s}\phi^\sigma_t \cdot (\phi^s_t)^{-1}$ of* $\mathrm{Lie}(\mathcal{G})$ *(note that $Y^s_0 \equiv 0$ and $Y^s_1 \equiv 0$). Then*

$$\frac{\partial}{\partial s}\mathrm{Ad}_{(\phi^s_t)^{-1}} X^s_t = \mathrm{Ad}_{(\phi^s_t)^{-1}}\frac{\partial}{\partial t} Y^s_t \quad \text{and} \quad \frac{\partial}{\partial t}\mathrm{Ad}_{(\phi^s_t)^{-1}} Y^s_t = \mathrm{Ad}_{(\phi^s_t)^{-1}}\frac{\partial}{\partial s} X^s_t.$$

Proof. The differential homotopy formula says $\frac{\partial}{\partial s}X^s_t = \frac{\partial}{\partial t}Y^s_t + [X^s_t, Y^s_t]$. Differentiating $\frac{\partial}{\partial s}\mathrm{Ad}_{(\phi^s_t)^{-1}}X^s_t$ we obtain $\mathrm{Ad}_{(\phi^s_t)^{-1}}([Y^s_t, X^s_t] + \frac{\partial}{\partial s}X^s_t)$, which by the differential homotopy formula equals $\mathrm{Ad}_{(\phi^s_t)^{-1}}\frac{\partial}{\partial t}Y^s_t$. The other equality is obtained in the same way. Both are equivalent to the original differential homotopy formula. □

Now

$$\begin{aligned}
-\frac{\partial}{\partial s}|_s \mathcal{A}_\mu(\phi^s_t) &= \frac{\partial}{\partial s}\int_0^1 \mu(X^s_t)(\overline{\phi^s_t}x)\,dt - \frac{\partial}{\partial s}\int_{D^s_t}\Omega \\
&= \int_0^1 \mu(\frac{\partial}{\partial s}X^s_t)(\overline{\phi^s_t}x)\,dt + \int_0^1 \iota_{\Upsilon^s_t(\overline{\phi^s_t}x)} d_{\overline{\phi^s_t}x}\mu(X^s_t)\,dt \\
&\qquad - \int_0^1 \iota_{\Upsilon^s_t(\overline{\phi^s_t}x)}\Omega(\Xi^s_t(\overline{\phi^s_t}x)) \\
&= \int_0^1 \mu(\mathrm{Ad}_{(\phi^s_t)^{-1}}\frac{\partial}{\partial s}X^s_t)(x)\,dt + \int_0^1 \Omega(\Xi^s_t(\overline{\phi^s_t}x), \Upsilon^s_t(\overline{\phi^s_t}x))\,dt \\
&\qquad - \int_0^1 \Omega(\Upsilon^s_t(\overline{\phi^s_t}x), \Xi^s_t(\overline{\phi^s_t}x)) \\
&= \int_0^1 \mu(\frac{\partial}{\partial t}\mathrm{Ad}_{(\phi^s_t)^{-1}}Y^s_t)(x)\,dt = \int_0^1 \frac{\partial}{\partial t}\mu(\mathrm{Ad}_{(\phi^s_t)^{-1}}Y^s_t)(x)\,dt \\
&= \mu(\mathrm{Ad}_{(\phi^s_1)^{-1}}Y^s_1)(x) - \mu(\mathrm{Ad}_{(\phi^s_0)^{-1}}Y^s_0)(x) \\
&= \mu(Y^s_1)(x) - \mu(Y^s_0)(x) = 0.
\end{aligned}$$

This yields the desired equality. Here D^s_t is obtained by gluing D and $C(s,t) = \overline{\phi^s_t}x$ along ϕ_x. The vector fields Ξ^s_t, Υ^s_t are the infinitesimal actions of X^s_t and Y^s_t.

At last, let us prove that $\mathcal{A}_\mu$ defines a homomorphism $\pi_1(\mathcal{G}) \to \mathbb{R}/\mathcal{P}_\Omega$. Indeed, take two loops $\phi = \{\phi_t\}$, $\psi = \{\psi_t\}$ based at Id. Consider their concatenation $\chi = \psi * \phi$. Then $\chi_x = \phi_x * \psi_x$. Moreover we can choose a spanning disk of χ_x that factors through the topological wedge (we refer to [43] for the definition of wedge and for related notations) of the spanning disks D_ϕ, D_ψ of ϕ_x, ψ_x. That is $D\colon \mathbb{D} \to \mathfrak{X}$ factors as $D\colon \mathbb{D} \longrightarrow \mathbb{D}\bigvee_1\mathbb{D} \xrightarrow{D_\phi \vee_x D_\psi} \mathfrak{X}$. Hence $\mathcal{A}_\mu(\chi) = \int_{D_\phi}\Omega - \int_0^1\int_0^1 \mu(X_t)(\overline{\phi_t}x) + \int_{D_\psi}\Omega - \int_0^1 \mu(Y_t)(\overline{\psi_t\phi_1}x)\,dt = \int_{D_\phi}\Omega - \int_0^1\int_0^1 \mu(X_t)(\overline{\phi_t}x) + \int_{D_\psi}\Omega - \int_0^1 \mu(Y_t)(\overline{\psi_t}x)\,dt = \mathcal{A}_\mu(\phi) + \mathcal{A}_\mu(\psi)$. Here X_t and Y_t are the elements of $\mathrm{Lie}(\mathcal{G})$ corresponding to ϕ and ψ. The penultimate equality follows from the fact that $\phi_1 = \mathrm{Id}$.

2.2. The quasimorphism on Hamiltonian–Hermitian groups. We now prove Theorem 1 on the construction of quasimorphisms on Hamiltonian–Hermitian groups.

Proof. First, the independence on the disk follows trivially, since $\pi_2(\mathcal{X}) = 0$ and Ω is closed.

We proceed to show that the map is independent upon homotopies of $\{g_t\}_{t=0}^1$ with fixed endpoints. Let g_t^s be a homotopy with fixed endpoints $g_0^s \equiv \mathrm{Id}$, $g_1^s \equiv g_1$ of $\vec{g} = \{g_t^0 = g_t\}_{t=0}^1$ to $\vec{h} = \{g_t^1 = h_t\}_{t=0}^1$. Note that in this situation the concatenation $q = \{g_t\}_{t=0}^1 \# \overline{\{h_t\}_{t=0}^1}$ is a contractible loop in $\mathcal{G}$ based at Id. Denote by C the disk $C = \{g_t^s \cdot x\}_{0\le s,t\le 1}$. Choose the disks of integration as follows. When computing for $\{h_t\}_{t=0}^1$ choose an arbitrary disk $D_{\{h_t\}_{t=0}^1}$ and for $\{g_t\}_{t=0}^1$ choose $D_{\{g_t\}_{t=0}^1} = D_{\{h_t\}_{t=0}^1} \cup C$ where the gluing is over the common path $\{h_t \cdot x\}_{t=0}^1$. Then

$$\nu_x(\vec{g}) - \nu_x(\vec{h}) = \int_C \Omega - \Big(\int_0^1 \mu(X_t)(g_t \cdot x) - \int_0^1 \mu(Y_t)(h_t \cdot x)\Big) = \mathcal{A}_\mu(q) = 0$$

since $\mathcal{A}_\mu$ is a homomorphism on $\pi_1(\mathcal{G})$ and q is a contractible loop. Here $\{X_t\}_{t=0}^1$ and $\{Y_t\}_{t=0}^1$ are the paths in Lie$(\mathcal{G})$ corresponding to $\{g_t\}_{t=0}^1$ and $\{h_t\}_{t=0}^1$

We now show the quasimorphism property of ν_x. Take two paths $\vec{g} = \{g_t\}_{0\le t\le 1}$, $\vec{h} = \{h_t\}_{0\le t\le 1}$ representing elements $\tilde{g}, \tilde{h}$ of $\tilde{\mathcal{G}}$. Denote by g, h their endpoints. We would like to compare $\nu_x(\tilde{g}\tilde{h})$ with $\nu_x(\tilde{g}) + \nu_x(\tilde{h})$. Note that $\tilde{g}\tilde{h}$ is represented by the path $\vec{g} \# g_1\vec{h}$, where $g_1\vec{h} = \{g_1 h_t\}_{t=0}^1$. Hence we will compare

$$\nu_x(\vec{g} \# g_1\vec{h}) \quad \text{to} \quad \nu_x(\vec{g}) + \nu_x(\vec{h}).$$

The definition of ν_x involves two summands – one involving the symplectic area and one involving the moment map. We first show that the terms involving the moment maps are equal. And indeed

$$\begin{aligned}\int_0^1 \mu(X_t)(g_t \cdot x)dt &+ \int_0^1 \mu(\mathrm{Ad}_{g_1} Y_t)(g_1 \cdot h_t \cdot x)dt \\ &= \int_0^1 \mu(X_t)(g_t \cdot x)dt + \int_0^1 \mu(Y_t)(h_t \cdot x)dt,\end{aligned} \tag{12}$$

by the equivariance of the moment map.

Now we show that the terms involving symplectic area agree up to the function

$$\int_{\Delta(x, g\cdot x, gh\cdot x)} \Omega \tag{13}$$

which is bounded by a constant $C_{\mathfrak{X}}$ depending only on the Domic–Toledo space $(\mathcal{X}, \Omega, \mathcal{K})$. Indeed choosing arbitrary disks of integration $D_{\vec{g}}$ for $\vec{g}$ and $D_{\vec{h}}$ for $\vec{h}$, choose for $\vec{g} \# g_1\vec{h}$ the disk $D_{\vec{g}\# g_1\vec{h}} = (D_{\vec{g}} \cup g_1 \cdot D_{\vec{h}}) \cup \Delta(x, g\cdot x, gh\cdot x)$ where the gluing is over the common path $[x, g\cdot x] \cup g[x, h\cdot x]$, which equals $[x, g\cdot x] \cup [g\cdot x, gh\cdot x]$ by preservation of $\mathcal{K}$. Hence

$$\begin{aligned}\int_{D_{\vec{g}\# g_1\vec{h}}} \Omega &= \int_{D_{\vec{g}}} \Omega + \int_{g_1\cdot D_{\vec{h}}} \Omega + \int_{\Delta(x,g\cdot x,gh\cdot x)} \Omega \\ &= \int_{D_{\vec{g}}} \Omega + \int_{D_{\vec{h}}} \Omega + \int_{\Delta(x,g\cdot x,gh\cdot x)} \Omega,\end{aligned} \tag{14}$$

by preservation of Ω by the action. This finishes the proof of the quasimorphism property.

