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Some remarks on orbit sets of unimodular rows

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Abstract. Let A be a d-dimensional smooth algebra over a perfect field of characteristic not 2. Let $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ be the set of unimodular rows of length $n+1$ up to elementary transformations. If $n \geq (d+2)/2$, it carries a natural structure of group as discovered by van der Kallen. If $n = d \geq 3$, we show that this group is isomorphic to a cohomology group $H^d(A, G^{d+1})$. This extends a theorem of Morel, who showed that the set $\mathrm{Um}_{d+1}(A)/\mathrm{SL}_{d+1}(A)$ is in bijection with $H^d(A, G^{d+1})/\mathrm{SL}_{d+1}(A)$. We also extend this theorem to the case $d = 2$. Using this, we compute the groups $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)$ when A is a real algebra with trivial canonical bundle and such that $\mathrm{Spec}(A)$ is rational. We then compute the groups $\mathrm{Um}_{d+1}(A)/\mathrm{SL}_{d+1}(A)$ when d is even, thus obtaining a complete description of stably free modules of rank d on these algebras. We also deduce from our computations that there are no stably free non free modules of top rank over the algebraic real spheres of dimension 3 and 7.

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1. Introduction

Let A be a commutative noetherian ring and P, Q be two projective A-modules which are stably isomorphic, i.e., $P \oplus A^n \simeq Q \oplus A^n$. The question is to know in which situations this implies $P \simeq Q$. A celebrated theorem of Bass and Schanuel states that this is always the case if P is of rank strictly bigger than the Krull dimension of the ring A (see [4, Theorem 9.3], or [5, Theorem 2]). If A is an algebra over an algebraically closed field, then Suslin showed that the result can be extended to projective modules whose rank is equal to the dimension of the ring ([31]). In general, this result is wrong as shown by the example of the tangent bundle over the algebraic real two-sphere.

As a special case of the question, the stably free modules were extensively studied. Let d denote the Krull dimension of A. By Bass–Schanuel's cancellation theorem, the study of stably free modules reduces to the case $P \oplus A \simeq A^{d+1}$. Such modules correspond to unimodular rows of length $d+1$. In general, let $\mathrm{Um}_{n+1}(A)$ denote the set of unimodular rows of length $n+1$. One sees that $\mathrm{GL}_{n+1}(A)$ acts on the right on this set, and so does its subgroup $\mathrm{E}_{n+1}(A)$ generated by elementary matrices. It is not hard to see that a unimodular row $(a_1,\dots,a_{n+1})$ yields a free module if and only if it is the first row of a matrix in $\mathrm{GL}_{n+1}(A)$. This observation led to the study of the sets $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ and $\mathrm{Um}_{n+1}(A)/\mathrm{GL}_{n+1}(A)$ (which is the same as $\mathrm{Um}_{n+1}(A)/\mathrm{SL}_{n+1}(A)$). An important step was the discovery by Vaserstein that $\mathrm{Um}_3(A)/\mathrm{E}_3(A)$ was carrying a natural structure of abelian group under some conditions on A ([35, Theorem 5.2]). These conditions are for example satisfied when A is of Krull dimension 2. Inspired by this case, van der Kallen put a structure of abelian group on $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ (under some hypothesis on A) which coincides with the previous one when $n=2$. This structure comes from the following observation: If $A=C(X)$ is the ring of continuous real functions on some nice CW-complex X, then the set of maps from X to $\mathbb{R}^{d+1}\setminus\{0\}$ up to homotopy is the cohomotopy group $\pi^d(X)$. In [33], van der Kallen showed that the group law was in some sense algebraic, thus leading to the group structure on $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ for any reasonable ring A. The problem is now to actually compute this group and its quotient $\mathrm{Um}_{n+1}(A)/\mathrm{SL}_{n+1}(A)$.

In his recent preprint [23], Morel showed that the group $\mathrm{Um}_{d+1}(A)/\mathrm{SL}_{d+1}(A)$ has a cohomological interpretation when A is a d-dimensional smooth algebra over a field k. Indeed, let K_{d+1}^{MW} be the unramified Milnor–Witt sheaf. Then a very easy computation shows that $H^d(\mathbb{A}^{d+1}\setminus\{0\},K_{d+1}^{\mathrm{MW}})=\mathrm{GW}(k)$, the Grothendieck–Witt group of k. Any unimodular row $(a_1,\dots,a_{d+1})$ can be seen as a morphism $f\colon \mathrm{Spec}(A)\to\mathbb{A}^{d+1}\setminus\{0\}$ and one can consider the pull back $f^*(\langle 1\rangle)$ in $H^d(A,K_{d+1}^{\mathrm{MW}})$, where $\langle 1\rangle$ denotes the unit in $\mathrm{GW}(k)$. Let $\mathcal{H}(k)$ be the $\mathbb{A}^1$-homotopy category of smooth k-schemes. One of the main theorems in [23] states that this map induces a bijection between $\mathrm{Hom}_{\mathcal{H}(k)}(A,\mathbb{A}^{d+1}\setminus\{0\})$ and $H^d(A,K_{d+1}^{\mathrm{MW}})$. Furthermore, the natural action of $\mathrm{GL}_{d+1}(A)$ on $\mathrm{Hom}_{\mathcal{H}(k)}(\mathrm{Spec}(A),\mathbb{A}^{d+1}\setminus\{0\})$ gives an action on $H^d(A,K_{d+1}^{\mathrm{MW}})$, which reduces to an action of $\mathrm{SL}_{d+1}(A)$. The quotient $H^d(A,K_{d+1}^{\mathrm{MW}})/\mathrm{SL}_{d+1}(A)$ is then in bijection with the set of stably free modules of rank d. Thus the above map induces a bijection $\mathrm{Um}_{d+1}(A)/\mathrm{SL}_{d+1}(A)\to H^d(A,K_{d+1}^{\mathrm{MW}})/\mathrm{SL}_{d+1}(A)$. For some technical reasons, Morel has to assume that $d\geq 3$ to prove this theorem. Observe also that if the field k is of characteristic different from 2, the group $H^d(A,K_{d+1}^{\mathrm{MW}})$ coincide with the group $H^d(A,G^{d+1})$ as defined in [11, Chapter 10] (following the original idea of [3]).

Our first goal in this paper is the following theorem (Theorem 4.9 in the text):

Theorem. *Let A be a smooth k-algebra of dimension d. Suppose that k is perfect. Then the map $\phi\colon \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)\to H^d(A,G^{d+1})$ is an isomorphism for $d\geq 3$.*

This result is also true if $d = 2$ and the field k is not perfect of characteristic different from 2. This will be treated in [13] using different methods. Our strategy is the following: First we show that $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ is nothing but the set of morphisms from $\mathrm{Spec}(A)$ to $\mathbb{A}^{n+1} \setminus \{0\}$ up to naive homotopy. Here we say that two morphisms $f, g\colon \mathrm{Spec}(A) \to \mathbb{A}^{n+1} \setminus \{0\}$ are naively homotopic if there exists a morphism $F\colon \mathrm{Spec}(A[t]) \to \mathbb{A}^{n+1} \setminus \{0\}$ whose evaluations in 0 and 1 are f and g respectively. Then we show that there is an exact sequence of pointed sets

$$\begin{array}{ccccc} \mathrm{SL}_n(A)/\mathrm{E}_n(A) \longrightarrow & \mathrm{SL}_{n+1}(A)/\mathrm{E}_{n+1}(A) & & & \\ & \downarrow & & & \\ & \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A) & \longrightarrow & \mathrm{Um}_{n+1}(A)/\mathrm{SL}_{n+1}(A) & \longrightarrow 0. \end{array}$$

which turns out to be an exact sequence of groups in some situations. Next we show that the set $\mathrm{GL}_n(A)/\mathrm{E}_n(A)$ is nothing else than $\mathrm{Hom}_{\mathcal{H}(k)}(\mathrm{Spec}(A), \mathrm{Sing}^{\bullet}\mathbb{G}\mathrm{L}_n)$ if $n \geq 3$. This is one of the results of [23], but we spend some lines to explain it in Section 4. The theorem is an obvious consequence of this fact.

Our next result extends the theorem of Morel to the case $d = 2$ (Theorem 4.11).

Theorem. *Let A be a smooth k-algebra of dimension 2, where k is a field of characteristic 0. The homomorphism ϕ induces an isomorphism*

$$\bar{\phi}\colon \mathrm{Um}_3(A)/\mathrm{SL}_3(A) \simeq H^2(A, G^3)/\mathrm{SL}_3(A).$$

The idea to prove this result is to use a result of Bhatwadekar and Sridharan relating $\mathrm{Um}_3(A)/\mathrm{SL}_3(A)$ with the Euler class group $E(A)$ and the weak Euler class group $E_0(A)$ (see [8]). Namely, there is an exact sequence

$$0 \longrightarrow \mathrm{Um}_3(A)/\mathrm{SL}_3(A) \xrightarrow{\psi} E(A) \longrightarrow E_0(A) \longrightarrow 0.$$

We then use the fact that if A is of smooth of dimension 2 then $E(A)$ coincide with the Chow–Witt group $\widetilde{\mathrm{CH}}^2(A)$ and $E_0(A)$ is just the Chow group $\mathrm{CH}^2(A)$. A comparison of exact sequences then yields the result.

Next we compute the group $H^d(A, G^{d+1})$ where A is a real algebra satisfying some extra conditions:

Theorem. *Let A be a smooth $\mathbb{R}$-algebra of dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then*

$$H^d(X, G^{d+j}) \simeq H^d(X, I^{d+j}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$$

where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).

We also show that when A is even-dimensional, then GL_{d+1} acts trivially on $H^d(A, G^{d+1})$ and we can completely compute the set of stably free modules of rank d in that case.