Now we discuss the independence of the homogeneization

$$\nu(\tilde{g}) = \lim_{k\to\infty} \frac{1}{k} \nu_x(\tilde{g}^k)$$

on the basepoint x. Take two basepoints x and x' and let $\{x_s\}_{s=0}^{1}$, $x_0 = x$, $x_1 = x'$ be a path in $\mathcal{X}$ connecting them. Note that it is enough for us to show that ν_x and $\nu_{x'}$ differ by a bounded function $\widetilde{\mathcal{G}} \to \mathbb{R}$. Let us compare $\nu_x(\vec{g})$ and $\nu_{x'}(\vec{g})$. Let $\delta := \vec{g}\cdot x \# [g\cdot x, x]$, $\delta' := \vec{g}\cdot x' \# [g\cdot x', x']$ and let D, D' be their contracting discs. Define the disk $C: [0,1]\times[0,1] \to \mathcal{X}$ by $C(s,t) = g_t\cdot x_s$. Moreover define S^0 be the contracting disk of $\{x_s\} \# [x', x]$. Then by preservation of $\mathcal{K}$ by the action $S^1 = g\cdot S^0$ will be the contracting disk of $\{g\cdot x_s\} \# [g\cdot x', g\cdot x]$. Note that $g\cdot x_s \equiv C(s,1)$. At last, define an adapted contracting disk of $[x,x'] \cup [x, g\cdot x'] \cup [g\cdot x, x'] \cup [g\cdot g, g\cdot x']$ as the union $Q = \Delta_0 \cup \Delta_1$ for the two geodesic triangles Δ_0, Δ_1 on $\{x, x', g\cdot x\}$ and on $\{g\cdot x, x', g\cdot x'\}$. Note then that $\Sigma = \overline{D_0} \cup D_1 \cup \overline{S^0} \cup S^1 \cup C \cup Q$ where the gluings go along the overlapping paths, is a sphere. Therefore

$$\begin{aligned}0 &= \int_{\Sigma} \Omega \\ &= \int_{\overline{D_0}\cup D_1\cup\overline{S^0}\cup S^1\cup C\cup Q} \Omega \\ &= \int_{D_1} \Omega - \int_{D_0} \Omega - \int_{S^0} \Omega + \int_{g\cdot S^0} \Omega + \int_{Q} \Omega + \int_{C} \Omega \\ &= \int_{D_1} \Omega - \int_{D_0} \Omega + \int_{Q} \Omega + \int_{C} \Omega \\ &= \nu_x(\tilde{\phi}) - \nu_{x'}(\tilde{\phi}) + \int_0^1 \mu(X_t)(g_t\cdot x) - \int_0^1 \mu(X_t)(g_t\cdot x') + \int_C \Omega + \int_Q \Omega\end{aligned} \tag{15}$$

since the action of $\widetilde{\mathcal{G}}$ on $\mathcal{X}$ preserves Ω. Wherefrom

$$|\nu_x(\tilde{g}) - \nu_{x'}(\tilde{g})| \leq |Z| + 2C_{\mathcal{X}}, \tag{16}$$

for $Z = \int_0^1 \mu(X_t)(g_t \cdot x) - \int_0^1 \mu(X_t)(g_t \cdot x') + \int_C \Omega$. We now show that Z equals zero, finishing the proof. And indeed letting $\{X_t\}_{t=0}^1$ be the path in Lie($\mathcal{G}$) corresponding to $\vec{g}$ and $\Xi_t = \bar{X}_t$, we have

$$\begin{aligned}\int_C \Omega &= \int_0^1 \int_0^1 \Omega(\partial_s C(s,t), \partial_t C(s,t)) ds dt \\ &= \int_0^1 \int_0^1 \Omega(\partial_s C, \Xi_t(C(s,t))) ds dt \\ &= \int_0^1 \int_0^1 d_{C(s,t)} \mu(X_t)(\partial_s C) ds dt \\ &= \int_0^1 \int_0^1 \partial_s \mu(X_t)(C(s,t)) ds dt \\ &= \int_0^1 \mu(X_t)(C(1,t)) - \mu(X_t)(C(0,t)) dt \\ &= \int_0^1 \mu(X_t)(g_t \cdot x')\, dt - \int_0^1 \mu(X_t)(g_t \cdot x) dt. \qquad \square\end{aligned}$$

We also prove Proposition 1.5.1 on the transformation of ν_x under conjugation with respect to a suitable normal extension.

Proof. Consider a path $\vec{g} = \{g_t\}_{t=0}^1$ representing $\tilde{g} \in \widetilde{\mathcal{G}}$. Then for an element $h \in \mathcal{H}$ the path $h\vec{g}h^{-1} = \{hg_t h^{-1}\}_{t=0}^1$ will represent $h\tilde{g}h^{-1}$. By definition

$$\nu_x(h\vec{g}h^{-1}) = \int_{D_{h\vec{g}h^{-1}}} \Omega - \int_0^1 \mu(\mathrm{Ad}_h X_t)(hg_t h^{-1} \cdot x) dt$$

for a disk $D_{h\vec{g}h^{-1}}$ with boundary $\delta_x = \vec{g} \cdot x \# [g \cdot x, x]$, and noting that by preservation of $\mathcal{K}$ we have the relation $h \cdot \delta_{h^{-1} \cdot x} = \delta_x$ for $\delta_{h^{-1} \cdot x} = \vec{g} \cdot (h^{-1} \cdot x) \# [g \cdot h^{-1} \cdot x, h^{-1} \cdot x]$ so that the disk D satisfying

$$h \cdot D = D_{h\vec{g}h^{-1}}$$

has boundary $\delta_{h^{-1} \cdot x}$, so that by preservation of Ω and by equivariance of μ with respect to $\mathcal{H}$ we have

$$\nu_x(h\vec{g}h^{-1}) = \int_D \Omega - \int_0^1 \mu(X_t)(g_t h^{-1} \cdot x) dt = \nu_{h^{-1} \cdot x}(\vec{g}),$$

which proves the proposition. Note that for every $\tilde{h} \in \widetilde{\mathcal{H}}$ with endpoint h we have $\tilde{h}\tilde{g}\tilde{h}^{-1} = h\tilde{g}h^{-1}$ since the paths $\{h_t g_t h_t^{-1}\}_{t=0}^1$ and $h\vec{g}h^{-1}$ are homotopic with fixed endpoints. $\square$

2.3. Finite-dimensional examples and Guichardet–Wigner quasimorphisms. In this section we define the Guichardet–Wigner quasimorphisms and prove Proposition 1.9.1 on reconstructing these through moment maps.

We remark that as we have assumed that G has finite center, there are no homogenous quasimorphisms on G (cf. [85], [18] and [7] for the group $\mathrm{Sp}(2n,\mathbb{R})$). Moreover it is known that the all homogenous quasimorphisms on $\widetilde{G}$ are proportional to ϱ_G (cf. [85], [18], [7]). From these two remarks it follows that it is enough to show the equality of ν_G and ϱ_G on $\pi_1(G) \cong \pi_1(K)$. In fact, ϱ_G is defined as the unique homogenous quasimorphism $\widetilde{G} \to \mathbb{R}$ such that its pullback $\varrho_G|_{\widetilde{K}}\colon \widetilde{K} \to \mathbb{R}$ to $\widetilde{K}$ by the natural map $\widetilde{K} \to \widetilde{G}$ coincides with the lift $\tilde{v}\colon \widetilde{K} \to \mathbb{R}$ of the canonical (up to powers) character $v\colon K \to S^1$, constructed in either one of several ways. The first way is as follows. The Lie algebra $\mathfrak{k}$ of K satisfies $\mathfrak{k} = \mathfrak{z} + [\mathfrak{k},\mathfrak{k}]$ where $\mathfrak{z}$ is the center of $\mathfrak{k}$ (Corollaries 4.25 and 1.56 in [60]). In the case when G is a simple Hermitian symmetric Lie group, $\mathfrak{z}$ is one-dimensional by [60], p. 513. Hence the center Z of K is one-dimensional. Take the identity component $Z_0 \cong S^1$ of Z. Then by Theorem 4.29 in [60] $K = (Z_0)K_{\mathrm{ss}}$, for K_{ss} the analytic subgroup with Lie algebra $[\mathfrak{k},\mathfrak{k}]$. The group K_{ss} has a finite center, therefore by taking quotients by K_{ss} we get a homomorphism $v\colon K \to Q \cong S^1$ from K to the quotient $Q \cong S^1$ of $Z_0 \cong S^1$ by a finite subgroup.

Example 1. For $G = \mathrm{Sp}(2n,\mathbb{R})$ we have $K \cong \mathrm{U}(n)$ and $K_{\mathrm{ss}} \cong \mathrm{SU}(n)$. Therefore the first construction gives the homomorphism $v\colon K \to \mathrm{U}(n)/\mathrm{SU}(n) \cong S^1$ is simply $v(k) = \det_{\mathbb{C}}(k)$.

The second way to construct $v\colon K \to S^1$ is by use of the action of G on the Hermitian symmetric space $X = G/K$ – it is shown in [55] that v equals the determinant of the linearization of the natural action of $K \subset G$ at the fixed point $x = [\mathrm{Id}] \in X = G/K$. Note that the two constructions of v agree up to the power $-2\dim_{\mathbb{C}}(X)/\#(Z_0 \cap K_{\mathrm{ss}})$ since the determinant of a scalar matrix equals the scalar raised to the power of the dimension of the space (cf. [58] – proof of Theorem 6.1 and [55] – proof of Théorème 2).

Example 2. For $G = \mathrm{Sp}(2n,\mathbb{R})$ we have $K_{\mathrm{ss}} \cong \mathrm{SU}(n)$, $Z_0 \cong D$ – the subgroup of diagonal matrices in $\mathrm{U}(n)$ and $\#(Z_0 \cap K_{\mathrm{ss}}) = n$. As in this case $\dim_{\mathbb{C}}(X) = n(n+1)/2$, the second construction gives the homomorphism $v(k) = \det_{\mathbb{C}}^{-(n+1)}(k)$.

We use the second way to define ρ_G now. Proposition 1.9.1 is then demonstrated as follows.

Proof. Consider the point $x = [\mathrm{Id}] \in X = G/K$. It is a fixed point under the natural action of $K \subset G$. By the construction of ν_x and of the equivariant moment map $\mu\colon \mathrm{Lie}(G) \times X \to \mathbb{R}$ for a path $\vec{k} = \{k_t\}_{t=0}^1$ in K with $k_0 = \mathrm{Id}$ representing $\tilde{k} \in \widetilde{K}$, we have

$$\begin{aligned}
\nu_x(\tilde{k}) &= -\int_0^1 \mu(\eta_t)(x)dt = -\int_0^1 (\alpha_X)_y(\hat{\eta}_y)dt \\
&= -\frac{1}{i}\int_0^1 \frac{d}{dt'}|_{t'=t} \det{}_{\mathbb{C}}((k_{t'})_{*_x}) \det{}_{\mathbb{C}}((k_t)_{*_x})^{-1} dt \\
&= -\text{varangle}(\{\det{}_{\mathbb{C}}((k_t)_{*_x})\}_{t=0}^1) \\
&= -\tilde{v}(\tilde{k}).
\end{aligned}$$

Hence ν_x equals $\tilde{v}$ on $\widetilde{K}$, and consequently ν_x equals ϱ_G on $\pi_1(G) \cong \pi_1(K)$. Therefore the homogeneization ν_G of ν_x equals $-\varrho_G$ on $\pi_1(G)$ and this confers the equality $\nu_G = -\varrho_G$ on the whole group $\widetilde{G}$. □

2.4. The equality of the homomorphisms $\mathcal{A}$ and I_{c_1} on $\pi_1(\mathrm{Ham}(M,\omega))$. Now we prove Theorem 2.

Proof. First we note the following equality due to Fujiki [44]. Given the bundle $\mathcal{Z}$ over $\mathcal{J}$ which has $\mathcal{Z}_J := (M, J)$ for the fiber over J, denote by $T_{\mathcal{Z},\mathcal{J}}$ the vertical bundle and take $c_1(K)$ to be the Chern form of the vertical canonical bundle K relative to the Hermitian metric given by $h(J) = g(J) - i\omega$ in the fiber over $J \in \mathcal{J}$, then

$$\Omega = \int_{\text{fiber}} c_1(K)\ p^*\omega^n, \tag{17}$$

for $p\colon \mathcal{Z} \to M$ the smooth projection map.

The Hamiltonian fiber bundle over S^2 corresponding to a loop $\gamma = \{\phi_t\}_{t=0}^1$ in $\mathcal{G}$ based at the identity can be described (cf. [76]) as $P_\gamma = M \times D_- \cup_\Phi M \times D_+$, where D_- and D_+ are two copies of the disk $\mathbb{D}$ and the gluing map $\Phi\colon \partial(M \times D_-) \cong M \times S^1 \to M \times S^1 \cong \partial(M \times D_+)$ is given by $\Phi\colon (x,t) \mapsto (\phi_t x, t)$.

Note that given a Hamiltonian loop $\gamma = \{\phi_t\}_{0\le t\le 1}$ the bundle $P = P_\gamma$ with a vertical compatible complex structure is obtained by a map $D\colon \mathbb{D} \to \mathcal{J}$ representing a relative homotopy class in $\pi_2(\mathcal{J}, \mathcal{G}_{J_0})$ corresponding to the loop $\gamma^{-1} = \{\psi_t = (\phi_t)^{-1}\}_{0\le t\le 1}$ – that is $\partial D\colon S^1 \to \mathcal{J}$ is given by $\{(\psi_t)_* J_0\}_{t=0}^1$. Note that $P|_{D_-}$ with its fiberwise complex structure is equal to $D^*\mathcal{Z}$. We denote by H_t the zero-mean-normalized Hamiltonian for γ and by G_t the zero-mean-normalized Hamiltonian for γ^{-1}. The two are related by the formula $G_t(x) = -H_t(\phi_t x)$.

Moreover

$$-I_{c_1}(\gamma) = \int_{\mathbb{D}} \int_{\text{fiber}} D^* c_1(K) u^n.$$

Since the coupling class u is represented by the form $\Upsilon := \{\omega \text{ on } M \times D_+;\ \omega + d(\psi(r)H_t(\phi_t x)dt)) \text{ on } M \times D_-\}$, we have

$$\begin{aligned} -I_{c_1} &= \int_{\mathbb{D}} \int_{\text{fiber}} D^* c_1(K)(\omega^n + n d(\psi(r) H_t(\phi_t x)\, dt\, \omega^{n-1})) \\ &= \int_{\mathbb{D}} \int_{\text{fiber}} D^* c_1(K)\omega^n + n \int_D \int_{\text{fiber}} D^* c_1(K) d(\psi(r) H_t(\phi_t x)\, dt\, \omega^{n-1}). \end{aligned}$$

By the result of Fujiki the first summand equals $\int_{\mathbb{D}} D^*\Omega$. It is therefore enough to show that the second summand equals $\int_0^1 dt \int_M S(\psi_t \cdot J_0) G_t(x)\omega^n$. The second summand satisfies

$$\begin{aligned} n \int_{\mathbb{D}} \int_{\text{fiber}} & D^* c_1(K) d(\psi(r) H_t(\phi_t x)\, dt\, \omega^{n-1}) \\ &= n \int_{M \times \mathbb{D}} d(D^* c_1(K) \psi(r) H_t(\phi_t x) dt \omega^{n-1}) \\ &= n \int_{M \times S^1} H_t(\phi_t x) D^* c_1 \omega^{n-1} dt \end{aligned}$$

and by Equation (6) we have

$$= \int_0^1 \int_M S(\psi_t \cdot J_0) H_t(\phi_t x)\omega^n(x) dt = -\int_0^1 \int_M S(\psi_t \cdot J_0) G_t(x) \omega^n(x) dt.$$

Consequently we have $I_{c_1}(\gamma) = -\mathcal{A}_\mu(\gamma^{-1}) = A(\gamma)$. □

2.5. The finite-dimensional case $G = \mathrm{Sp}(2n, \mathbb{R})$ and the Maslov quasimorphism. In this section we would like to write out the finite-dimensional example more explicitly in the case $G = \mathrm{Sp}(2n, \mathbb{R})$ – for later use in particular. When $G = \mathrm{Sp}(2n, \mathbb{R})$ the maximal compact subgroup is $K \cong \mathrm{U}(n)$ and the space $X = G/K$ has several guises. First it can be considered as the Siegel upper half-space [86] $\mathcal{S}_n = \{X + iY \mid X, Y \in \mathrm{Mat}(n, \mathbb{R}),\ X = X^t, Y = Y^t,\ Y > 0\} \subset \mathrm{Mat}(n, \mathbb{C})$. Here there is a natural Kähler form $\sigma_{\text{Siegel}} = \mathrm{trace}(Y^{-1}dX \wedge Y^{-1}dY)$ where the complex structure comes from the one on $\mathrm{Mat}(n, \mathbb{C})$. This form is Kähler–Einstein with cosmological constant $\lambda = -\frac{n+1}{2}$ [86], that is,

$$\mathrm{Ric}(\sigma_{\text{Siegel}}) = -\frac{n+1}{2}\sigma_{\text{Siegel}}. \tag{18}$$

From which, since proportional metrics have equal Ricci forms, we have

$$\sigma_{\text{Siegel}} = \frac{2}{n+1}\sigma_{\text{Bergman}}, \tag{19}$$

for the Bergman Kähler form σ_{Bergman} on X.

Second, the space $X = G/K$ can be considered as the space J_c of ω_{std}-compatible complex structures on the symplectic vector space $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$. In this model, a natural symplectic form σ_{trace} is given by $(\sigma_{\mathrm{trace}})_J(A, B) = \frac{1}{4}\mathrm{trace}(JAB)$ for $J \in \mathrm{J}_c$ and $A, B \in T_J(\mathrm{J}_c)$. A short computation based on the fact that all G-invariant 2-forms on X are proportional gives

$$\sigma_{\mathrm{trace}} = \frac{1}{2}\sigma_{\mathrm{Siegel}}, \tag{20}$$

under the natural isomorphisms $\mathrm{J}_c \cong X$, $\mathcal{S}_n \cong X$.

By Examples 1 and 2 the Maslov quasimorphism $\tau_{\mathrm{Lin}} \colon \widetilde{G} \to \mathbb{R}$ restricting on $\widetilde{K}$ to $\tilde{v}$ for $v = \det^2_{\mathbb{C}}$ on $K \cong \mathrm{U}(n)$ can be written as

$$\tau_{\mathrm{Lin}} = \frac{2}{n+1}\nu_{G,\mathrm{Bergman}} \tag{21}$$

in terms of ν_G for $\sigma_X = \sigma_{\mathrm{Bergman}}$. Therefore by Equation (19)

$$\tau_{\mathrm{Lin}} = \nu_{G,\mathrm{Siegel}} \tag{22}$$

for $\sigma_X = \sigma_{\mathrm{Siegel}}$, and by Equation (20)

$$\frac{1}{2}\tau_{\mathrm{Lin}} = \nu_{G,\mathrm{trace}} \tag{23}$$

for $\sigma_X = \sigma_{\mathrm{trace}}$.

Note that by [86] σ_{Siegel} and consequently σ_{trace} has non-positive sectional curvature. Moreover σ_{trace} is Kähler–Einstein with cosmological constant $-(n+1)$.

Now consider $X \cong \mathrm{J}_c$ with $\sigma_X = \sigma_{\mathrm{trace}}$. By Equation (23), Lemma 1.4.1 and by the definition of ν_x we have $-\frac{1}{2}\tau_{\mathrm{Lin}} = \nu_G \simeq \nu_x(\vec{g}) = \int_{D_{\vec{g}}} \sigma_X - \int_0^1 \mu(\xi_t)(g_t \cdot x)dt$. Hence

$$\int_{D_{\vec{g}}} \sigma_X \simeq \frac{1}{2}\tau_{\mathrm{Lin}}(\vec{g}) + \int_0^1 \mu(\xi_t)(g_t \cdot x)dt. \tag{24}$$

For later calculations we will want the moment map summand in this formula more explicit. We write a formula for μ using the fact that it is an equivariant moment map for the action of the semisimple Lie group $\mathrm{Sp}(2n, \mathbb{R})$ (cf. [60]) on $\mathcal{S}_n$. Note that an equivariant moment map for the action of a semisimple Lie group on a symplectic manifold is unique [66]. Hence it is enough to show the following.

Lemma 2.5.1. *Consider the action of* $\mathrm{Sp}(2n, \mathbb{R})$ *on* $\mathcal{S}_n \cong \mathrm{J}_c$ *with the invariant Kähler form* σ_{trace}. *Then it is Hamiltonian with the equivariant moment map* $\mu_{\mathcal{S}_n} \colon \mathcal{S}_n \times \mathfrak{sp}(2n, \mathbb{R}) \to \mathbb{R}$ *given by* $\mu_{\mathcal{S}_n}(J)(\Xi) = -\frac{1}{2}\mathrm{trace}(\Xi J)$.

Proof. The symplectic form σ on $\mathcal{S}_n$ can be described as $\sigma(A, B) = \frac{1}{4}\mathrm{trace}(JAB)$ using the isomorphism $\mathcal{S}_n \cong \mathrm{J}_c$ – the space of complex structures on $\mathbb{R}^{2n}$ compatible with the standard symplectic form. Let us first compute the vector field $\overline{\Xi}$ generated by the infinitesimal action of Ξ. At a point $J \in \mathcal{S}_n$, denoting $\Phi_t = \exp(t\Xi) \in \mathrm{Sp}(2n, \mathbb{R})$ we have $\overline{\Xi}_J = \frac{d}{dt}|_{t=0}\Phi_t \cdot J = \frac{d}{dt}|_{t=0}\Phi_t J \Phi_t^{-1} = \Xi J - J\Xi = -[J, \Xi]$. Then for $B \in T_J\mathcal{S}_n$ we compute

$$d_J(\mathrm{trace}(\Xi J))(B) = \mathrm{trace}(\Xi B).$$

Finally, for $B \in T_J\mathcal{S}_n$ we have

$$\begin{aligned}(i_{\overline{\Xi}}\sigma)_J(B) &= \sigma_J(\overline{\Xi}_J, B) = -\sigma_J([J,\Xi], B) = -\frac{1}{4}\mathrm{trace}(J[J,\Xi]B) \\ &= -\frac{1}{4}\mathrm{trace}(-J\Xi JB + J^2\Xi B) = \frac{1}{4}\mathrm{trace}(\Xi JBJ + \Xi B) \\ &= \frac{1}{2}\mathrm{trace}(\Xi B).\end{aligned}$$

The last expression equals $-d_J(-\frac{1}{2}\mathrm{trace}(\Xi J))(B)$ as we have computed, and we are done. □

2.6. The local type of the quasimorphism on $\widetilde{\mathbf{Ham}}(M,\omega)$. We shall now describe the local behaviour of the quasimorphism $\mathfrak{S}$ – we compute its restriction to subgroups $\mathcal{G}_B \subset \mathcal{G}$ of diffeomorphisms supported in embedded balls B in M, proving Theorem 3.

Definition 2.6.1 (Embedded balls). We denote by $\mathcal{U}$ the set of embedded balls in M.

Given a symplectic manifold (M,ω) with an almost complex structure $J_0 \in \mathcal{J}$ with Hermitian scalar curvature $S(J_0)$ of mean $c = n\int_M c_1(TM,\omega)\omega^{n-1}/\int_M \omega^n$, and $B \in \mathcal{U}$ an embedded ball in M, we will show that the restriction $\nu_B = \mathfrak{S}|_{\mathcal{G}_B}$ of the quasimorphism $\mathfrak{S}$ to $\mathcal{G}_B = \mathrm{Ham}_c(B, \omega|_B)$ satisfies

$$\nu_B = \frac{1}{2}\tau_B - c\mathrm{Cal},$$

where τ_B is the Barge–Ghys Maslov quasimorphism on $\mathcal{G}_B = \mathrm{Ham}_c(B^{2n}, \omega_{\mathrm{std}})$ and Cal is the Calabi homomorphism.

Since the quasimorphism $\mathfrak{S}$ is homogenous and its distance from $\mathfrak{S}_{J_0}$ is bounded we can make calculations with $\mathfrak{S}_{J_0}$ allowing for an error term that vanishes under homogenization. The proof consists of writing ν_B (using Section 2.5) as the sum of $\frac{1}{2}\tau$ and a remainder term. Then we use some differential geometry to show that the remainder term equals a multiple of the Calabi homomorphism. For the differential geometry part we would like to use the canonical connection on the Hermitian manifold (M,ω,J_0) that is defined by the following of its properties. It preserves ω and J_0 and its torsion has vanishing $(1,1)$-component:

$$\nabla J_0 = 0, \nabla\omega = 0, T_\nabla^{(1,1)} \equiv 0. \tag{25}$$

This connection has an equivalent definition in terms of $\bar{\partial}$-operators on complex vector bundles, which is the one used in [30]. It is sometimes called "the Chern connection", and sometimes "the second canonical connection of Ehresmann–Liebermann" (cf. [51], Section 2, [61] and [87], Section 2).