Theorem. *Let A be a smooth $\mathbb{R}$-algebra of even dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then the set of stably free modules of rank d is isomorphic to $\bigoplus_{C\in\mathcal{C}} \mathbb{Z}$, where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).*

In odd dimension, things are more complicated. If S^3 and S^7 denote the real algebraic spheres of dimension 3 and 7, we show that all the stably free modules of top rank on these spheres are free.

1.1. Conventions. Throughout the article, k will be a commutative field of characteristic different from 2. All k-algebras are commutative and essentially of finite type over k. If A is such an algebra and $\mathfrak{p}$ is any prime ideal in A, we denote by $k(\mathfrak{p})$ the residue field in $\mathfrak{p}$. If $\mathfrak{p}$ is of height n, we denote by $\omega_{\mathfrak{p}}$ the $k(\mathfrak{p})$-vector space $\mathrm{Ext}^n_{A_{\mathfrak{p}}}(k(\mathfrak{p}), A_{\mathfrak{p}})$ (which is of dimension 1 if the ring is regular). When we write $\widetilde{W}(k(\mathfrak{p}))$, we always mean the Witt group of $k(\mathfrak{p})$-vector spaces endowed with symmetric isomorphisms for the duality $\mathrm{Hom}_{k(\mathfrak{p})}(\_, \omega_{\mathfrak{p}})$. The Witt group $\widetilde{W}(k(\mathfrak{p}))$ is a module over the classical Witt ring $W(k(p))$ of $k(\mathfrak{p})$. If $\langle\alpha\rangle$ denotes the class of $\alpha \in k(\mathfrak{p})^\times$ in the classical Witt group, and ξ is any element of $\widetilde{W}(k(\mathfrak{p}))$, we denote by $\langle\alpha\rangle \cdot \xi$ the product of $\langle\alpha\rangle$ and ξ.

2. Unimodular rows and naive homotopies of maps

2.1. Naive homotopies. Let A be a k-algebra, where k is a field. For any $m, n \in \mathbb{N}$ such that $m \leq n$, let $\mathrm{Um}_{m,n}(A)$ be the set of surjective homomorphisms $A^n \to A^m$. Let $\mathrm{E}_n(A)$ be the subgroup of $\mathrm{SL}_n(A)$ generated by the elementary matrices. This group acts (on the right) on $\mathrm{Um}_{m,n}(A)$ and we denote the set of orbits by $\mathrm{Um}_{m,n}(A)/\mathrm{E}_n(A)$. In particular, when $m = 1$ we get the set of unimodular rows under elementary transformations, and when $m = n$ we get the set $\mathrm{GL}_n(A)/\mathrm{E}_n(A)$, which is a group when $n \geq 3$.

For any m, n as above, denote by $V(m,n)$ the ideal of $\mathbb{A}^{mn}$ (seen as the set of $m\times n$ matrices) generated by the $m \times m$ minors. Denote by $D(m,n)$ the open subscheme $\mathbb{A}^{mn} \setminus V(m,n)$ of $\mathbb{A}^{mn}$. In particular, $D(1,n) = \mathbb{A}^n \setminus \{0\}$ and $D(n,n) = \mathrm{GL}_n(k)$.

Let X, Y be two schemes over k. We say that two homomorphisms $f, g\colon X \to Y$ are naively homotopic if there exists a morphism $F\colon X\times\mathbb{A}^1 \to Y$ such that $F(0) = f$ and $F(1) = g$ where $F(i)$ denotes the evaluation in $i = 0, 1$. We consider the equivalence relation generated by naive homotopies and we denote by $\mathrm{Hom}_{\mathbb{A}^1}(X, Y)$

the set of equivalence classes of morphism from X to Y. If $X = \mathrm{Spec}(A)$, observe that $\mathrm{Hom}(X, D(m,n)) = \mathrm{Um}_{m,n}(A)$ and we can identify the naive homotopy classes as follows:

Theorem 2.1. *Let A be a smooth k-algebra and $X = \mathrm{Spec}(A)$. Then*

$$\mathrm{Hom}_{\mathbb{A}^1}(X, D(m,n)) = \mathrm{Um}_{m,n}(A)/\mathrm{E}_n(A)$$

for any m, n.

Proof. First notice that any elementary matrix is naively homotopic to the identity. Let L and L' be two elements of $\mathrm{Um}_{m,n}(A)$. Suppose that there is an element M in $\mathrm{Um}_{m,n}(A[t])$ such that $M(0) = L$ and $M(1) = L'$. Consider the exact sequence

$$0 \longrightarrow P \longrightarrow A[t]^n \xrightarrow{M} A[t]^m \longrightarrow 0$$

where P is the kernel of M. Notice that P is projective, and therefore it is extended from A by [18] (or more generally [27] and [28]), i.e., $P = P(0)[t]$. But $P(0)$ is defined by the following sequence

$$0 \longrightarrow P(0) \longrightarrow A^n \xrightarrow{L} A^m \longrightarrow 0.$$

Comparing the two (split) exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P(0)[t] & \longrightarrow & A[t]^n & \xrightarrow{M} & A[t]^m & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & P(0)[t] & \longrightarrow & A[t]^n & \xrightarrow{L} & A[t]^m & \longrightarrow & 0, \end{array}$$

we see that there exists an automorphism ψ of $A[t]^n$ such that the diagram commutes. Observe that $\psi(0) = \mathrm{Id}$. By [37], $\psi \in \mathrm{E}_n(A[t])$ (here, the referee pointed out that Vorst's results can be greatly generalized using the work of Popescu, see [27] and [28] again). Evaluating at $t = 1$, we get $L' = L\psi(1)$. Thus the result is proved. $\square$

2.2. The group structure on $\mathrm{Um}_n(A)/\mathrm{E}_n(A)$. The universal weak Mennicke symbol on the set $\mathrm{Um}_n(A)/\mathrm{E}_n(A)$ is the free group $\mathrm{WMS}_n(A)$ with generators $\mathrm{wms}(v)$ for all $v \in \mathrm{Um}_n(A)$ and relations

(i) $\mathrm{wms}(v) = \mathrm{wms}(vg)$ for any $g \in \mathrm{E}_n(A)$;

(ii) if $(x, v_2, \dots, v_n)$ and $(1-x, v_2, \dots, v_n)$ are both unimodular, then

$$\mathrm{wms}(1-x, v_2, \dots, v_n)\,\mathrm{wms}(x, v_2, \dots, v_n) = \mathrm{wms}(x(1-x), v_2, \dots, v_n).$$

Remark 2.2. The reader familiar with weak Mennicke symbols might have remarked that this definition is different from the original one (see [33, §3.2]). However, both definitions coincide when $n \geq (\dim(A) + 4)/2$ by [34, Theorem 3.3].

By definition, there is a map wms: $\mathrm{Um}_n(A)/\mathrm{E}_n(A) \to \mathrm{WMS}_n(A)$. In [33, Theorem 4.1], it is proven that this map is a bijection under certain conditions. In the same paper, it is shown that $\mathrm{WMS}_n(A)$ is abelian in that case ([33, Theorem 3.6]). We condense these informations in the next result:

Theorem 2.3 (van der Kallen). *Let A be a commutative ring of Krull dimension $d \geq 2$. Then the map* wms: $\mathrm{Um}_n(A)/\mathrm{E}_n(A) \to \mathrm{WMS}_n(A)$ *is a bijection for any $n \geq (d+4)/2$. Moreover,* $\mathrm{WMS}_n(A)$ *is an abelian group.*

2.3. An exact sequence. For $n \geq 1$ Consider the morphism of algebraic groups $\mathrm{SL}_n \to \mathrm{SL}_{n+1}$ sending a matrix M to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix}\right)$. Consider also the morphism $\mathrm{SL}_{n+1} \to \mathbb{A}^{n+1} \setminus \{0\}$ sending a matrix to its first row. We get a sequence

$$\mathrm{SL}_n \longrightarrow \mathrm{SL}_{n+1} \longrightarrow \mathbb{A}^{n+1} \setminus \{0\}.$$

If A is a smooth k-algebra, we apply the functor $\mathrm{Hom}_{\mathbb{A}^1}(A, \_)$ to this sequence to get a sequence of pointed sets (where $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ is pointed by $[1, 0, \dots, 0]$)

$$\mathrm{SL}_n(A)/\mathrm{E}_n(A) \longrightarrow \mathrm{SL}_{n+1}/\mathrm{E}_{n+1}(A) \longrightarrow \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A).$$

This sequence of pointed sets is exact for quite general rings A:

Proposition 2.4. *Let A be a commutative ring of dimension d. For $n \geq 2$, the sequence of pointed sets*

$$\mathrm{SL}_n(A)/\mathrm{E}_n(A) \longrightarrow \mathrm{SL}_{n+1}/\mathrm{E}_{n+1}(A) \longrightarrow \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$$

is exact. If moreover $n = d$ and $d \geq 3$, then it is an exact sequence of groups.

Proof. We begin by proving the first assertion. Notice first that the sequence is clearly a complex. Let $M \in \mathrm{SL}_{n+1}(A)$ be such that there exists $E \in \mathrm{E}_{n+1}(A)$ with $ME = \left(\begin{smallmatrix} 1 & 0 \\ \star & M' \end{smallmatrix}\right)$ for some $M' \in \mathrm{GL}_n(A)$. There is then a matrix $F \in \mathrm{E}_{n+1}(A)$ such that $FME = \left(\begin{smallmatrix} 1 & 0 \\ 0 & M' \end{smallmatrix}\right)$. Now $M^{-1}FM$ is in $\mathrm{E}_{n+1}(A)$ since the latter is normal in $\mathrm{SL}_{n+1}(A)$ for $n \geq 2$ by [32]. Therefore $M(M^{-1}FM)E$ comes from $\mathrm{SL}_n(A)$ and the sequence is exact.