Consider B as a smooth embedding $B\colon B^{2n} \to M$ from the standard ball

$$B^{2n} = \{(z_1, \dots, z_n) \mid \Sigma_{j=1}^n |z_j|^2 < 1\} \subset \mathbb{C}^n$$

to M. For purposes of trivialization and estimates choose for each two points $x, y \in B$ a path $\gamma_{x,y}$ starting at x and ending at y that depends continuously on $(x, y) \in B \times B$ where $\gamma_{x,x}$ is the constant path at x for all $x \in B$. This can be achieved for example by taking linear segments in B^{2n}. Then we have the following lemma.

Lemma 2.6.1. *Let $B_- \Subset B$ be any closed ball compactly contained in B. Then the following two statements hold by continuity and compactness of $B_- \times B_-$.*

(1) *For every one-form $\lambda \in \Omega^1(B)$ the function $B_- \times B_- \to \mathbb{R}$ given by $(x, y) \mapsto \int_{\gamma_{x,y}} \lambda$ is bounded by a constant depending only on B, B_-, λ.*

(2) *Given any connection ∇' preserving ω and any fixed symplectic trivialization $TB \cong V \times B$ for a symplectic vector space (V, ω_V), the map $B_- \times B_- \to \mathrm{Sp}(V, \omega_V)$ obtained by means of the trivialization by the parallel transport $\Gamma_{\gamma_{x,y}}\colon T_xB \to T_yB$ with respect to ∇' has a compact image in $\mathrm{Sp}(V, \omega_V)$.*

Take a path $\{\phi_t\}_{t=0}^1 \subset \mathcal{G}_B$ with $\phi_0 = \mathrm{Id}$. We shall now unwind the definition of $\nu_B(\{\phi_t\}_{t=0}^1)$. Over each $x \in B$ we have the fiber $\mathcal{S}_x$ of the bundle $\mathcal{S} \to B$. In $\mathcal{S}_x$ we have the path $(\phi_t \cdot J_0)_x$. Now we shall define a path $\Phi(x)_t$ in $\mathrm{Sp}(T_xM, \omega_x)$ associated to $(\phi_t)_{*x}$ such that under the action of $\mathrm{Sp}(T_xM, \omega_x)$ on $\mathcal{S}_x$, we have $\Phi(x)_t \cdot (J_0)_x = (\phi_t \cdot J_0)_x = (\phi_{t*\phi_t^{-1}x})(J_0)_{\phi_t^{-1}x}(\phi_{t*\phi_t^{-1}x})^{-1}$.

Indeed consider for each $t \in [0, 1]$ the path $\gamma_{x,\phi_t^{-1}x}$. The parallel transport along this path preserves J_0 and maps $\Gamma_{\gamma_{x,\phi_t^{-1}x}}\colon T_xM \to T_{\phi_t^{-1}x}M$. Then $\Phi(x)_t = (\phi_{t*\phi_t^{-1}x}) \circ \Gamma_{\gamma_{x,\phi_t^{-1}x}}\colon T_xM \to T_xM$ is the required map. Indeed

$$\begin{aligned}
\Phi(x)_t \cdot (J_0)_x &= \Phi(x)_t (J_0)_x \Phi(x)_t^{-1} \\
&= (\phi_{t*\phi_t^{-1}x})\Gamma_{\gamma_{x,\phi_t^{-1}x}}(J_0)_x(\Gamma_{\gamma_{x,\phi_t^{-1}x}})^{-1}(\phi_{t*\phi_t^{-1}x})^{-1} \\
&= (\phi_{t*\phi_t^{-1}x})(J_0)_{\phi_t^{-1}x}(\phi_{t*\phi_t^{-1}x})^{-1} \\
&= (\phi_t \cdot J_0)_x,
\end{aligned}$$

by preservation of J_0. Henceforth we omit the subscript z in $(J_0)_z$ whenever this is determined by the context.

Then for all $x \in B$ we have the loop $\delta(x) = \{\Phi(x)_t \cdot J_0\}_{t=0}^1 \# \overline{[J_0, \Phi(x)_1 \cdot J_0]}$. We then for all $x \in B$ choose a disk $D(x)$ that bounds $\delta(x)$ – in fact one can construct $D(x)$ as the geodesic join of $\{\Phi(x)_t \cdot J_0\}_{t=0}^1$ with J_0 – that is $D(x) = \bigcup_t \overline{[J_0, \Phi(x)_t \cdot J_0]}$ properly parametrized. Denote $\gamma_t(x) = \overline{[J_0, \Phi(x)_t \cdot J_0]}$. Denote by $\beta_t(x)$ the path $\{\Phi(x)_{t'} \cdot J_0\}_{t'=0}^{t'=t}$.

Recall from Section 1.4 that $a \simeq b$ denotes the equality of the functions a, b up to a function that is bounded by a constant that does not depend on their arguments. Compute

$$\begin{aligned} \nu_B(\{\phi_t\}_{t=0}^1) &\simeq \int_D \Omega - \int_0^1 \mu(X_t)(\phi_t \cdot J_0) \\ &= \int_B \Big(\int_{D(x)} \sigma_x \Big) \omega^n(x) - \int_0^1 \int_M S(\phi_t \cdot J_0) H_t(x) \omega^n(x). \end{aligned} \tag{26}$$

Now note that by Equation (24) and the definition of the moment map for the action of $G = \mathrm{Sp}(2n)$ on $X = G/K$,

$$\int_{D(x)} \sigma_x \simeq \frac{1}{2} \tau_{\mathrm{Lin}}(\{\Phi(x)_t\}_{t=0}^1) + \int_0^1 h(x)_t(\Phi(x)_t \cdot (J_0)_x) dt, \tag{27}$$

where the function $h(x)_t(\cdot) = \mu_{S_x}(\Xi(x)_t)(\cdot)$, for $\Xi(x)_t = \frac{d}{d\tau}|_{\tau=t} \Phi(x)_\tau \circ \Phi(x)_t^{-1}$, is the contact Hamiltonian for the canonical lifting of $\Phi(x)_t$ to the principal S^1-bundle of unit vectors in $\Lambda^N T\mathcal{S}_x$ simply by use of the differential (cf. Equation (10)). As a side remark it may be said, following [32], that this finite-dimensional moment map is the main reason for the existence of the corresponding infinite-dimensional moment map.

Consequently, integrating over B with respect to the form ω^n, we have

$$\begin{aligned} \nu_B(\{\phi_t\}_{t=0}^1) \simeq \frac{1}{2} \cdot \int_B & \tau_{\mathrm{Lin}}(\{\Phi(x)_t\}_{t=0}^1) \omega^n(x) \\ &+ \int_B \int_0^1 h(x)_t(\Phi(x)_t \cdot J_0)\, dt\, \omega^n(x) \\ &- \int_0^1 \int_M S(\phi_t \cdot J_0) H_t(x) \omega^n(x). \end{aligned} \tag{28}$$

By the definition of the Barge–Ghys Maslov quasimorphism on $\mathcal{G}$ and Lemma 2.6.1, the first term homogenizes to $\frac{1}{2}\tau_B$. Our goal is hence to compute the sum of the second and the third terms.

By Lemma 2.5.1 we rewrite the second term in Equation (27) as

$$\int_0^1 h(x)_t(\Phi(x)_t \cdot J_0) = - \int_0^1 \frac{1}{2} \mathrm{trace}(\Xi(x)_t(\Phi(x)_t \cdot J_0)) dt, \tag{29}$$

for $\Xi(x)_t = \frac{d}{d\tau}|_{\tau=t} \Phi(x)_\tau \circ \Phi(x)_t^{-1}$.

Now note that instead of using the parallel transport along $\gamma_{x,\phi_t^{-1}x}$ to define $\Phi(x)_t : T_x B \to T_x B$ we could use the one along $p_{x,t} = \{\phi_{t'}^{-1}x\}_{t'=0}^t$ to define the map $\Psi(x)_t = (\phi_{t*\phi_t^{-1}x}) \circ \Gamma_{p_{x,t}} : T_x B \to T_x B$. Then we have

$$\Phi(x)_t = \Psi(x)_t U(x,t), \tag{30}$$

for the unitary map $U(x,t) = \Gamma_{p_{x,t}}^{-1} \circ \Gamma_{\gamma_{x,\phi_t^{-1}x}} : T_x B \to T_x B$. Form $\Upsilon(x)_t = \frac{d}{d\tau}|_{\tau=t}\Psi(x)_\tau \circ \Psi(x)_t^{-1}$ and $\Theta(x,t) = \frac{d}{d\tau}|_{\tau=t}U(x,\tau) \circ U(x,t)^{-1}$. Then by Equation (30) we have

$$\Xi(x)_t = \Psi(x)_t \Theta(x,t) \Psi(x)_t^{-1} + \Upsilon(x)_t, \tag{31}$$

and

$$\begin{aligned} \Phi(x)_t \cdot J_0 &= \Phi(x)_t J_0 \Phi(x)_t^{-1} = \Psi(x)_t U(x,t) J_0 U(x,t)^{-1} \Psi(x)_t{}^{-1} \\ &= \Psi(x)_t J_0 \Psi(x)_t{}^{-1} = \Psi(x)_t \cdot J_0, \end{aligned} \tag{32}$$

because $U(x,t)$ is J_0-linear.

Therefore, by Equation (29) and noting that

$$\mathrm{trace}(\Psi(x)_t \Theta(x,t) \Psi(x)_t^{-1}(\Psi(x)_t \cdot J_0)) = \mathrm{trace}(\Theta(x,t) J_0)$$

we have

$$\begin{aligned} &\int_0^1 h(x)_t(\Phi(x)_t \cdot J_0) \\ &\quad = -\int_0^1 \frac{1}{2}\mathrm{trace}(\Theta(x,t) \cdot J_0)dt - \int_0^1 \frac{1}{2}\mathrm{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0))dt. \end{aligned} \tag{33}$$

Note additionally that

$$\frac{1}{2}\mathrm{trace}(\Theta(x,t) \cdot J_0) = -\frac{1}{i}\mathrm{trace}_{\mathbb{C}}(\Theta(x,t)),$$

considering $\Theta(x,t)$ as a skew-Hermitian operator on the complex Hermitian space $(T_x B, J_0, \omega_x)$. Moreover,

$$\mathrm{trace}_{\mathbb{C}}(\Theta(x,t)) = \Theta^n(x,t),$$

where

$$\Theta^n(x,t) = \frac{d}{d\tau}|_{\tau=t} U^n(x,\tau) \circ U^n(x,t)^{-1},$$

for $U^n(x,t) = (\Gamma_{p_{x,t}}^n)^{-1} \circ \Gamma_{\gamma_{x,\phi_t^{-1}x}}^n : \Lambda_{\mathbb{C}}^n T_x B \to \Lambda_{\mathbb{C}}^n T_x B$, for the naturally induced parallel translations on the Hermitian complex line bundle $\Lambda_{\mathbb{C}}^n TB$, endowing TB with the Hermitian structure (J_0, ω) and the connection ∇.