If $n = d$ and $d \geq 3$ the terms in the sequence are groups. Moreover, the map $\mathrm{SL}_{n+1}(A)/\mathrm{E}_{n+1}(A) \to \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ is a homomorphism of groups by [33, Theorem 5.3 (ii)]. □

Now the cokernel of the map $\mathrm{SL}_{n+1}(A)/\mathrm{E}_{n+1}(A) \to \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ is just $\mathrm{Um}_{n+1}(A)/\,\mathrm{SL}_{n+1}(A)$ which is the set of isomorphism classes of stably free modules of rank n over A. The following result is an obvious consequence of the above proposition, but we state it for further reference.

Theorem 2.5. *Let A be a commutative ring of dimension d. For any $n \geq 2$, there is an exact sequence of pointed sets*

$$\begin{array}{ccccc} \mathrm{SL}_n(A)/\mathrm{E}_n(A) \longrightarrow \mathrm{SL}_{n+1}(A)/\mathrm{E}_{n+1}(A) & & & & \\ \downarrow & & & & \\ \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A) \longrightarrow \mathrm{Um}_{n+1}(A)/\,\mathrm{SL}_{n+1}(A) \longrightarrow 0. & & & & \end{array}$$

If $n = d$ and $d \geq 3$, this is an exact sequence of groups.

3. Computations of some cohomology groups

3.1. The sheaf G. In this section, we briefly recall the definition and first properties of the sheaf G^j (for any $j \in \mathbb{Z}$) defined in [10, Definition 3.25]. More precisely, we will exhibit a flasque resolution of G^j, which will facilitate further computations.

If X is a regular scheme over k, consider the Gersten–Witt complex ([2, Theorem 7.2], recall our conventions about $\widetilde{W}$)

$$\cdots \longrightarrow \bigoplus_{x_p \in X^{(p)}} \widetilde{W}(k(x_p)) \xrightarrow{d} \bigoplus_{x_{p+1} \in X^{(p+1)}} \widetilde{W}(k(x_{p+1})) \longrightarrow \cdots.$$

Choosing a generator of ω_p for any x_p, we obtain isomorphisms $W(k(p)) \to \widetilde{W}(k(p))$. Consider the fundamental ideal $I(k(p))$ of even dimensional quadratic forms in $W(k(p))$, and its powers $I^j(k(p))$ for any $j \in \mathbb{Z}$ where $I^j(k(p)) = W(k(p))$ if $j < 0$ by convention. For any $j \in \mathbb{Z}$, we denote by $\widetilde{I}^j(k(p))$ the image of $I^j(k(p))$ under the isomorphism $W(k(p)) \to \widetilde{W}(k(p))$. Notice that this definition is independent of the choice of the isomorphism $W(k(p)) \to \widetilde{W}(k(p))$ ([11, Lemma E.1.12]).

It turns out that the differential d respects the subgroups $\widetilde{I}^j(k(x_p))$ ([11, Theorem 9.2.4] or [15, Theorem 6.4]) and therefore for any $j \in \mathbb{Z}$ we get a complex $C(X, I^j)$:

$$\cdots \longrightarrow \bigoplus_{x_p \in X^{(p)}} \widetilde{I}^{j-p}(k(x_p)) \xrightarrow{d_I} \bigoplus_{x_{p+1} \in X^{(p+1)}} \widetilde{I}^{j-p-1}(k(x_{p+1})) \longrightarrow \cdots.$$

This complex can be seen as a flasque resolution of a sheaf I^j on X, which is the sheaf associated to the presheaf $\mathcal{I}^j$ defined on any open subset $U \subset X$ by $\mathcal{I}^j(U) = H^0(C(U, I^j))$ ([10, §3]).

For any x_p and any $n \in \mathbb{Z}$, consider the group

$$\bar{I}^n(k(x_p)) := I^n(k(x_p))/I^{n+1}(k(x_p)).$$

It is easily seen that $\bar{I}^n(k(x_p)) := \tilde{I}^n(k(x_p))/\tilde{I}^{n+1}(k(x_p))$ ([11, Lemma E.1.13]). Therefore we obtain a complex $C(X, \bar{I}^j)$:

$$\cdots \longrightarrow \bigoplus_{x_p \in X^{(p)}} \bar{I}^{j-p}(k(x_p)) \xrightarrow{-d_I} \bigoplus_{x_{p+1} \in X^{(p+1)}} \bar{I}^{j-p-1}(k(x_{p+1})) \longrightarrow \cdots$$

which fits in an exact sequence of complexes

$$0 \longrightarrow C(X, I^{j+1}) \longrightarrow C(X, I^j) \longrightarrow C(X, \bar{I}^j) \longrightarrow 0$$

for any $j \in \mathbb{Z}$ (observe that if $j < 0$, the right hand side is trivial). If $\bar{I}^j$ is the sheaf associated to the complex $C(X, \bar{I}^j)$, then by definition we obtain an exact sequence of sheaves on X:

$$0 \longrightarrow I^{j+1} \longrightarrow I^j \longrightarrow \bar{I}^j \longrightarrow 0.$$

Now there is a complex in Milnor K-theory $C(X, K_j^M)$ ([16, Proposition 1]):

$$\cdots \longrightarrow \bigoplus_{x_p \in X^{(p)}} K_{j-p}^M(k(x_p)) \xrightarrow{d_K} \bigoplus_{x_{p+1} \in X^{(p+1)}} K_{j-p-1}^M(k(x_{p+1})) \longrightarrow \cdots.$$

Again, this complex can be seen as a flasque resolution of a sheaf K_j^M on X. For any x_p and any $n \in \mathbb{N}$, there is a homomorphism $s_n : K_n^M(k(x_p)) \to \bar{I}^n(k(x_p))$ defined by mapping an elementary symbol $\{a_1, \ldots, a_n\}$ to the class of the n-fold Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ modulo $I^{n+1}(k(x_p))$ ([20, Theorem 4.1]). These homomorphisms yield a morphism of complexes $C(X, K_j^M) \to C(X, \bar{I}^j)$ for any $j \in \mathbb{N}$ ([11, Theorem 10.2.6]). We can therefore take the fibre product of the complexes $C(X, K_j^M)$ and $C(X, I^j)$ over $C(X, \bar{I}^j)$ to get a complex $C(X, G^j)$

$$\cdots \longrightarrow \bigoplus_{x_p \in X^{(p)}} \tilde{G}^{j-p}(k(x_p)) \xrightarrow{d_G} \bigoplus_{x_{p+1} \in X^{(p+1)}} \tilde{G}^{j-p-1}(k(x_{p+1})) \longrightarrow \cdots$$

which is a flasque resolution of a sheaf G^j on X. Here the groups $\widetilde{G}^{j-p}(k(x_p))$ are the fibre products

$$\begin{array}{ccc} \widetilde{G}^{j-p}(k(x_p)) & \longrightarrow & \widetilde{I}^{j-p}(k(x_p)) \\ \downarrow & & \downarrow \\ K^M_{j-p}(k(x_p)) & \longrightarrow & \bar{I}^{j-p}(k(x_p)). \end{array}$$

Notice that the group $\widetilde{G}^{j-p}(k(x_p))$ is also twisted by the vector space ω_p. When the vector space is canonically isomorphic to $k(x_p)$, we drop the twiddle. By definition, we get an exact sequence of sheaves on X

$$0 \longrightarrow I^{j+1} \longrightarrow G^j \longrightarrow K^M_j \longrightarrow 0$$

for any $j \in \mathbb{Z}$.

If A be a smooth k-algebra of dimension d, the above sequence of sheaves gives an exact sequence

$$H^d(A, I^{j+1}) \longrightarrow H^d(A, G^j) \longrightarrow H^d(A, K^M_j) \longrightarrow 0$$

for any $j \in \mathbb{N}$. The natural map of sheaves $G^{j+1} \to I^{j+1}$ gives a surjective homomorphism $H^d(A, G^{j+1}) \to H^d(A, I^{j+1})$ and we get an exact sequence

$$H^d(A, G^{j+1}) \longrightarrow H^d(A, G^j) \longrightarrow H^d(A, K^M_j) \longrightarrow 0$$

for any $j \in \mathbb{N}$. By definition, $H^d(A, G^d)$ is the Chow–Witt group $\widetilde{\mathrm{CH}}^d(A)$ as defined in [3] or [11, Definition 10.2.14] and $H^d(A, K^M_d)$ is the Chow group $\mathrm{CH}^d(A)$. Putting everything together, we have:

Proposition 3.1. *Let A be a smooth k-algebra of dimension d. There is an exact sequence*

$$H^d(A, G^{d+1}) \longrightarrow \widetilde{\mathrm{CH}}^d(A) \longrightarrow \mathrm{CH}^d(A) \longrightarrow 0.$$

3.2. The sheaf K^{MW}. First recall the following definition from [21, Definition 5.1]:

Definition 3.2. Let F be a field (possibly of characteristic 2). Let $K^{\mathrm{MW}}_*(F)$ be the (unitary, associative) $\mathbb{Z}$-graded ring freely generated by the symbols $[a]$ of degree 1 with $a \in F^\times$ and a symbol η of degree -1 subject to the following relations:

1. $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in F^\times$.
2. $[a][1-a] = 0$ for any $a \in F^\times - \{1\}$.

3. $\eta(\eta[-1]+2)=0$.
4. $\eta[a]=[a]\eta$ for any $a\in F^\times$.

There is a natural homomorphism $K_*^{\mathrm{MW}}(F)\to K_*^M(F)$ such that $[a]\mapsto\{a\}$ and $\eta\mapsto 0$. For any $n\in\mathbb{Z}$ there is also a natural homomorphism $K_n^{\mathrm{MW}}(F)\to I^n(F)$ such that $[a_1,\dots,a_n]\mapsto\langle -1,a_1\rangle\otimes\dots\otimes\langle -1,a_n\rangle$ and $\eta\mapsto\langle 1\rangle\in I^{-1}(F)=W(F)$ (this definition is also meaningful in characteristic 2, see [22, §2.1]). These homomorphisms coincide on $\bar{I}^n(F)$ and therefore yield a homomorphism $K_n^{\mathrm{MW}}(F)\to G^n(F)$ for any $n\in\mathbb{Z}$. The expected result holds ([21, Theorem 5.3] if F is of characteristic different from 2, and [22, Remark 2.12] in characteristic 2):

Theorem 3.3. *The homomorphism $K_n^{\mathrm{MW}}(F)\to G^n(F)$ is an isomorphism.*

One can also define a Gersten complex in Milnor–Witt K-theory (twisting these groups accordingly, see [22, Remark 2.21]), and obtain a complex $C(X,K_j^{\mathrm{MW}})$ for any $j\in\mathbb{Z}$ which coincides (under the homomorphisms of Theorem 3.3) with the complex $C(X,G^j)$ for any smooth X over a field of characteristic different from 2.

In view of this, one has the choice to work either with the complex in Milnor–Witt K-theory or with the complex $C(X,G^j)$. This is mostly a question of point of view. On the one hand, Milnor–Witt K-theory appears very naturally in $\mathbb{A}^1$-homotopy, as we will see below. On the other hand, the complex $C(X,G^j)$ puts more emphasis on the Gersten–Witt complex and seems closer to higher Grothendieck–Witt groups (also known as Hermitian K-theory). In particular, lots of concrete computations are available. Of course this distinction is artificial, since both complexes are the same! At the end, I decided to work with the complex G^j because of my personal preference for the latter.

3.3. A useful computation. In this section, we compute the cohomology groups of the sheaf G^j on $\mathbb{A}^{n+1}-\{0\}$ for any $j\in\mathbb{N}$. For the forthcoming results, there are a few useful facts to know:

1. The functor $H^i(\_,G^j)$ is contravariant on the category of smooth schemes over k ([10, Definition 7.1]).
2. The projection $p\colon X\times\mathbb{A}^n\to X$ induces an isomorphism
$$p^*\colon H^i(X,G^j)\to H^i(X\times\mathbb{A}^n,G^j)$$
for any $i,j\in\mathbb{Z}$ ([11, Theorem 11.2.9]).
3. For any $j\in\mathbb{Z}$ and any open subscheme $\iota\colon U\to X$, with closed complement $Y=X-U$, there is a long exact sequence of localization
$$\cdots\longrightarrow H_Y^i(X,G^j)\longrightarrow H^i(X,G^j)\xrightarrow{\iota^*}H^i(U,G^j)\xrightarrow{\partial}H_Y^{i+1}(X,G^j)\longrightarrow\cdots$$

where $H^i_Y(X, G^j)$ denotes the cohomology group with support on Y ([11, Lemma 10.4.7]).

In particular, let $U = \mathbb{A}^{n+1} - \{0\}$. The groups $H^i(U, G^j)$ fit in the localization sequence

$$\cdots \longrightarrow H^i_{\{0\}}(\mathbb{A}^{n+1}, G^j) \longrightarrow H^i(\mathbb{A}^{n+1}, G^j)$$
$$\longrightarrow H^i(U, G^j) \xrightarrow{\partial} H^{i+1}_{\{0\}}(\mathbb{A}^{n+1}, G^j) \longrightarrow \cdots$$

for any $j \in \mathbb{Z}$. The cohomology groups $H^i_{\{0\}}(\mathbb{A}^{n+1}, G^j)$ are by definition the cohomology groups of the complex with only the group $\widetilde{G}^{j-n-1}(k(\mathfrak{q}))$ in degree $n+1$, where $\mathfrak{q}$ is the prime ideal $(x_1, \ldots, x_{n+1}) \subset k[x_1, \ldots, x_{n+1}]$. Hence $k(\mathfrak{q}) = k$ and ω_p is the k-vector space generated by the Koszul complex $\mathrm{Kos}(x_1, \ldots, x_{n+1})$ associated to the regular sequence $(x_1, \ldots, x_{n+1})$. Therefore $H^i_{\{0\}}(\mathbb{A}^{n+1}, G^j) = 0$ if $i \neq n+1$ and $H^{n+1}_{\{0\}}(\mathbb{A}^{n+1}, G^j) = \widetilde{G}^{j-n-1}(k)$.

Using homotopy invariance, we obtain $H^0(\mathbb{A}^{n+1}, G^j) = H^0(k, G^j) = G^j(k)$ and $H^i(\mathbb{A}^{n+1}, G^j) = 0$ if $i > 0$. We therefore get the following computation:

$$H^i(U, G^j) = \begin{cases} G^j(k) & \text{if } i = 0, \\ 0 & \text{if } 0 < i < n, \\ \widetilde{G}^{j-n-1}(k) & \text{if } i = n, \end{cases}$$

where the last line is given by the isomorphism $\partial\colon H^n(U, G^j) \to H^{n+1}_{\{0\}}(\mathbb{A}^{n+1}, G^j)$, which is $H^0(k, G^0)$-linear (i.e., $\mathrm{GW}(k)$-linear). Since we use it in the sequel, we give an explicit description of ∂ for $j = n+1$.

Let $B = k[x_1, \ldots, x_{n+1}]$ and consider the Koszul complex $\mathrm{Kos}(x_2, \ldots, x_{n+1})$ associated to the regular sequence $x_2, \ldots, x_{n+1}$. We get an isomorphism

$$\psi_{x_2,\ldots,x_{n+1}}\colon B/(x_2, \ldots, x_{n+1}) \simeq \mathrm{Ext}^n_B(B/(x_2, \ldots, x_{n+1}), B)$$

given by $1 \mapsto \mathrm{Kos}(x_2, \ldots, x_{n+1})$. Localizing at $\mathfrak{p} = (x_2, \ldots, x_{n+1})$, it becomes an isomorphism $\psi_{x_2,\ldots,x_{n+1}}\colon k(x_1) \simeq \mathrm{Ext}^n_{B_\mathfrak{p}}(k(x_1), B_\mathfrak{p})$. Observe that $x_1 \in B_\mathfrak{p}^\times$ and consider the couple $(x_1, \langle -\psi_{x_2,\ldots,x_{n+1}}, x_1\psi_{x_2,\ldots,x_{n+1}}\rangle)$ in the fibre product

$$\begin{array}{ccc} \widetilde{G}^1(k(x_1)) & \longrightarrow & \widetilde{I}(k(x_1)) \\ \downarrow & & \downarrow \\ K^M_1(k(x_1)) & \longrightarrow & I(k(x_1))/I^2(k(x_1)). \end{array}$$

It defines an element ξ of $H^n(U, G^{n+1})$ which is mapped under ∂ to the generator (as $\mathrm{GW}(k)$-module) of $\widetilde{G}^0(k)$ given by the Koszul complex $\mathrm{Kos}(x_1, \ldots, x_{n+1})$ (see [1, §9]).

4. The homomorphism $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A) \to H^n(A, G^{n+1})$

4.1. The homomorphism. Let A be a smooth k-algebra and $X = \mathrm{Spec}(A)$. We define a map

$$\phi\colon \mathrm{Hom}(X, \mathbb{A}^{n+1} - \{0\}) \to H^n(A, G^{n+1})$$

by $\phi(f) = f^*(\xi)$, where $f^*\colon H^n(\mathbb{A}^{n+1} - \{0\}, G^{n+1}) \to H^n(A, G^{n+1})$ is the pull-back induced by f ([10, Definition 7.2]). Because of the homotopy invariance of $H^n(A, G^{n+1})$, we get a map

$$\phi\colon \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A) \to H^n(A, G^{n+1}).$$

Theorem 4.1. *Let A be a smooth k-algebra. Then the map*

$$\phi\colon \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A) \to H^n(A, G^{n+1})$$

induces a homomorphism

$$\Phi\colon \mathrm{WMS}_{n+1}(A) \to H^n(A, G^{n+1})$$

for any $n \geq 2$.

Proof. Since $H^n(A, G^{n+1})$ is a group and the relation (i) in $\mathrm{WMS}_{n+1}(A)$ is clearly satisfied in $H^n(A, G^{n+1})$, it is enough to verify that relation (ii) is also satisfied. We start with a simple computation in $G^1(k(t))$. Using [17, Chapter I, Proposition 5.1], we have $\langle t, 1-t\rangle = \langle 1, t(t-1)\rangle$ in $I(k(t))$ because both forms represent 1 and they have the same discriminant. Adding $\langle -1, -1\rangle$ on both sides, we get $\langle -1, t\rangle + \langle -1, 1-t\rangle = \langle -1, t(1-t)\rangle$ in $I(k(t))$. Therefore we have an equality

$$(t, \langle -1, t\rangle) + (1-t, \langle -1, 1-t\rangle) = (t(1-t), \langle -1, t(1-t)\rangle) \tag{1}$$

in $G^1(k(t))$ (note that this is obvious in $K_1^{\mathrm{MW}}(k(t))$).

Suppose now that $(x, v_2, \dots, v_{n+1})$ and $(1-x, v_2, \dots, v_{n+1})$ are unimodular rows in A. Observe then that $(x(1-x), v_2, \dots, v_{n+1})$ is also unimodular. Performing if necessary elementary operations on this unimodular line, we can suppose that the sequence $(v_2, \dots, v_{n+1})$ is regular.