Therefore

$$\int_0^1 h(x)_t(\Phi(x)_t \cdot J_0) = \int_{D_B(x)} \rho - \frac{1}{2}\int_0^1 \operatorname{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0)), \tag{34}$$

where $i\rho$ is the curvature two-form of the connection ∇^n on $\Lambda^n_{\mathbb{C}}(TM, J_0)$ naturally induced from ∇ on (TM, J_0) and $D_B(x)$ is the disk spanned by $\bigcup_{t=0}^1 \gamma_{x,\phi_t^{-1}x}$. Note that $\partial D_B(x) = p_{x,1} \# \overline{\gamma_{x,\phi_1^{-1}x}}$. Now $\rho|_B \in \Omega^2_{\text{closed}}(B,\mathbb{R})$ has by the Poincaré lemma a primitive $\alpha \in \Omega^1(B,\mathbb{R})$. Hence by Stokes' formula we have

$$\int_{D_B(x)} \rho = \int_{p_{x,1}} \alpha - \int_{\gamma_{x,\phi_1^{-1}x}} \alpha. \tag{35}$$

Choosing $B_- \Subset B$ such that $\operatorname{supp}(\phi_t) \subset B_-$ for all $t \in [0,1]$, we have $\gamma_{y,\phi_1^{-1}y} \equiv y$ for all $y \in B \setminus B_-$, hence by Lemma 2.6.1 we have the following uniform estimate for the second term $|\int_{\gamma_{x,\phi_1^{-1}x}} \alpha| \leq C(B_-, B)$, for a constant $C(B_-, B, \alpha)$ depending only on α, $B_- \supset \bigcup_{t=0}^1 \operatorname{supp}(\phi_t)$ and on B.

Now denote $\psi_t = \phi_t^{-1}$. Denote Y_t the Hamiltonian vector generating ψ_t. Recall that $p_{x,1} = \{\phi_t^{-1}x\}_{t=0}^1 = \{\psi_t x\}_{t=0}^1$. Hence the first term in Equation (35) satisfies

$$\int_{p_{x,1}} \alpha = \int_0^1 ((\psi_t)^* i_{Y_t}\alpha)_x dt.$$

Hence integrating Equation (34) over B we express

$$\int_B \int_0^1 h(x)_t(\Phi(x)_t \cdot J_0)\, dt\, \omega^n(x)$$

as

$$\begin{aligned}&\int_0^1 \int_B (\psi_t)^* i_{Y_t}\alpha\, \omega^n dt \\ &\quad - \frac{1}{2}\int_B \int_0^1 \operatorname{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0)) dt\, \omega^n(x) + \operatorname{Bdd}(\{\phi_t\}_{t=0}^1)\end{aligned} \tag{36}$$

for a function $\operatorname{Bdd}(\{\phi_t\}_{t=0}^1)$ that satisfies

$$|\operatorname{Bdd}(\{\phi_t\}_{t=0}^1)| \leq C_1(B_-, B, \alpha)$$

for a constant $C_1(B_-, B)$ depending only on α, $B_- \supset \bigcup_{t=0}^1 \operatorname{supp}(\phi_t)$ and on B.

We shall now show that the first term in Equation (36) corrected by the moment map term $\int_0^1 \mu(X_t)(\phi_t \cdot J_0) dt$ in the definition of the quasimorphism is proportional to the Calabi homomorphism. After that we will show that the second term vanishes.

Let G_t (for each $t \in [0,1]$) be the function that vanishes near ∂B and satisfies $i_{Y_t}\omega = -dG_t$. Then $i_{Y_t}\alpha\ \omega^n = n\alpha\ i_{Y_t}\omega\ \omega^{n-1} = -n\alpha\ dG_t\ \omega^{n-1}$. Hence

$$\begin{aligned}\int_B (\psi_t)^* i_{Y_t}\alpha\,\omega^n &= \int_B i_{Y_t}\alpha\,\omega^n = n\int_B dG_t\ \alpha\ \omega^{n-1}\\ &= -n\int_B G_t d\alpha\omega^{n-1} = -n\int_B G_t d\alpha\omega^{n-1} = -n\int_B G_t\rho\omega^{n-1}\end{aligned}$$

and by definition of the Hermitian scalar curvature we have

$$= -\int_B G_t S(J_0)\omega^n$$

and, denoting H_t^0 (for each $t \in [0,1]$) the function that vanishes near ∂B and satisfies $i_{X_t}\omega = -dH_t^0$, and noting that by the cocycle formula [74] $G_t(x) = -H_t^0(\phi_t x)$, we have

$$= \int_B S(J_0)H_t^0(\phi_t x)\omega^n(x).$$

Hence

$$\begin{aligned}\int_0^1 dt \int_B (\psi_t)^* i_{Y_t}\alpha\,\omega^n - \int_0^1 dt\int_M S(\phi_t\cdot J_0)H_t(x)\omega^n(x)\\ = \int_0^1 dt\int_M S(\phi_t\cdot J_0)(H_t^0 - H_t)\omega^n,\end{aligned}$$

where we extend H_t^0 by zero from B to M, and noting that $H_t^0 - H_t$ depends on t only and equals the mean $\int_B H_t^0\omega^n / \int_M \omega^n$ we have

$$= -\Big(\int_M S(\phi_t\cdot J_0)\omega^n \Big/ \int_M \omega^n\Big)\int_0^1 dt\int_B H_t^0\omega^n = -c\cdot \mathrm{Cal}_B(\{\phi_t\}_{t=0}^1). \quad (37)$$

Now it remains to show that $\int_0^1\int_B \mathrm{trace}(\Upsilon(x)_t(\Psi(x)_t\cdot J_0))\omega^n(x)dt$ vanishes. First we would like to note that since the $(1,1)$-component of the torsion T of ∇ vanishes, we have

$$T(X, J_0Y) = T(J_0X, Y) \quad (38)$$

for all vector fields X, Y on M. Moreover since ∇ preserves J_0 we have

$$J_0\nabla_\bullet X = \nabla_\bullet(J_0X) \quad (39)$$

for all vector fields X on M, where for a vector field Z on M, we denote by $\nabla_\bullet Z$ the endomorphism of TM given by $Y \mapsto \nabla_Y Z$.

For a vector field Z on M define then the endomorphism A_Z of TM by $A_Z = L_Z - \nabla_Z$. Then by [62], Vol. 1, Appendix 6, page 292, we have

$$A_Z = -\nabla_\bullet Z - T(Z,\cdot) \quad (40)$$

and

$$-\operatorname{trace} A_Z = \operatorname{div}_{\omega^n}(Z), \tag{41}$$

where $\operatorname{div}_{\omega^n}(Z) \in C_0^\infty(M, \mathbb{R})$ is defined by

$$\operatorname{div}_{\omega^n}(Z)\, \omega^n = L_Z \omega^n.$$

Now we prove a formula relating the action of J_0 on TM and the tensor A_Z. We claim that for all vector fields X on M we have

$$\operatorname{trace}(A_X J_0) = \operatorname{trace}(A_{J_0 X}). \tag{42}$$

Indeed

$$\begin{aligned} -\operatorname{trace}(A_X J_0) &= \operatorname{trace}(\nabla_\bullet X \circ J_0 + T(X, J_0 \cdot)) && \text{by Equation (40)} \\ &= \operatorname{trace}(J_0 \nabla_\bullet X + T(J_0 X, \cdot)) && \text{by Equation (38)} \\ &= \operatorname{trace}(\nabla_\bullet J_0 X + T(J_0 X, \cdot)) = -\operatorname{trace} A_{J_0 X} && \text{by Equation (39).} \end{aligned}$$

Let us now compute $\Upsilon(x)_t = \frac{d}{d\tau}|_{\tau=t} \Psi(x)_\tau \circ \Psi(x)_t^{-1}$ in terms of the connection and of the vector field X_t generating the path of diffeomorphisms $\{\phi_t\}_{t=0}^1$. Recalling that $\Psi(x)_t = (\phi_{t*\phi_t^{-1}x}) \circ \Gamma_{p_{x,t}}$ we have

$$\frac{d}{d\tau}|_{\tau=t} \Psi(x)_\tau = (\phi_{t*\phi_t^{-1}x})(L_{X_t} - \nabla_{X_t})_{\phi_t^{-1}(x)} \Gamma_{p_{x,t}}.$$

Consequently,

$$\Upsilon(x)_t = (\phi_{t*\phi_t^{-1}x})(A_{X_t})_{\phi_t^{-1}(x)} (\phi_{t*\phi_t^{-1}x})^{-1} \tag{43}$$

for the endomorphism A_{X_t} of TM. Then

$$\begin{aligned} \operatorname{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0)) &= \operatorname{trace}((\phi_{t*\phi_t^{-1}x})(A_{X_t})_{\phi_t^{-1}(x)} (\phi_{t*\phi_t^{-1}x})^{-1} \\ &\qquad ((\phi_{t*\phi_t^{-1}x})\Gamma_{p_{x,t}}(J_0)_x \Gamma_{p_{x,t}}^{-1} (\phi_{t*\phi_t^{-1}x})^{-1})) \\ &= \operatorname{trace}((A_{X_t})_{\phi_t^{-1}(x)} \Gamma_{p_{x,t}}(J_0)_x \Gamma_{p_{x,t}}^{-1}) \\ &= \operatorname{trace}(A_{X_t} J_0)(\phi_t^{-1}(x)) \\ &= \operatorname{trace}(A_{J_0 X_t})(\phi_t^{-1}(x)), \end{aligned} \tag{44}$$

by Equation (42). Hence

$$\begin{aligned} &\int_0^1 \int_B \operatorname{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0))\omega^n(x) dt \\ &\quad = \int_0^1 \int_B \operatorname{trace}(A_{J_0 X_t})(\phi_t^{-1}(x))\omega^n(x) dt \\ &\quad = \int_0^1 \int_B \operatorname{trace}(A_{J_0 X_t})\omega^n dt \\ &\quad = -\int_0^1 \int_B \operatorname{div}(J_0 X_t)\omega^n dt = 0. \end{aligned} \tag{45}$$

Therefore, assembling Equations (26), (36), (37), (45) and Definition 1.7.3 we have

$$\nu_B(\{\phi_t\}_{t=0}^1) = \frac{1}{2} \cdot \tau_B(\{\phi_t\}_{t=0}^1) - c \cdot \mathrm{Cal}_B(\{\phi_t\}_{t=0}^1) + \mathrm{Bdd}_2(\{\phi_t\}_{t=0}^1),$$

for a function $\mathrm{Bdd}_2(\{\phi_t\}_{t=0}^1)$ bounded by a constant $C_2(B_-, B, \alpha)$ that depends only on B, α and $B_- \supset \bigcup_{t=0}^1 \mathrm{supp}(\phi_t)$. Noting that $\mathrm{supp}({\phi_t}^k) \subset \mathrm{supp}(\phi_t)$ for every $t \in [0,1]$, $k \in \mathbb{Z}$ and homogenizing, we finish the proof.

2.7. The restriction to the Py quasimorphism. In this section we prove the first point of Theorem 4 on the equality of the Py quasimorphism of Definition 1.7.4 and the general quasimorphism from Corollary 1 when the symplectic manifold (M, ω) is monotone – that is $c_1(TM, \omega) = \kappa[\omega]$ where $\kappa \neq 0$. The computation is somewhat similar to that of the local type – with the exception that there is no trivialization involved really.