Now the pull back of ξ under the map $f\colon \mathrm{Spec}(A) \to \mathbb{A}^{n+1} - \{0\}$ given by $(x, v_2, \dots, v_{n+1})$ is precisely the cycle $(x, \langle -1, x\rangle)$ supported on $A/(v_2, \dots, v_{n+1})$. Since $(1-x, v_2, \dots, v_{n+1})$ is also unimodular by assumption, we obtain a cycle $(1-x, \langle -1, 1-x\rangle)$ also supported on $A/(v_2, \dots, v_{n+1})$. Because of relation 1 above, we see that the relation (ii) in $\mathrm{WMS}_{n+1}(A)$ is also satisfied in $H^n(A, G^{n+1})$ and the theorem is proved. □

Applying Theorem 2.3, we get the following corollary:

Corollary 4.2. *Let A be a smooth k-algebra of dimension d. For any $n \geq (d+2)/2$ the map $\phi\colon \mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A) \to H^n(A, G^{n+1})$ is a homomorphism of groups.*

There is an elementary proof of the fact that ϕ is surjective in some non trivial situations. Let $\mathfrak{m}$ be any maximal ideal in A and put $d = \dim(A)$. Then there is a regular sequence $(v_1, \dots, v_d)$ such that $A/(v_1, \dots, v_d)$ is a finite length A-module and $A_\mathfrak{m}/(v_1, \dots, v_d)A_\mathfrak{m} = A/\mathfrak{m}$ (use [9, Corollary 2.4]). The primary decomposition of this ideal is $(v_1, \dots, v_d) = \mathfrak{m} \cap M_1 \cap \dots \cap M_r$ for some $\mathfrak{m}_i$-primary ideals M_i (where $\mathfrak{m}_i$ are comaximal maximal ideals). Thus

$$A/(v_1, \dots, v_d) \simeq A/\mathfrak{m} \times A/M_1 \times \dots \times A/M_r.$$

Let $\alpha \in (A/\mathfrak{m})^\times$. Then there exists an element $a \in A$ such that its class modulo $(v_1, \dots, v_d)$ is $(\alpha, 1, \dots, 1)$ under the above isomorphism. Therefore $(a, v_1, \dots, v_d)$ is unimodular. Consider the Koszul complex $\mathrm{Kos}(v_1, \dots, v_d)$ associated to the regular sequence $(v_1, \dots, v_d)$. As in Section 3, we get an isomorphism

$$\psi_{v_1,\dots,v_d}\colon A/(v_1, \dots, v_d) \to \mathrm{Ext}^d_A(A/(v_1, \dots, v_d), A)$$

defined by $\psi_{v_1,\dots,v_d}(1) = \mathrm{Kos}(v_1, \dots, v_d)$. Consider $(a, \langle -\psi_{v_1,\dots,v_d}, a\psi_{v_1,\dots,v_d}\rangle)$ in $\bigoplus_{\mathfrak{q} \in \mathrm{Spec}(A)^{(d)}} G^1(A/\mathfrak{q})$. By construction, it vanishes outside $\mathfrak{m}$ and, as α varies, generates $G^1(A/\mathfrak{m})$ because any $(ab, \langle a\psi_{v_1,\dots,v_d}, b\psi_{v_1,\dots,v_d}\rangle)$ is equal to

$$(a, \langle -\psi_{v_1,\dots,v_d}, a\psi_{v_1,\dots,v_d}\rangle) - (-b, \langle -\psi_{v_1,\dots,v_d}, -b\psi_{v_1,\dots,v_d}\rangle)$$

in G^1. We have proven:

Proposition 4.3. *Let A be a smooth k-algebra of dimension d. Then the homomorphism $\phi\colon \mathrm{Um}_{d+1}/\mathrm{E}_{d+1}(A) \to H^d(A, G^{d+1})$ is surjective.*

Our next goal in the next section is to show that ϕ is in fact an isomorphism when $d \geq 3$, independently of the dimension d of the algebra. The case $d = 2$ will be treated in the sequel.

4.2. The case $d \geq 3$. In this section, we will use results of Morel ([23]). We will have to first recall some definitions and results in $\mathbb{A}^1$-homotopy theory. Our reference here will be [24]. Consider the category Sm/k of smooth schemes over k, endowed with the Nisnevich topology. The category of simplicial sheaves of sets on Sm/k (in the Nisnevich topology) is endowed with a model structure ([24, Definition 1.2, Theorem 1.4]), and we denote by $\mathcal{H}_s(k)$ its homotopy category. If F, G are two simplicial sheaves, we denote by $\mathrm{Hom}_{\mathcal{H}_s(k)}(F, G)$ the set of homomorphisms in this category.

Let X be a smooth scheme over k and consider the simplicial sheaf $\mathrm{Sing}^\bullet(X)$ defined at the level $n \in \mathbb{N}$ by $U \mapsto (X \times \Delta^n)(U)$ for any smooth scheme U. Here Δ^n denotes the usual n-simplex over k, i.e., $\Delta^n = \mathrm{Spec}(k[x_0, \dots, x_n]/\sum x_i - 1)$. Observe that there is a canonical map of simplicial sheaves $X \to \mathrm{Sing}^\bullet(X)$ (where X is seen as a simplicially constant sheaf). If moreover X is an algebraic group, then the above map is a map of simplicial sheaves of groups.

For any simplicial sheaf F, there exists a fibrant simplicial sheaf RF and a trivial cofibration $F \to RF$. Such an association can be done functorially. If X is a smooth scheme, then $\mathrm{Hom}_{\mathcal{H}_s(k)}(X, F) = \pi_0(RF(X))$ by definition. One of the results of [23] is that the map of simplicial sheaves $\mathbb{G}\mathrm{L}_n \to \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n$ induces an isomorphism $\mathrm{GL}_n(A)/\mathrm{E}_n(A) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(A, \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)$ for $n \geq 3$. The idea is to show that the map $\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n \to R\,\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n$ induces for any affine smooth scheme $\mathrm{Spec}(A)$ a weak-equivalence of simplicial sets $(\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)(A) \to (R\,\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)(A)$ for $n \geq 3$. The explanation of the proof first requires a definition (see [23]).

Definition 4.4. Let F be a presheaf of simplicial sets over Sm/k.

1) We say that F satisfies the affine B.G. property in the Nisnevich topology if for any smooth k-algebra A, any étale A-algebra $A \to B$ and any $f \in A$ such that $A/f \to B/f$ is an isomorphism, the diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(A_f) & \longrightarrow & F(B_f) \end{array}$$

is homotopy cartesian.

2) We say that F satisfies the $\mathbb{A}^1$-invariance property if for any smooth k-algebra A the map $F(A) \to F(A[t])$ induced by the inclusion $A \to A[t]$ is a weak equivalence.

The following theorem is a particular case of a theorem proved by Morel. Its proof is done in [23].

Theorem 4.5. *Let k be a perfect field. Let F be a simplicial sheaf of groups on* Sm/k *(for the Nisnevich topology). Suppose that F satisfies the affine B.G. property in the Nisnevich topology and the $\mathbb{A}^1$-invariance property. Then for any smooth k-algebra A the map $F(A) \to RF(A)$ is a weak equivalence.*

Corollary 4.6. *Let k be a perfect field and let A be a smooth k-algebra. Then the map of simplicial sheaves $\mathbb{G}\mathrm{L}_n \to \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n$ induces an isomorphism*

$$\mathrm{GL}_n(A)/\mathrm{E}_n(A) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(A, \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)$$

for $n \geq 3$.

Proof. We first prove that $\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n$ satisfies the properties of Definition 4.4. If F is any sheaf on Sm/k, then it is not hard to see that $\mathrm{Sing}^\bullet F$ is $\mathbb{A}^1$-invariant (see [23]). The affine B.G. property is also proven in [23] and requires $n \geq 3$. Theorem 4.5 shows then that $(\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)(A)$ is weak-equivalent to $(R\,\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)(A)$. Therefore $\pi_0((\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)(A)) \simeq \pi_0((R\,\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)(A)$. The left-hand term is just $\mathrm{GL}_n(A)/\mathrm{E}_n(A)$ by Theorem 2.1 and the other term is $\mathrm{Hom}_{\mathcal{H}_s(k)}(A, \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)$ by definition. □

Let now $\mathcal{H}(k)$ be the $\mathbb{A}^1$-homotopy category of smooth schemes over k. It can be seen as the full subcategory of $\mathbb{A}^1$-local objects in $\mathcal{H}_s(k)$ ([24, Theorem 3.2]). It turns out that $\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n$ is $\mathbb{A}^1$-local for $n \neq 2$. So $\mathrm{Hom}_{\mathcal{H}_s(k)}(A, \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n) = \mathrm{Hom}_{\mathcal{H}(k)}(A, \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n)$.

Consider the (pointed) map of simplicial sheaves $\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n \to \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_{n+1}$ induced by the inclusion $\mathbb{G}\mathrm{L}_n \to \mathbb{G}\mathrm{L}_{n+1}$ sending M to $\left(\begin{smallmatrix}1 & 0\\ 0 & M\end{smallmatrix}\right)$. It is a cofibration whose cofiber is $\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_{n+1}/\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n$, and it is not hard to see that the latter is isomorphic to $\mathrm{Sing}^\bullet(\mathbb{G}\mathrm{L}_{n+1}/\mathbb{G}\mathrm{L}_n)$. Moreover, the map of simplicial sheaves $\mathrm{Sing}^\bullet(\mathbb{G}\mathrm{L}_{n+1}/\mathbb{G}\mathrm{L}_n) \to \mathrm{Sing}^\bullet(\mathbb{A}^{n+1} \setminus \{0\})$ is a weak equivalence in $\mathcal{H}(k)$ and the following sequence

$$\mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_n \longrightarrow \mathrm{Sing}^\bullet\mathbb{G}\mathrm{L}_{n+1} \longrightarrow \mathrm{Sing}^\bullet(\mathbb{A}^{n+1} \setminus \{0\})$$

is a fibration sequence in $\mathcal{H}(k)$ ([23]). This is one of the ingredients of the proof of the following theorem of Morel ([23] again):

Theorem 4.7 (F. Morel). *Let A be a smooth k-algebra and let $n \geq 3$. Suppose that A is of dimension $d \leq n$. Then the natural map*

$$\mathrm{Hom}_{\mathcal{H}(k)}(A, \mathrm{Sing}^\bullet(\mathbb{A}^{n+1} \setminus \{0\})) \to H^n(A, G^{n+1})$$

is a bijection. This induces a bijection between the set of stably free modules of rank n and $H^n(A, G^{n+1})/\mathrm{GL}_{n+1}(A)$. Moreover, $A^\times$ acts trivially on $H^n(A, G^{n+1})$ and therefore $H^n(A, G^{n+1})/\mathrm{GL}_{n+1}(A) = H^n(A, G^{n+1})/\,\mathrm{SL}_{n+1}(A)$.

Remark 4.8. Notice that if $d < n$ then the set of stably free modules of rank n and $H^n(A, G^{n+1})$ are both trivial.

This allows to prove the following theorem:

Theorem 4.9. *Let A be a smooth k-algebra of dimension d. Suppose that k is perfect. Then the map $\phi\colon \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) \to H^d(A, G^{d+1})$ is an isomorphism for $d \geq 3$.*

Proof. By Theorem 2.5, there is an exact sequence of groups

$$\begin{array}{ccccc} \mathrm{SL}_d(A)/\mathrm{E}_d(A) & \longrightarrow & \mathrm{SL}_{d+1}(A)/\mathrm{E}_{d+1}(A) & & \\ & & \downarrow & & \\ & & \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) & \longrightarrow & \mathrm{Um}_{d+1}(A)/\,\mathrm{SL}_{d+1}(A) \longrightarrow 0. \end{array}$$

Because

$$\mathrm{Sing}^\bullet \mathbb{GL}_d \longrightarrow \mathrm{Sing}^\bullet \mathbb{GL}_{d+1} \longrightarrow \mathrm{Sing}^\bullet(\mathbb{A}^{d+1} \setminus \{0\})$$

is a fibration sequence and because of Theorem 4.7, we have an exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{H}(k)}(A, \mathrm{Sing}^\bullet \mathbb{GL}_d) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}(k)}(A, \mathrm{Sing}^\bullet \mathbb{GL}_{d+1}) & & \\ & & \downarrow & & \\ & & H^d(A, G^{d+1}) & \longrightarrow & H^d(A, G^{d+1})/\mathrm{GL}_{d+1}(A) \\ & & & & \downarrow \\ & & & & 0. \end{array}$$

Using the definition of ϕ, as well as Corollary 4.6, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{SL}_d(A)/\mathrm{E}_d(A) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}(k)}(A, \mathrm{Sing}^\bullet \mathbb{GL}_d) \\ \downarrow & & \downarrow \\ \mathrm{SL}_{d+1}(A)/\mathrm{E}_{d+1}(A) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}(k)}(A, \mathrm{Sing}^\bullet \mathbb{GL}_{d+1}) \\ \downarrow & & \downarrow \\ \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) & \longrightarrow & H^d(A, G^{d+1}) \\ \downarrow & & \downarrow \\ \mathrm{Um}_{d+1}(A)/\,\mathrm{SL}_{d+1}(A) & \longrightarrow & H^d(A, G^{d+1})/\mathrm{GL}_{d+1}(A) \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

The two top homomorphisms are injective with cokernel $A^\times$. We conclude by applying Theorem 4.7. □

Remark 4.10. As in the previous theorem, observe that if $n > d$, then $H^n(A, G^{n+1})$ and $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ are both trivial.

4.3. The case $d = 2$. We first recall some definitions. Let A be a k-algebra of dimension d, where k is of characteristic 0. Then one can define the *Euler class group* $E(A)$ of A ([8, §4]) and the *weak Euler class group* $E_0(A)$ of A ([8, §6]). In short, $E(A)$ is the group generated by pairs (J, ω_J), where $J \subset A$ is an ideal of height d such that J/J^2 is generated by d elements and ω_J is an equivalent class of surjections $(A/J)^d \to J/J^2$, modulo relations similar to rational equivalence. The group $E_0(A)$ is generated by elements (J), where J is an ideal of height d as above. There is a natural surjection $E(A) \to E_0(A)$. If d is even, there is an exact sequence ([8, Theorem 7.6])

$$\mathrm{Um}_{d+1}(A)/\,\mathrm{SL}_{d+1}(A) \xrightarrow{\psi} E(A) \longrightarrow E_0(A) \longrightarrow 0$$

where ψ is defined as follows:

Let $(a_1, \dots, a_{d+1})$ be a unimodular row. By performing if necessary elementary operations, we can suppose that the ideal $J = (a_2, \dots, a_{d+1})$ is of height d. Let $e_2, \dots, e_{d+1}$ be a basis of $(A/J)^d$ and let $\omega_J : (A/J)^d \to J/J^2$ be the surjection defined by $\omega_J(e_i) = a_i$ for any i. Because $(a_1, \dots, a_{d+1})$ is unimodular and $(a_2, \dots, a_{d+1})$ is of height d, $a_1 \in (A/J)^\times$ and we can define ψ by $\psi(a_1, \dots, a_{d+1}) = (J, a_1\omega_J)$ in $E(A)$. The proof that this is well defined is done in [8, §7] and this is where we need that A contains $\mathbb{Q}$.

Suppose now that A is of dimension 2. Then the above sequence is exact on the left also, i.e., we have a short exact sequence ([8, Proposition 7.3, Proposition 7.5])

$$0 \longrightarrow \mathrm{Um}_3(A)/\,\mathrm{SL}_3(A) \xrightarrow{\psi} E(A) \longrightarrow E_0(A) \longrightarrow 0.$$

If A is smooth over k, then $\phi\colon \mathrm{Um}_3(A)/\mathrm{E}_3(A) \to H^2(A, G^3)$ gives a homomorphism $\mathrm{SL}_3(A)/\mathrm{E}_3(A) \to H^2(A, G^3)$ (after composition with the homomorphism $\mathrm{SL}_3(A)/\mathrm{E}_3(A) \to \mathrm{Um}_3(A)/\mathrm{E}_3(A)$).

Theorem 4.11. *Let A be a smooth k-algebra of dimension* 2, *where k is a field of characteristic* 0. *The homomorphism ϕ induces an isomorphism*

$$\bar{\phi}\colon \mathrm{Um}_3(A)/\,\mathrm{SL}_3(A) \simeq H^2(A, G^3)/\,\mathrm{SL}_3(A).$$

Proof. Observe first that $\bar{\phi}$ is surjective by Proposition 4.3. Now there are surjective homomorphisms $E(A) \to \widetilde{\mathrm{CH}}^2(A)$ and $E_0(A) \to \mathrm{CH}^2(A)$ ([11, Proposition 17.2.10]) making the following diagram commutative:

$$\begin{array}{ccc} E(A) & \longrightarrow & E_0(A) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{CH}}^2(A) & \longrightarrow & \mathrm{CH}^2(A). \end{array}$$

Because $\dim(A) = 2$, the homomorphism $E(A) \to \widetilde{\mathrm{CH}}^2(A)$ is an isomorphism ([11, Theorem 15.3.11] and [8, Theorem 7.2]). We then get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Um}_3(A)/\,\mathrm{SL}_3(A) & \xrightarrow{\psi} & E(A) & \longrightarrow & E_0(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \simeq} & & \downarrow & & \\ & & H^2(A, G^3)/\,\mathrm{SL}_3(A) & \longrightarrow & \widetilde{\mathrm{CH}}^2(A) & \longrightarrow & \mathrm{CH}^2(A) & \longrightarrow & 0. \end{array}$$

Hence there exists a homomorphism $f\colon H^2(A, G^3)/\,\mathrm{SL}_3(A) \to \mathrm{Um}_3(A)/\,\mathrm{SL}_3(A)$ such that $f\bar{\phi} = \mathrm{Id}$. So $\bar{\phi}$ is also injective. □

5. Computations for real varieties

5.1. Computation of $H^d(A, G^{d+j})$. From now on, A is a smooth $\mathbb{R}$-algebra of dimension $d \geq 2$ with trivial orientation, i.e., $\omega_{A/\mathbb{R}} \simeq A$. Put $X = \mathrm{Spec}(A)$. First we compute $H^d(X, I^{d+j})$ for any $j \geq 0$.