As in the computation of the local type, we use the parallel transport along $p_{x,t} = \{\phi_{t'}^{-1} x\}_{t'=0}^t$ to define the map $\Psi(x)_t = (\phi_{t*\phi_t^{-1}x}) \circ \Gamma_{p_{x,t}} : T_x B \to T_x B$. Then $\Upsilon(x)_t = \frac{d}{d\tau}|_{\tau=t} \Psi(x)_\tau \circ \Psi(x)_t^{-1}$ will satisfy

$$\Upsilon(x)_t = (\phi_{t*\phi_t^{-1}x})(A_{X_t})_{\phi_t^{-1}(x)} (\phi_{t*\phi_t^{-1}x})^{-1}$$

for the endomorphism A_{X_t} of TM, for $A_{X_t} = L_{X_t} - \nabla_{X_t}$ as in Equation (43). Then

$$\mathrm{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0)) = \mathrm{trace}(A_{J_0 X_t})(\phi_t^{-1}(x)),$$

as before in Equation (44). Moreover, identically to Equation (45) we have

$$\int_M \mathrm{trace}(\Upsilon(x)_t(\Psi(x)_t \cdot J_0))\omega^n(x) = 0. \tag{46}$$

We shall now rewrite $\mathfrak{S}_{J_0}(\{\phi_t\}_{t=0}^1)$ via $\Psi_t(x)$. For all $x \in B$ we have the loop $\delta(x) = \{\Psi(x)_t \cdot J_0\}_{t=0}^1 \# \overline{[J_0, \Phi(x)_1 \cdot J_0]}$. We then for all $x \in B$ choose a disk $D(x)$ that bounds $\delta(x)$ – in fact one can construct $D(x)$ as the geodesic join of $\{\Psi(x)_t \cdot J_0\}_{t=0}^1$ with J_0 – that is $D(x) = \bigcup_t \overline{[J_0, \Psi(x)_t \cdot J_0]}$ properly parametrized. Denote $\gamma_t(x) = \overline{[J_0, \Psi(x)_t \cdot J_0]}$. Denote by $\beta_t(x)$ the path $\{\Psi(x)_{t'} \cdot J_0\}_{t'=0}^{t'=t}$.

Compute

$$\begin{aligned} \mathfrak{S}_{J_0}(\{\phi_t\}_{t=0}^1) &= \int_D \Omega - \int_0^1 \mu(X_t)(\phi_t \cdot J_0) \\ &= \int_B \Big(\int_{D(x)} \sigma_x \Big) \omega^n(x) - \int_0^1 \int_M S(\phi_t \cdot J_0) H_t(x) \omega^n(x). \end{aligned} \tag{47}$$

Now as before by Equation (24) and the definition of the moment map for the action of $G = \mathrm{Sp}(2n)$ on $X = G/K$

$$\int_{D(x)} \sigma_x \simeq \frac{1}{2}\tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1) - \int_0^1 f(x)_t(\Psi(x)_t \cdot J_0)dt \tag{48}$$

where the function

$$f(x)_t(J) = -\frac{1}{2}\mathrm{trace}(\Upsilon(x)_t J) \tag{49}$$

is the contact Hamiltonian for the canonical lifting of $\Psi(x)_t$ to the principal S^1-bundle of unit vectors in $\Lambda_{\mathbb{C}}^N T\mathcal{S}_x$, $N = n(n+1)/2$ (cf. Equation (10)). Hence by Equations (47), (48), (49) and (46) we have

$$\mathfrak{S}_{J_0}(\{\phi_t\}_{t=0}^1) \simeq \frac{1}{2}\int_M \tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1)\omega^n(x) - \int_0^1\int_M S(\phi_t \cdot J_0)H_t(x)\omega^n(x). \tag{50}$$

We shall now rewrite the Py quasimorphism S_2 from Definition 1.7.4 via $\Psi(x)_t$. Then comparing the effect of the difference in connections with the second term in Equation (50) we shall establish the equality.

First we note that the connection ∇ gives us a parallel transport on $\mathcal{L}(TM,\omega)$ and on P^2, since it preserves J_0 and ω. Moreover, since the map $\det^2 \colon \mathcal{L}(TM,\omega) \to P^2$ is defined using only J_0 and ω the following diagram commutes:

$$\begin{tikzcd} \mathcal{L}(TM,\omega)_{(\phi_t)^{-1}x} \arrow[r, "\Gamma_{\overline{p_{x,t}}}"] \arrow[d, "\det^2"'] & \mathcal{L}(TM,\omega)_x \arrow[d, "\det^2"] \\ P^2_{(\phi_t)^{-1}x} \arrow[r, "\Gamma_{\overline{p_{x,t}}}"] & P^2_x. \end{tikzcd} \tag{51}$$

In other words for $L_0 \in \mathcal{L}_{(\phi_t)^{-1}x}(TM,\omega)$ we have $\det^2(\Gamma_{\overline{p_{x,t}}}L_0) = \Gamma_{\overline{p_{x,t}}}\det^2(L_0)$. It will be more convenient to compute S_2 on the inverse path $\{\psi_t = \phi_t^{-1}\}_{t=0}^1$. Indeed consider the paths $\det^2(\psi_{t*_x}L)$ and $\hat{\psi}_t(\det^2(L))$ in P^2 for $L \in \mathcal{L}(TM,\omega)_x$. These paths differ by an angle as follows

$$\det^2(\psi_{t*_x}(L)) = e^{i2\pi\vartheta(t)}\hat{\psi}_t(\det^2(L)).$$

Then the paths $\Gamma_{\overline{p_{x,t}}}\det^2(\psi_{t*_x}L) = \det^2(\Gamma_{\overline{p_{x,t}}}\psi_{t*_x}L)$ (here we use Equation (51)) and $\Gamma_{\overline{p_{x,t}}}\hat{\psi}_t(\det^2(L))$ in $(P^2)_x$ also differ by the same angle. And since these are paths in one fiber, we have

$$\begin{aligned} \mathrm{angle}(L,\{\psi_t\}_{t=0}^1) &= \mathrm{varangle}(\{e^{i2\pi\vartheta(t)}\}_{t=0}^1) \\ &= \mathrm{varangle}(\{\det^2(\Gamma_{\overline{p_{x,t}}}\psi_{t*_x}L)\}_{t=0}^1) - \mathrm{varangle}(\{\Gamma_{\overline{p_{x,t}}}\hat{\psi}_t(\det^2(L))\}_{t=0}^1). \end{aligned} \tag{52}$$

Note that the second term in Equation (52) does not depend on the choice of $L \in \mathcal{L}(TM,\omega)_x$, since both $\Gamma_{\overline{p_{x,t}}}$ and $\hat{\psi}_t$ commute with rotations of the fibers. Therefore the function

$$\mathrm{angle}(x, \{\psi_t\}_{t=0}^1) = \inf_{L\in\mathcal{L}(TM,\omega)_x} \mathrm{angle}(L, \{\psi_t\}_{t=0}^1)$$

satisfies

$$\begin{aligned}\mathrm{angle}(x, \{\psi_t\}_{t=0}^1) = \inf_{L\in\mathcal{L}(TM,\omega)_x} &\big(\mathrm{varangle}(\{\det{}^2(\Gamma_{\overline{p_{x,t}}}\psi_{t*_x}L)\}_{t=0}^1)\big)\\ &- \mathrm{varangle}(\{\Gamma_{\overline{p_{x,t}}}\hat{\psi}_t y\}_{t=0}^1),\end{aligned} \tag{53}$$

for any $y \in (P^2)_x$. Note first that $\psi_{t*_x} = (\phi_{t*(\phi_t)^{-1}x})^{-1}$ and therefore $\Gamma_{\overline{p_{x,t}}}\psi_{t*_x} = \Psi(x)_t^{-1}$. Then note that

$$\begin{aligned}\mathrm{varangle}(\{\det{}^2(\Gamma_{\overline{p_{x,t}}}\psi_{t*_x}L)\}_{t=0}^1) &\simeq \tau_{\mathrm{Lin}}(\{\Psi(x)_t^{-1}\}_{t=0}^1)\\ &= -\tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1)\end{aligned} \tag{54}$$

by the construction of the Maslov quasimorphism on the universal cover of the linear symplectic group using its action on the Lagrangian Grassmannian [7]. Therefore

$$\inf_{L\in\mathcal{L}(TM,\omega)_x} (\mathrm{varangle}(\{\det{}^2(\Gamma_{\overline{p_{x,t}}}\psi_{t*_x}L)\}_{t=0}^1)) \simeq -\tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1). \tag{55}$$

Now it remains to interpret the integral over M with respect to ω^n of the term $\mathrm{varangle}(\{\Gamma_{\overline{p_{x,t}}}\hat{\psi}_t y\}_{t=0}^1)$ in Equation (53) via the Hermitian scalar curvature. For this purpose consider the two connection one-forms α and λ on P^2 – where $d\alpha = 2\tilde{\omega}$ and λ comes from the connection ∇ on TM and therefore satisfies $d\lambda = 2\tilde{\rho}$ (for a form η on M we denote by $\tilde{\eta}$ its lift by the natural projection $P^2 \to M$). These connection one-forms differ by $\tilde{\theta} = \alpha - \lambda$ for a one-form θ on M. Then denoting by Y_t the Hamiltonian vector field generating $\{\psi_t\}$ with normalized Hamiltonian G_t (by the zero mean condition), and by $\hat{Y}_t$ the vector field generating $\{\hat{\psi}_t\}$ we have

$$\mathrm{varangle}(\{\Gamma_{\overline{p_{x,t}}}\hat{\psi}_t y\}_{t=0}^1) = \int_0^1 (\hat{\psi}_t)^* i_{\hat{Y}_t}\tilde{\theta}(x)dt = \int_0^1 (\psi_t)^* i_{Y_t}\theta(x)dt. \tag{56}$$

We now compute as follows:

$$\int_M \int_0^1 (\psi_t)^* i_{Y_t}\theta(x)dt\,\omega^n(x) = \int_0^1 \int_M (\psi_t)^* i_{Y_t}\theta\omega^n dt = \int_0^1 \int_M i_{Y_t}\theta\omega^n dt. \tag{57}$$

It is therefore sufficient to compute the integrand

$$
\begin{aligned}
\int_M i_{Y_t}\theta\omega^n &= n\int_M \theta i_{Y_t}\omega\omega^{n-1}\\
&= -n\int_M \theta dG_t\omega^{n-1} = -n\int_M d\theta G_t\omega^{n-1}\\
&= -2n\int_M (\omega-\rho)G_t\omega^{n-1} = -2n\int_M G_t\omega^n + 2n\int_M G_t\rho\omega^{n-1}\\
&= 2n\int_M G_t\rho\omega^{n-1},
\end{aligned}
$$

by the definition of the Hermitian scalar curvature

$$= 2\int_M G_t S(J_0)\omega^n,$$

since $G_t(x) = -H_t(\phi_t x)$ by the cocycle formula

$$= -2\int_M S(J_0)H_t(\phi_t x)\omega^n(x). \tag{58}$$

Therefore by Equations (53), (55), (56), (58) we have from the definition of S_2 (Definition 1.7.4) that

$$- S_2(\{\psi_t\}_{t=0}^1) \simeq -\int_M \tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1)\omega^n(x) + 2\int_M S(J_0)H_t(\phi_t x)\omega^n(x). \tag{59}$$

Therefore by Lemma 1.4.1 we have

$$
\begin{aligned}
-S_2(\{\phi_t\}_{t=0}^1) &\simeq S_2(\{\psi_t\}_{t=0}^1)\\
&= \int_M \tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1)\omega^n(x) - 2\int_M S(J_0)H_t(\phi_t x)\omega^n(x).
\end{aligned} \tag{60}
$$

From Equations (50) and (60) we conclude that

$$2\mathfrak{S}_{J_0} \simeq -S_2,$$

which by homogenizing gives

$$2\mathfrak{S} = -\mathfrak{S}_{\mathrm{Py}}$$

finishing the proof.

2.8. The restriction to the Entov quasimorphism. Here we prove the second point of Theorem 4 on the agreement of the general quasimorphism of Corollary 1 and the quasimorphism of Entov [36] from Definition 1.7.5. First we give an alternative definition of Entov's quasimorphism along the lines of the definition of Py's quasimorphism, which will more easily be shown to agree with the general quasimorphism.