Proposition 5.1. *For any $j \geq 0$, we have $H^d(X, I^{d+j}) \simeq \bigoplus_{C\in\mathcal{C}} \mathbb{Z}$ where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$. More precisely, choose a real point x_C for any C in $\mathcal{C}$ and a generator ξ_{x_C} of $\mathrm{Ext}^d_A(\mathbb{R}(x_C), A)$. Then the generators are the classes of the forms $(\langle 1, 1\rangle)^j \cdot \xi_{x_C}$ in $I^j(\mathbb{R}(x_C))$.*

Proof. For $j = 0$, this is [11, Theorem 16.3.8]. We prove the result by induction on j. Consider the form $\langle 1, 1\rangle \in I(\mathbb{R})$. It can be seen as an element of $H^0(\mathbb{R}, I)$. The multiplication by this element yields a homomorphism

$$\cdot\langle 1, 1\rangle\colon H^d(A, I^{d+j}) \to H^d(A, I^{d+j+1}).$$

Now the homomorphism of sheaves $I^{d+j+1} \to I^{d+j}$ induces a homomorphism $H^d(A, I^{d+j+1}) \to H^d(A, I^{j+d})$. It is easy to check that the composition of these two homomorphism is the multiplication by 2 from $H^d(A, I^{j+d})$ to itself. By induction $H^d(A, I^{j+d})$ is a sum of copies of $\mathbb{Z}$, and therefore the multiplication by 2 is injective. So the homomorphism $\cdot\langle 1, 1\rangle\colon H^d(A, I^{d+j}) \to H^d(A, I^{d+j+1})$ is injective. But the multiplication by $\langle 1, 1\rangle$ is surjective as a map from $\bigoplus_{x\in X^{(d)}} I^j(\mathbb{R}(x))$ to $\bigoplus_{x\in X^{(d)}} I^{j+1}(\mathbb{R}(x))$ because all residue fields are $\mathbb{R}$ or $\mathbb{C}$. Therefore the multiplication by $\langle 1, 1\rangle$ is also surjective on cohomology groups. □

Remark 5.2. If the canonical module $\omega_{A/\mathbb{R}}$ is non trivial, Proposition 5.1 is already wrong for $j = 0$ (see [7, Corollary 6.3]). More precisely, let A be a smooth $\mathbb{R}$-algebra of dimension d and let $X = \mathrm{Spec}(A)$. Then $H^d(A, I^d)$ is a finitely generated abelian group, with a free part corresponding to the compact connected components of $X(\mathbb{R})$

where the canonical module is trivial and a $\mathbb{Z}/2$-vector space corresponding to the compact connected components of $X(\mathbb{R})$ where the canonical module is not trivial. This can be deduced from [6, Theorem 4.21].

At the moment, I do not know how to compute $H^d(X, I^{d+j})$ for $j > 0$ for general smooth real algebras. Further work should clarify this.

The next result is an obvious consequence of the proposition.

Corollary 5.3. *For any $j \geq 0$, we have $H^d(X, \bar{I}^{d+j}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}/2\mathbb{Z}$ and an exact sequence of cohomology groups*

$$0 \longrightarrow H^d(A, I^{d+j+1}) \longrightarrow H^d(A, I^{d+j}) \longrightarrow H^d(A, \bar{I}^{d+j}) \longrightarrow 0.$$

Next we exhibit some exact sequence which will be useful for the computation of $H^d(X, G^{d+1})$. We first prove a preliminary result. Let $f: X \otimes \mathbb{C} \to X$ be the finite morphism induced by the inclusion $\mathbb{R} \subset \mathbb{C}$. For any $j \geq 0$, it yields a morphism $f_*: H^d(X \otimes \mathbb{C}, K^M_{d+j}) \to H^d(X, K^M_{d+j})$. Moreover, the natural projection gives a homomorphism $H^d(X, K^M_{d+j}) \to H^d(X, K^M_{d+j}/2K^M_{d+j})$.

Proposition 5.4. *For any $j \geq 1$, the sequence*

$$H^d(X \otimes \mathbb{C}, K^M_{d+j}) \xrightarrow{f_*} H^d(X, K^M_{d+j}) \longrightarrow H^d(X, K^M_{d+j}/2K^M_{d+j}) \longrightarrow 0$$

is exact.

Proof. It suffices to show that the sequence of groups

$$\bigoplus_{x \in (X\otimes\mathbb{C})^{(d)}} K^M_j(\mathbb{R}(x)) \xrightarrow{f_*} \bigoplus_{y \in X^{(d)}} K^M_j(\mathbb{R}(y)) \longrightarrow \bigoplus_{y \in X^{(d)}} K^M_j(\mathbb{R}(y))/2K^M_j(\mathbb{R}(y)) \longrightarrow 0$$

is exact. We have two distinct cases, depending on whether y is a complex point or a real point. Suppose first that y is a complex point. Then there are two points x_1 and x_2 in $(X \otimes \mathbb{C})^{(d)}$ over y and the above sequence becomes

$$K^M_j(\mathbb{C}) \oplus K^M_j(\mathbb{C}) \xrightarrow{f_*} K^M_j(\mathbb{C}) \longrightarrow K^M_j(\mathbb{C})/2K^M_j(\mathbb{C}) \longrightarrow 0$$

where f_* is just the sum (which is surjective). Since $j \geq 1$, $K^M_j(\mathbb{C})$ is 2-divisible and therefore $K^M_j(\mathbb{C})/2K^M_j(\mathbb{C}) = 0$.

Suppose now that y is a real point. There is only a complex point over y and the sequence becomes

$$K_j^M(\mathbb{C}) \xrightarrow{f_*} K_j^M(\mathbb{R}) \longrightarrow K_j^M(\mathbb{R})/2K_j^M(\mathbb{R}) \longrightarrow 0.$$

Here f_* is just the transfer map given by the inclusion $\mathbb{R} \subset \mathbb{C}$. But $K_j^M(\mathbb{R})$ is just the direct sum of a 2-divisible group D generated by symbols $\{a_1, \dots, a_j\}$ with $a_i > 0$ and a factor $\mathbb{Z}/2\mathbb{Z}$ generated by $\{-1, \dots, -1\}$. Now f_* is surjective on D (use [19, Proposition 14.64]) and 0 on the subgroup generated by $\{-1, \dots, -1\}$ because $K_j^M(\mathbb{C})$ is 2-divisible. So the sequence is exact. □

As a corollary, we get:

Proposition 5.5. *Let X be a real smooth affine variety with trivial canonical bundle. Then for any $j \geq 0$, the sequence*

$$H^d(X \otimes \mathbb{C}, G^{d+j}) \xrightarrow{f_*} H^d(X, G^{d+j}) \longrightarrow H^d(X, I^{d+j}) \longrightarrow 0$$

is split exact, where the first homomorphism is induced by the finite morphism $f \colon X \otimes \mathbb{C} \to X$ and the second by the map of sheaves $G^{d+j} \to I^{d+j}$. Moreover, the morphism of sheaves $G^{d+j} \to K_{d+j}^M$ induces an isomorphism $H^d(X \otimes \mathbb{C}, G^{d+j}) \to H^d(X \otimes \mathbb{C}, K_{d+j}^M)$.

Proof. If $j = 0$, this is [11, Theorem 16.6.4] and [11, Remark 10.2.16]. We suppose now that $j \geq 1$. First observe that, since $I(\mathbb{C}) = 0$, we have $G^j(\mathbb{C}) = K_j^M(\mathbb{C})$. This proves the last assertion of the theorem. This also proves that the composition

$$H^d(X \otimes \mathbb{C}, G^{d+j}) \xrightarrow{f_*} H^d(X, G^{d+j}) \longrightarrow H^d(X, I^{d+j})$$

is zero since the groups $G^j(\mathbb{R}(x))$ are the fibre products of $K_j^M(\mathbb{R}(x))$ and $I^j(\mathbb{R}(x))$ over $\bar{I}^j(\mathbb{R}(x))$ for any $x \in X^{(d)}$. Using the definition of the corresponding sheaves, it is not hard to see that there is a commutative diagram of sheaves whose rows are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{d+j+1} & \longrightarrow & G^{d+j} & \longrightarrow & K_{d+j}^M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{d+j+1} & \longrightarrow & I^{d+j} & \longrightarrow & \bar{I}^{d+j} & \longrightarrow & 0. \end{array}$$

This yields the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H^d(X\otimes\mathbb{C},G^{d+j}) & \longrightarrow & H^d(X\otimes\mathbb{C},K^M_{d+j}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^d(X,I^{d+j+1}) & \longrightarrow & H^d(X,G^{d+j}) & \longrightarrow & H^d(X,K^M_{d+j}) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^d(X,I^{d+j+1}) & \longrightarrow & H^d(X,I^{d+j}) & \longrightarrow & H^d(X,\bar{I}^{d+j}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are exact. A simple chase in the diagram shows that it suffices to prove that the right column is exact to finish. Proposition 5.4 gives an exact sequence

$$H^d(X\otimes\mathbb{C},K^M_{d+j}) \xrightarrow{f_*} H^d(X,K^M_{d+j}) \longrightarrow H^d(X,K^M_{d+j}/2K^M_{d+j}) \longrightarrow 0.$$

But the homomorphisms s_n of Section 3.1 yield a homomorphism

$$H^d(X,K^M_{d+j}/2K^M_{d+j}) \to H^d(X,\bar{I}^{d+j})$$

which is in fact an isomorphism by [36, Theorem 7.4] and [26, Theorem 4.1]. □

Next we prove that $H^d(X\otimes\mathbb{C},K^M_{d+j})=0$ for some interesting algebras. Recall that a real variety X is said to be rational if $X\otimes\mathbb{C}$ is birational to $\mathbb{P}^d$.

Proposition 5.6. *Let A be a smooth $\mathbb{R}$-algebra of dimension d. Suppose that $X=$ $\mathrm{Spec}(A)$ is rational. Then $H^d(X\otimes\mathbb{C},K^M_{d+j})=0$ for any $j\geq 0$.*

Proof. Suppose first $j=0$. Then $H^d(X\otimes\mathbb{C},K^M_d)=\mathrm{CH}^d(X\otimes\mathbb{C})=0$ because $X\otimes\mathbb{C}$ is rational. Using [25, Corollary 3.4, Theorem 2.11] (see also [29] and [30]), this shows that any maximal ideal $\mathfrak{m}$ in $A\otimes\mathbb{C}$ is complete intersection. Let $\{a_1,\dots,a_j\}$ be an element of $K^M_j(\mathbb{C})=K^M_j((A\otimes\mathbb{C})/\mathfrak{m})$. Let $(f_1,\dots,f_d)$ be a regular sequence generating $\mathfrak{m}$. Consider the symbol $\{f_d,a_1,a_2,\dots,a_j\}$ defined on the residue fields of the generic points of $(A\otimes\mathbb{C})/(f_1,\dots,f_{d-1})$. It defines an element of $\bigoplus_{x\in\mathrm{Spec}(A\otimes\mathbb{C})^{(d-1)}}K^M_{j+1}(\mathbb{R}(x))$ whose boundary is $\{a_1,\dots,a_j\}$. □

Finally, we get:

Theorem 5.7. *Let A be a smooth $\mathbb{R}$-algebra of dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then*

$$H^d(X, G^{d+j}) \simeq H^d(X, I^{d+j}) \simeq \bigoplus_{C \in \mathcal{C}} \mathbb{Z}$$

for $j \geq 0$, where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).