Definition 2.8.1 (A second definition of the quasimorphism $\mathfrak{S}_{\mathrm{En}}$). Given a symplectic manifold (M,ω) with $c_1(TM,\omega) = 0$ one first trivializes the top exterior power $\Lambda^n_{\mathbb{C}}(TM,J) \cong \mathbb{C} \times M$ of (TM,ω,J) for $J \in \mathcal{J}$ as a Hermitian line bundle. The square P^2 of the unit frame bundle $S^1 \times M \cong P \xrightarrow{S^1} M$ of $L = \Lambda^n_{\mathbb{C}}(TM,J,\omega)$ – that is the unitary frame bundle P^2 of $L^{\otimes 2}$ – admits a natural map $\det^2\colon \mathcal{L}(TM,\omega) \to P^2$ from the Lagrangian Grassmannian bundle $\mathcal{L}(TM,\omega)$, since $\mathcal{L}(TM)_x = \mathrm{U}(TM_x,\omega_x,J_x)/\mathrm{O}(n)$. For a path $\vec{\phi} = \{\phi_t\}_{t=0}^1$ in $\mathcal{G}$ with $\phi_0 = \mathrm{Id}$, choosing a point $L \in \mathcal{L}(TM,\omega)_x$ we have the curve $\{\phi_{t*x}(L)\}_{0\le t\le 1}$ in $\mathcal{L}(TM,\omega)$, and consequently the curve $\{\det^2(\phi_{t*x}(L))\}_{0\le t\le 1}$ in P^2. By means of the induced trivialization $P^2 \cong S^1 \times M$ this gives a continuous curve $e^{i2\pi\vartheta(t)}\colon [0,1] \to S^1$. Define

$$\mathrm{angle}(L,\vec{\phi}) = \mathrm{varangle}(\{e^{i2\pi\vartheta(t)}\}_{t=0}^1) = \vartheta(1) - \vartheta(0),$$

and then the function

$$\mathrm{angle}(x,\vec{\phi}) = \inf_{L\in\mathcal{L}(TM,\omega)_x} \mathrm{angle}(L,\vec{\phi})$$

is measurable and bounded on M and

$$R_1(\vec{\phi}) = \int_M \mathrm{angle}(x,\vec{\phi})\omega^n(x)$$

does not depend on homotopies of $\vec{\phi}$ with fixed endpoints, defining a quasimorphism

$$R_1\colon \widetilde{\mathcal{G}} \to \mathbb{R}.$$

Its homogeneization $\mathfrak{S}_{\mathrm{En}}\colon \widetilde{G} \to \mathbb{R}$, defined by $\mathfrak{S}_{\mathrm{En}}(\tilde{\phi}) := \lim_{k\to\infty} \frac{R_1(\tilde{\phi}^k)}{k}$ is a homogenous quasimorphism on $\widetilde{G}$ that is independent of the non-canonical choices of trivialization, and of the almost complex structure J.

Proposition 2.8.1. *Definitions 1.7.5 and 2.8.1 for the Entov quasimorphism are equivalent.*

Proof (Sketch). Following Appendix C in [84] one notes that the trivialization of (TM,ω,J) over $U = M \setminus Z$ can be chosen to agree with the restriction from M to U of a given trivialization of $\Lambda^n_{\mathbb{C}}(TM,J)$. Then given a path $\vec{\phi}$ one immediately has $\simeq$ equality of the two $\mathrm{angle}(x,\vec{\phi})$ functions on $U_{\vec{\phi}} = M \setminus Z_{\vec{\phi}}$ by the construction of the Maslov quasimorphism on the universal cover of the linear symplectic group using its action on the Lagrangian Grassmannian [7] and the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{L}(TM,\omega)|_U \arrow[r, "\det^2"] \arrow[d, "\sim"] & (\Lambda^n_{\mathbb{C}}(TM,J))^{\otimes 2}_U \arrow[r] \arrow[d, "\sim"] & {}^b\mathbb{C}|_U \arrow[d, equal] \\
\mathcal{L}({}^b\mathbb{C}^n,\omega_{\mathrm{std}})|_U \arrow[r, "\det^2"] & (\bigwedge^n {}^b\mathbb{C}^n)^{\otimes 2}|_U \arrow[r, equal] & {}^b\mathbb{C}|_U,
\end{tikzcd}$$

where ${}^b\mathbb{C}$ is the trivial complex line bundle $\mathbb{C} \times M$ over M, and all vector bundles are complex and Hermitian. $\square$

Now we turn to showing the equality $\mathfrak{S} = \mathfrak{S}_{\mathrm{En}}$. The proof is very similar to the one for the first point of Theorem 4 and is even somewhat easier. Therefore we mostly outline the main steps and leave out details that are identical to those in Section 2.7.

First we recall Equation (50)

$$\mathfrak{S}_{J_0}(\{\phi_t\}_{t=0}^1) \simeq \frac{1}{2}\int_M \tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1)\omega^n(x) - \int_0^1 \int_M S(\phi_t \cdot J_0) H_t(x)\omega^n(x).$$

We also recall the commutation relation of Equation (51):

$$\begin{tikzcd}
\mathcal{L}(TM,\omega)_{(\phi_t)^{-1}x} \arrow[r, "\Gamma_{\overline{p_{x,t}}}"] \arrow[d, "\det^2"'] & \mathcal{L}(TM,\omega)_x \arrow[d, "\det^2"] \\
P^2_{(\phi_t)^{-1}x} \arrow[r, "\Gamma_{\overline{p_{x,t}}}"] & P^2_x.
\end{tikzcd}$$

That is for $L_0 \in \mathcal{L}_{(\phi_t)^{-1}x}(TM,\omega)$ we have $\det^2(\Gamma_{\overline{p_{x,t}}} L_0) = \Gamma_{\overline{p_{x,t}}} \det^2(L_0)$.

It will be more convenient to compute R_1 on the inverse path $\{\psi_t = \phi_t^{-1}\}_{t=0}^1$. Indeed the path $\det^2(\psi_{t*_x} L)$ in P^2 gives by the trivialization a smooth angle function $e^{i2\pi\vartheta(t)}\colon [0,1] \to S^1$. The path $\Gamma_{\overline{p_{x,t}}}\colon (P^2)_{(\phi_t)^{-1}x} \to (P^2)_x$ also gives by the trivialization a smooth angle function $e^{i2\pi\varphi(x,t)}\colon [0,1] \to S^1$. Noting the relation $\Gamma_{\overline{p_{x,t}}} \det^2(\psi_{t*_x} L) = \det^2(\Gamma_{\overline{p_{x,t}}} \psi_{t*_x} L)$ (by Equation (51)), we have

$$\begin{aligned}
\operatorname{angle}(L, \{\psi_t\}_{t=0}^1) &= \operatorname{varangle}(\{e^{i2\pi\vartheta(t)}\}_{t=0}^1) \\
&= \operatorname{varangle}(\{\det^2(\Gamma_{\overline{p_{x,t}}} \psi_{t*_x} L)\}_{t=0}^1) \\
&\quad - \operatorname{varangle}(\{e^{i2\pi\varphi(x,t)}\}_{t=0}^1).
\end{aligned} \tag{61}$$

Consequently, the function $\operatorname{angle}(x, \{\psi_t\}_{t=0}^1) = \inf_{L \in \mathcal{L}(TM,\omega)_x} \operatorname{angle}(L, \{\psi_t\}_{t=0}^1)$ satisfies

$$\begin{aligned}
&\operatorname{angle}(x, \{\psi_t\}_{t=0}^1) \\
&= \inf_{L \in \mathcal{L}(TM,\omega)_x} (\operatorname{varangle}(\{\det^2(\Gamma_{\overline{p_{x,t}}} \psi_{t*_x} L)\}_{t=0}^1)) - \operatorname{varangle}(\{e^{i2\pi\varphi(x,t)}\}_{t=0}^1).
\end{aligned} \tag{62}$$

Note first that $\psi_{t*_x} = (\phi_{t*(\phi_t)^{-1}x})^{-1}$ and therefore $\Gamma_{\overline{p_{x,t}}} \psi_{t*_x} = \Psi(x)_t^{-1}$. Then note that

$$\operatorname{varangle}(\{\det^2(\Gamma_{\overline{p_{x,t}}} \psi_{t*_x} L)\}_{t=0}^1) \simeq \tau_{\mathrm{Lin}}(\{\Psi(x)_t^{-1}\}_{t=0}^1) = -\tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1) \tag{63}$$

by the construction of the Maslov quasimorphism on the universal cover of the linear symplectic group using its action on the Lagrangian Grassmannian [7]. Therefore

$$\inf_{L \in \mathcal{L}(TM,\omega)_x} (\text{varangle}(\{\det^2(\Gamma_{\overline{p_{x,t}}} \psi_{t*_x} L)\}_{t=0}^1)) \simeq -\tau_{\text{Lin}}(\{\Psi(x)_t\}_{t=0}^1). \quad (64)$$

It remains now to interpret the integral over M with respect to ω^n of the term varangle$(\{e^{i2\pi\varphi(x,t)}\}_{t=0}^1)$ in Equation (62) via the Hermitian scalar curvature. For this purpose note that the trivialization $P^2 \cong S^1 \times M$ is equivalent to a flat connection α on P^2 without holonomy. Consider now the two connection one-forms α and λ on P^2 – where in particular $d\alpha = 0$ and λ comes from the connection ∇ on TM and therefore satisfies $d\lambda = 2\tilde{\rho}$ (for a form η on M we denote by $\tilde{\eta}$ its lift by the natural projection $P^2 \to M$). These connection one-forms differ by $\tilde{\theta} = \alpha - \lambda$ for a one-form θ on M. Then denoting by Y_t the Hamiltonian vector field generating $\{\psi_t\}$ with Hamiltonian G_t normalized by the zero mean condition, and by $\widehat{Y}_t$ the horizontal vector field that projects onto Y_t generating the path $\{\hat{\psi}_t\}$ of α-preserving diffeomorphism of P^2 (in other words $\hat{\psi}_t = \text{Id} \times \psi_t$ in the trivialization $P^2 \cong S^1 \times M$) we have

$$\text{varangle}(\{e^{i2\pi\varphi(x,t)}\}_{t=0}^1) = \int_0^1 (\hat{\psi}_t)^* i_{\widehat{Y}_t} \tilde{\theta}(x) dt = \int_0^1 (\psi_t)^* i_{Y_t} \theta(x)\, dt. \quad (65)$$

We now compute as follows:

$$\int_M \int_0^1 (\psi_t)^* i_{Y_t} \theta(x) dt \omega^n(x) = \int_0^1 \int_M (\psi_t)^* i_{Y_t} \theta \omega^n dt = \int_0^1 \int_M i_{Y_t} \theta \omega^n dt. \quad (66)$$

It is therefore sufficient to compute the integrand

$$\begin{aligned} \int_M i_{Y_t} \theta \omega^n &= n \int_M \theta i_{Y_t} \omega \omega^{n-1} = -n \int_M \theta d G_t \omega^{n-1} = -n \int_M d\theta G_t \omega^{n-1} \\ &= 2n \int_M \rho G_t \omega^{n-1} = 2n \int_M G_t \rho \omega^{n-1} = 2n \int_M G_t \rho \omega^{n-1}, \end{aligned}$$

by the definition of the Hermitian scalar curvature

$$= 2 \int_M G_t S(J_0) \omega^n,$$

since $G_t(x) = -H_t(\phi_t x)$ by the cocycle formula

$$= -2 \int_M S(J_0) H_t(\phi_t x) \omega^n(x). \quad (67)$$

Therefore by Equations (62), (64), (65), (67) we have from the definition of R_1 that

$$R_1(\{\psi_t\}_{t=0}^1) \simeq -\int_M \tau_{\text{Lin}}(\{\Psi(x)_t\}_{t=0}^1) \omega^n(x) + 2 \int_M S(J_0) H_t(\phi_t x) \omega^n(x). \quad (68)$$

Therefore by Lemma 1.4.1 we have

$$\begin{aligned} R_1(\{\phi_t\}_{t=0}^1) &\simeq -R_1(\{\psi_t\}_{t=0}^1) \\ &= \int_M \tau_{\mathrm{Lin}}(\{\Psi(x)_t\}_{t=0}^1)\omega^n(x) - 2\int_M S(J_0)H_t(\phi_t x)\omega^n(x). \end{aligned} \tag{69}$$

From Equations (50) and (69) we conclude that

$$2\mathfrak{S}_{J_0} \simeq R_1,$$

which by homogenizing gives

$$2\mathfrak{S} = \mathfrak{S}_{\mathrm{En}}$$

finishing the proof.