Proof. The first isomorphism is clear in view of Proposition 5.5 and Proposition 5.6. The second isomorphism is just Proposition 5.1. □

Remark 5.8. If $d \geq 3$ this shows that

$$\mathrm{Hom}_{\mathbb{A}^1}(X, \mathbb{A}^{d+1} \setminus \{0\}) = \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)$$

(which is isomorphic to $H^d(X, G^{d+1})$) is isomorphic to the cohomotopy group $\pi^d(X(\mathbb{R}))$. Observe that if the algebra is not rational, then the complex points may appear making this statement incorrect.

5.2. Stably free modules. The previous section allows to understand the structure of stably free modules over good real algebras. Before stating the result, we briefly recall the definition of the Euler class.

Let A be a smooth k-algebra of dimension d and let P be a projective module of rank d over A with trivial determinant. To such a module, one can associate an Euler class $\tilde{c}_d(P)$ in $\widetilde{\mathrm{CH}}^d(A)$ ([23] or [11, Chapter 13]) which satisfies the following property (proven in [23] if $d \geq 4$, in [14] if $d = 3$ and in [11] if $d = 2$): $\tilde{c}_d(P) = 0$ if and only if $P \simeq Q \oplus A$ (the same result holds for projective modules with non trivial determinant, but we do not use this fact here). When d is even, the Euler class allows to strengthen our results:

Theorem 5.9. *Let A be a smooth $\mathbb{R}$-algebra of even dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then the set of isomorphism classes of stably free modules of rank d is isomorphic to $\bigoplus_{C \in \mathcal{C}} \mathbb{Z}$, where $\mathcal{C}$ is the set of compact connected components of $X(\mathbb{R})$ (endowed with the Euclidian topology).*

Proof. By Proposition 3.1, there is an exact sequence

$$H^d(X, G^{d+1}) \longrightarrow \widetilde{\mathrm{CH}}^d(X) \longrightarrow \mathrm{CH}^d(X) \longrightarrow 0.$$

Theorem 5.7, shows that this sequence is exact on the left also.

Suppose that $d \geq 3$. Because of Theorem 4.9, we get a short exact sequence:

$$0 \longrightarrow \mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) \longrightarrow \widetilde{\mathrm{CH}}^d(X) \longrightarrow \mathrm{CH}^d(X) \longrightarrow 0$$

and $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) \simeq \bigoplus_{C\in\mathcal{C}}\mathbb{Z}$ by Theorem 5.7. Using [8, §7], we see that the homomorphism $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A) \to \widetilde{\mathrm{CH}}^d(X)$ associates to a stably free module P (representing a unimodular row) its Euler class. The Euler class of A^d being trivial, a unimodular row coming from $\mathrm{GL}_{d+1}(A)$ has therefore image 0 in $\widetilde{\mathrm{CH}}^d(X)$. The exact sequence above shows that $\mathrm{GL}_{d+1}(A)$ acts trivially on $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)$. This proves the result when $d \geq 4$.

Suppose now that $d = 2$. Because of Theorem 4.11, it suffices to compute $H^2(A, G^3)/\mathrm{SL}_3(A)$. The same argument as above shows that the action of $\mathrm{SL}_3(A)$ on $H^2(A, G^3)$ is trivial. This concludes the proof. □

Theorem 5.10. *Let A be a smooth $\mathbb{R}$-algebra of even dimension d with trivial canonical bundle. Suppose that $X = \mathrm{Spec}(A)$ is rational. Then a stably free module of rank d over A is free if and only if its Euler class is* 0.

Proof. Again, the exact sequence

$$H^d(X, G^{d+1}) \longrightarrow \widetilde{\mathrm{CH}}^d(X) \longrightarrow \mathrm{CH}^d(X) \longrightarrow 0.$$

is also exact on the left by Theorem 5.7 and the map $H^d(X, G^{d+1}) \to \widetilde{\mathrm{CH}}^d(X)$ in the exact sequence of Proposition 3.1 sends a stably free module to its Euler class. □

Remark 5.11. Observe that we heavily use the fact that A is of even dimension in the theorem in order to identify the homomorphism $H^d(X, G^{d+1}) \to \widetilde{\mathrm{CH}}^d(X)$ of Proposition 3.1. In odd dimension, this homomorphism cannot be the Euler class, since the Euler class of an odd dimensional stably free module is trivial. It is clear however that the homomorphism $H^d(X, G^{d+1}) \to \widetilde{\mathrm{CH}}^d(X)$ is in general non trivial! A consequence of this is that the action of $\mathrm{SL}_{d+1}(A)$ on $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)$ might be non trivial if d is odd. We will see below that this is the case for the real algebraic spheres S^3 and S^7.

The other hypotheses in the theorem are explained by the fact that we use Theorem 5.7 in the proof of the theorem. As already said in Remark 5.2, I do not know how to compute the groups involved when the canonical module is not trivial. If the algebra is not rational, then the group $\mathrm{Um}_{d+1}(A)/\mathrm{E}_{d+1}(A)$ might contain some non trivial subgroup generated by complex points. This subgroup will be contained in the kernel of the Euler class, but I do not see why the corresponding modules should be trivial. Again, this should be clarified in further work.

As an illustration of the theorem, let S^d denote the algebraic real sphere of dimension d, i.e., $S^d = \mathrm{Spec}(\mathbb{R}[x_1, \ldots, x_{d+1}]/\sum x_i^2 - 1)$.

Corollary 5.12. *The set of isomorphism classes of stably free modules of rank $2d$ over S^{2d} is isomorphic to $\mathbb{Z}$. It is generated by the tangent bundle.*

Proof. The first statement is an obvious corollary of Theorem 5.9, since the set of real maximal ideal is the real sphere of dimension 3. We prove next that the tangent bundle generates $H^{2d}(S^{2d}, G^{2d+1})$. By Theorem 5.7, it suffices to see that it generates $H^{2d}(S^{2d}, I^{2d+1})$. Consider the complete intersection ideal $\mathfrak{a} = (x_1, \dots, x_{2d})$ and the symmetric isomorphism

$$\psi_{x_1,\dots,x_{2d}} : A/\mathfrak{a} \to \mathrm{Ext}^{2d}_A(A/\mathfrak{a}, A)$$

defined by $1 \mapsto \mathrm{Kos}(x_1, \dots, x_{2d})$, where the latter is the Koszul complex associated to the regular sequence $(x_1, \dots, x_{2d})$. Since x_{2d+1} is invertible modulo $\mathfrak{a}$ we can consider the symmetric isomorphism $\langle -1, x_{2d+1}\rangle \cdot \psi_{x_1,\dots,x_{2d}}$ on the finite length module $A/\mathfrak{a}$.

Now we have a decomposition of the form $\mathfrak{a} = \mathfrak{m}_1 \cap \mathfrak{m}_{-1}$, where $\mathfrak{m}_1 = (x_1, \dots, x_{2d}, x_{2d+1} - 1)$ and $\mathfrak{m}_{-1} = (x_1, \dots, x_{2d}, x_{2d+1} + 1)$. This decomposition decomposes the finite length module $A/\mathfrak{a}$ (and the symmetric isomorphism $\langle -1, x_{2d+1}\rangle \cdot \psi_{x_1,\dots,x_{2d}}$). Since $x_{2d+1} \equiv 1$ modulo $\mathfrak{m}_1$ and $\langle -1, 1\rangle = 0$ in $I(\mathbb{R})$, we see that

$$(A/\mathfrak{a}, \langle -1, x_{2d+1}\rangle \cdot \psi_{x_1,\dots,x_{2d}}) = (A/\mathfrak{m}_{-1}, \langle -1, -1\rangle \cdot (\psi_{x_1,\dots,x_{2d}})_{\mathfrak{m}_{-1}})$$

holds in the group $H^{2d}(S^{2d}, I^{2d+1})$, where $(\psi_{x_1,\dots,x_{2d}})_{\mathfrak{m}_{-1}}$ is the localization of $\psi_{x_1,\dots,x_{2d}}$. The right hand term is a generator of $H^{2d}(S^{2d}, I^{2d+1})$ by Proposition 5.1, and the left hand term is the image of the unimodular row $(x_1, \dots, x_{2d+1})$ under the homomorphism

$$\phi\colon \mathrm{Um}_{2d+1}(S^{2d})/\mathrm{E}_{2d+1}(S^{2d}) \to H^{2d}(S^{2d}, I^{2d+1})$$

of Section 4.1. □

In odd dimension, the situation is a bit more complicated as illustrated by the following result:

Proposition 5.13. *All stably free modules of top rank on S^3 and S^7 are free.*

Proof. We do the proof for S^3, the case of S^7 being similar. The proof of the above corollary shows that $\mathrm{Um}_4(S^3)/\mathrm{E}_4(S^3) \simeq \mathbb{Z}$ with generator the tangent bundle. It is well known that the tangent bundle over S^3 is free and therefore its associated unimodular row comes from $\mathrm{GL}_4(S^3)$. This shows that $\mathrm{Um}_4(S^3)/\mathrm{GL}_4(S^3) = 0$. □

Remark 5.14. In the proposition, we restricted to S^3 and S^7 because in those cases the tangent bundle is actually free. In [12], we proved that all the projective modules on S^3 are free, while the analogue result on S^7 seems far out of range at the moment.

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