2.9. Calibrating the L_2^2 norm. Here we derive Equation (8).

Note that the second summand of $\mathfrak{S}_{J_0}(\vec{\phi}) = \int_{D_{\vec{\phi}}} \Omega - \int_0^1 S(J_0)H_t(\phi_t x)\omega^n(x)dt$ satisfies

$$\begin{aligned} &\left|\int_0^1 S(J_0)H_t(\phi_t x)\omega^n(x)dt\right| \\ &\qquad \leq \|S(J_0)\|_{L^q(M,\omega^n)} \cdot \int_0^1 \|H_t\|_{L^p(M,\omega^n)}dt \end{aligned} \tag{70}$$

where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$ and is therefore bounded by $C_p\|\vec{\phi}_t\|_{k,p}$ for every $k \geq 0$ and $1 \leq p \leq \infty$.

Let us turn to the first summand $\int_{D_{\vec{\phi}}} \Omega$. First note that since on the Siegel upper half-space $\mathcal{S}_n$ the natural invariant Kähler form $\sigma_{\mathcal{S}_n}$ has a primitive $\lambda_{\mathcal{S}_n}$ that is bounded by a constant $C(n)$ with respect to the metric induced by $(\sigma_{\mathcal{S}_n}, j_{\mathcal{S}_n})$ and vanishes on geodesics starting at $i\,\mathrm{Id}$, the infinite-dimensional space $(\mathcal{J}, \Omega, \mathbb{J})$ also has a primitive Λ for Ω that is bounded with respect to the metric induced by $(\Omega, \mathbb{J})$ by the constant $C(n,\omega) = C(n)\mathrm{Vol}(M,\omega^n)^{1/2}$ and vanishes on geodesics starting at J_0. That is

$$|\Lambda(\Upsilon)| \leq C(n)\mathrm{Vol}(M,\omega^n)^{1/2}\Omega(\Upsilon, \mathbb{J}\Upsilon)^{1/2},$$

for a vector $\Upsilon \in T_J\mathcal{J}$. In that case $\int_{D_{\vec{\phi}}} \Omega = \int_{\vec{\phi}\cdot J_0} \Lambda = \int_0^1 \Lambda_{\phi_t\cdot J_0}((\phi_t)_* L_{X_t} J_0)dt$

and consequently

$$\begin{aligned}
\Big|\int_{D_{\vec{\phi}}} \Omega\Big| &\leq C(n,\omega)\int_0^1 \Omega_{\phi_t\cdot J_0}((\phi_t)_* L_{X_t}J_0, (\phi_t\cdot J_0)((\phi_t)_* L_{X_t}J_0))^{1/2}dt \\
&\leq C'(n,\omega)\int_0^1 \Big(\int_M \operatorname{trace}((\phi_t)_*(L_{X_t}J_0)^2)\omega^n\Big)^{1/2}dt \\
&= C'(n,\omega)\int_0^1 \Big(\int_M \operatorname{trace}((L_{X_t}J_0)^2)\omega^n\Big)^{1/2}dt \\
&\leq C''(n,\omega,J_0)\int_0^1 \Big(\int_M (|X_t|^2 + |\nabla X_t|^2\Big)\omega^n)^{1/2}dt \\
&\leq C^{(3)}(n,\omega,J_0)\|\vec{\phi}\|_{2,2}.
\end{aligned} \tag{71}$$

Therefore by (71) and (70) we have for all $\tilde{\phi} \in \widetilde{\mathcal{G}}$ the estimate

$$\mathfrak{S}_{J_0}(\tilde{\phi}) \leq C(n,\omega,J_0)\|\tilde{\phi}\|_{2,2}. \tag{72}$$

3. Discussion

(1) It was shown by Donaldson in [31], [32] that $\mathcal{G}$ acts in a Hamiltonian way on additional spaces (e.g. spaces of submanifolds/cycles). These may yield more homomorphisms $\pi_1(\mathrm{Ham}) \to \mathbb{R}$ by the action-homomorphism construction for equivariant moment maps, and perhaps new quasimorphisms on $\widetilde{\mathcal{G}}$. Moreover, Futaki shows in [47] that the space $\mathcal{J}_{int} \subset \mathcal{J}$ of integrable almost complex structures can be endowed with additional symplectic structures that give different moment maps for the action of $\mathcal{G}$, from which the Bando–Futaki invariants F_{c_k} are obtained when restricting to the subgroup $\mathcal{G}_{J_0}$. It would be interesting to extend the methods of Futaki to all $\mathcal{J}$, taking care of the Nijenhuis tensor, and to check two things. First it is most likely that the corresponding action-homomorphisms on $\pi_1(\mathcal{G})$ will coincide with the invariants I_{c_k} (cf. [64]) obtained by integrating the k-th vertical Chern class times u^{n+1-k} in Definition 1.7.1. Second, it would be interesting to extend the perturbation of Futaki to incorporate such invariants as $I_{c_1c_2}$ corresponding to symmetric polynomials that are not elementary.

(2) It is interesting to note that the Entov quasimorphism (Definitions 1.7.5, 2.8.1) is defined on the extension $\mathcal{H} = \mathrm{Symp}(M,\omega)$ of the group $\mathcal{G} = \mathrm{Ham}(M,\omega)$, while the moment map picture is currently stated for the action of $\mathcal{G}$ on $\mathcal{J}$ only. It is therefore interesting to check whether in the case $c_1(TM,\omega) = 0$ the moment map for the action of $\mathcal{G}$ on $\mathcal{J}$ extends to a moment map for the action of $\mathcal{H}$ on $\mathcal{J}$ – along the lines of [32] for example. It would also be interesting to investigate the possibility of extending the moment map this way without conditions on $c_1(TM,\omega)$ to provide an

extension when it is possible and to investigate the obstructions to extending when the extension is not possible. This may well be related to the Flux homomorphism.

(3) It is interesting to investigate the restriction $\mathcal{A}_\mu$ of $\mathfrak{S}$ to $\pi_1\mathcal{G}$ for symplectic manifolds (M,ω) of finite volume that are not closed. Does this restriction have an interpretation like I_{c_1} in terms of characteristic numbers of the associated Hamiltonian vector bundle? It would also be interesting to say something new about the Entov quasimorphism in the new interpretation. Can it be computed for example for the new symplectic manifolds constructed by Fine and Panov ([42] and references therein)?

(4) It would be interesting to compare the general principle for generating quasimorphisms introduced in this paper with other general constructions of quasimorphisms. While the relation to the Burger–Iozzi–Wienhard construction of the rotation number from [17] is at least intuitively relatively simple to trace, the relation to the works of Ben Simon and Hartnick [11], [10], [12] (cf. Calegari [23]) is somewhat more mysterious, since there seems to be no straightforward analogue of the Shilov boundary for the space $(\mathcal{J},\Omega,\mathbb{J})$ of compatible almost complex structures on (M,ω). Hence it is an interesting question to exhibit a specific explicit invariant partial order or poset that gives the quasimorphism $\mathfrak{S}$ on $\widetilde{\mathrm{Ham}}(M,\omega)$.

(5) From a general philosophical point of view the action of $\mathcal{G} = \mathrm{Ham}(M,\omega)$ on $\mathcal{J}$ with Donaldson's equivariant moment map allows one to consider $\mathcal{G}$ in its C^1-topology as a generalized Hermitian Lie group with a generalized Hermitian symmetric space of non-compact type. In a way it behaves similarly to $\mathrm{Sp}(2n,\mathbb{R})$, which would be a "Hermitian" feature of $\mathcal{G}$. In comparison, the group $\mathcal{G}$ with the Hofer metric and related invariants is known to exhibit certain "hyperbolic features" (cf. [75]) – shared with Gromov-hyperbolic finitely generated groups. This approach can be used to study the representations into $\mathcal{G}$ of fundamental groups of compact Kähler manifolds, e.g. Riemann surfaces of genus at least 2. It is easy to construct the analogue of the Toledo invariant for representations of surface groups (using the bounded 2-cocycle of Reznikov [80], [81], [82] that equals the differential of $\mathfrak{S}_{J_0}$ which corresponds to the "bounded Kähler class") that satisfies a corresponding Milnor–Wood type inequality (this can for example be proven using the quasimorphism $\mathfrak{S}_{J_0}$). One could then check which values of the Toledo invariant can be attained – note that this value will be I_{c_1} on a certain loop γ_ρ associated with the representation ρ, and hence for Kähler–Einstein manifolds is conjectured to vanish [84] – this holds for example on $(\mathbb{C}P^n,\omega_{\mathrm{FS}})$ [37], [39]. These methods could possibly be used to obtain restrictions on Hamiltonian actions of such groups, which would be complementary to those established by Polterovich (cf. [75]), since surface groups are undistorted. In particular the notion of maximal representations (following works of Burger–Iozzi–Wienhard and others cf. [16] for a survey) could be defined and their properties studied. The above-mentioned works of Ben Simon and Hartnick could again be of some use.

Note also that while certain embeddings of right-angled Artin groups (and hence of most surface groups) into $\mathcal{G}$ of any symplectic manifold were constructed by Kapovich in [59] these representations will have zero Toledo invariant. Indeed these constructions either factor through the subgroup $\mathcal{G}_B$ of diffeomorphisms supported in a ball, where the restriction of the quasimorphism to π_1 is trivial (cf. definition 1.7.3 of the Barge–Ghys average Maslov quasimorphism) or take values in $\mathcal{G}$ of a surface of genus g, where the restriction I_{c_1} vanishes since $\pi_1(\mathcal{G})$ is trivial (or torsion for the sphere). The surface can also have boundary – the Toledo invariant will still vanish, by the embedding functoriality (Proposition 1.7.1). However, it is quite an easy fact that since $\widetilde{\mathrm{Ham}}(M,\omega)$ for closed M is perfect by a theorem of Banyaga [5], every element $\gamma \in \pi_1\mathrm{Ham}$ is of the form $\gamma = \gamma_\rho$ for some representation $\rho\colon \pi_1(\Sigma_g) \to \mathrm{Ham}$ (one can take g to be the commutator length of $\gamma \in \widetilde{\mathrm{Ham}}$). Hence for $M = \mathrm{Bl}_1(\mathbb{C}P^1)$, say, there is a nonzero Toledo invariant representation, the corresponding class in π_1 represented by a toric loop. It would therefore be interesting to write this class explicitly as a product of commutators in $\widetilde{\mathrm{Ham}}$.

(6) Another interesting computation to make is that of $\mathfrak{S}$ on Hamiltonian paths generated by a time-independent (autonomous) Hamiltonian. This would give a *quasi-state*-type functional (cf. e.g. [38], [77]) on $C^\infty(M,\mathbb{R})$ corresponding to the quasimorphism $\mathfrak{S}$. This functional would retain the properties of linearity on Poisson-commutative subspaces and $\mathrm{Symp}(M,\omega)$-invariance, however it would not be monotone (since this would imply continuity in the L^∞-norm) or vanish on functions with supports displaceable by Hamiltonian isotopies. In particular, it would be curious to find a formula for the value of this quasi-state on Morse functions on the manifold in terms of local data around the critical points, similarly to what was computed by Py in his thesis [79] for the case of the two-sphere S^2. Here Equation (50) could be very useful. One could also ask whether there are similar localization formulas for actions of other groups, e.g. $\mathbb{R}^k$ with tame fixed manifolds. For one, in the case when (M,ω) is toric the restriction I_{c_1} of $\mathfrak{S}$ to $\pi_1(\mathcal{G})$ has been computed on loops coming from the torus action (cf. [84] and references therein).

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