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# Arithmetic properties of $\varphi(n)/\lambda(n)$ and the structure of the multiplicative group modulo *n*

William D. Banks, Florian Luca and Igor E. Shparlinski

**Abstract.** For a positive integer *n*, we let  $\varphi(n)$  and  $\lambda(n)$  denote the Euler function and the Carmichael function, respectively. We define  $\xi(n)$  as the ratio  $\varphi(n)/\lambda(n)$  and study various arithmetic properties of  $\xi(n)$ .

#### Mathematics Subject Classification (2000). 11A25.

Keywords. Euler function, Carmichael function.

#### 1. Introduction and notation

Let  $\varphi(n)$  denote the *Euler function*, which is defined as usual by

$$\varphi(n) = #(\mathbb{Z}/n\mathbb{Z})^{\times} = \prod_{p^{\nu} \parallel n} p^{\nu-1}(p-1), \quad n \ge 1.$$

The *Carmichael function*  $\lambda(n)$  is defined for all  $n \ge 1$  as the largest order of any element in the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^{\times}$ . More explicitly, for any prime power  $p^{\nu}$ , one has

$$\lambda(p^{\nu}) = \begin{cases} p^{\nu-1}(p-1) & \text{if } p \ge 3 \text{ or } \nu \le 2, \\ 2^{\nu-2} & \text{if } p = 2 \text{ and } \nu \ge 3, \end{cases}$$

and for an arbitrary integer  $n \ge 2$ ,

$$\lambda(n) = \operatorname{lcm}\left(\lambda(p_1^{\nu_1}), \ldots, \lambda(p_k^{\nu_k})\right),\,$$

where  $n = p_1^{\nu_1} \dots p_k^{\nu_k}$  is the prime factorization of *n*. Clearly,  $\lambda(1) = 1$ .

Despite their many similarities, the functions  $\varphi(n)$  and  $\lambda(n)$  often exhibit remarkable differences in their arithmetic behavior, and a vast number of results about the growth rate and various arithmetical properties of  $\varphi(n)$  and  $\lambda(n)$  have been obtained; see for example [4], [5], [7], [8], [9], [11], [15]. In this paper, we consider the

arithmetical function defined by

$$\xi(n) = \frac{\varphi(n)}{\lambda(n)}, \quad n \ge 1,$$

and we study some of its arithmetic properties.

In particular, letting P(k) denote the largest prime factor of a positive integer k (with the convention that P(1) = 1), we study the behavior of  $P(\xi(n))$ . Our results imply that typically  $\xi(n)$  is much "smoother" than a random integer k of the same size. To make this comparison, it is useful to recall that Theorem 2 of [9] implies that the estimate

$$\xi(n) = \exp\left(\log_2 n \log_3 n + C \log_2 n + o(\log_2 n)\right) \tag{1}$$

holds on a set of positive integers *n* of asymptotic density 1 with some absolute constant C > 0. Here, and in the sequel, for a real number z > 0 and a natural number  $\ell$ , we write  $\log_{\ell} z$  for the recursively defined function given by  $\log_1 z = \max\{\log z, 1\}$ , where  $\log z$  denotes the natural logarithm of *z*, and  $\log_{\ell} z = \max\{\log(\log_{\ell-1} z), 1\}$  for  $\ell > 1$ . When  $\ell = 1$ , we omit the subscript (however, we still assume that all the logarithms that appear below are at least 1). Of course, when *z* is sufficiently large, then  $\log_{\ell} z$  is nothing more than the  $\ell$ -fold composition of the natural logarithm evaluated at *z*.

We also use  $\Omega(n)$  and  $\omega(n)$  with their usual meanings:  $\Omega(n)$  denotes the total number of prime divisors of n > 1 counted with multiplicity, while  $\omega(n)$  is the number of distinct prime factors of n > 1; as usual, we put  $\Omega(1) = \omega(1) = 0$ . In this paper, we also study the functions  $\Omega(\xi(n))$  and  $\omega(\xi(n))$ .

Observe that a prime *p* divides  $\xi(n)$  if and only if the *p-Sylow subgroup* of the group  $(\mathbb{Z}/n\mathbb{Z})^{\times}$  is *not* cyclic. Thus,  $P(\xi(n))$  and  $\omega(\xi(n))$  can be viewed as measures of "non-cyclicity" of this group. In particular,  $\omega(\xi(n))$  is the number of non-cyclic Sylow subgroups of  $(\mathbb{Z}/n\mathbb{Z})^{\times}$ .

We also remark that any prime  $p | \xi(n)$  has that property that  $p^2 | \varphi(n)$ . Thus, while studying the prime factors of  $\xi(n)$ , one is naturally lead to an associated question concerning the difference  $\Omega(\varphi(n)) - \omega(\varphi(n))$ , a question that we address here as well.

As usual, for a large number  $x, \pi(x)$  denotes the number of primes  $p \le x$ , and for positive integers a, k with  $gcd(a, k) = 1, \pi(x; k, a)$  denotes the number of primes  $p \le x$  with  $p \equiv a \pmod{k}$ .

We use the Vinogradov symbols  $\gg$ ,  $\ll$ ,  $\asymp$  as well as the Landau symbols O and o with their usual meanings. The implied constants in the symbols O,  $\gg$ ,  $\ll$  and  $\asymp$  are always absolute unless indicated otherwise.

Finally, we say that a certain property holds for "almost all" *n* if it holds for all  $n \le x$  with at most o(x) exceptions, as  $x \to \infty$ .

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#### **2.** Distribution of $P(\xi(n)), \omega(\xi(n))$ and $\Omega(\xi(n))$

In what follows, let us call a real-valued function  $\varepsilon(x)$  *admissible* if

- $\varepsilon(x)$  is a decreasing function, with limit 0 as  $x \to \infty$ ;
- $\varepsilon(x) \log_2 x$  is an increasing function, tending to  $\infty$  as  $x \to \infty$ .

We begin with the following statement, which may be of independent interest.

**Lemma 1.** For any admissible function  $\varepsilon(x)$  and any prime  $q \le \varepsilon(x) \log_2 x$ , every positive integer  $n \le x$  has at least  $(\log_2 n)/2q$  distinct prime factors  $p \equiv 1 \pmod{q}$ , with at most o(x) exceptions.

*Proof.* Let  $\omega(n, q)$  denote the number of distinct prime factors p of n such that  $p \equiv 1 \pmod{q}$ . For any real number  $y \ge 1$  and integer  $a \ge 1$ , put

$$S(y,a) = \sum_{\substack{p \le y \\ p \equiv 1 \pmod{a}}} \frac{1}{p}.$$
(2)

It is known (see Theorem 1 in [18] or Lemma 6.3 in [17]) that

$$S(y, a) = \frac{\log_2 y}{\varphi(a)} + O(1).$$
 (3)

In particular, the estimate

$$S(n,q) = \frac{\log_2 n}{q-1} + O(1) \gg \varepsilon(x)^{-1}$$

holds for all q in the stated range and all  $n > x^{1/2}$ , once x is sufficiently large. By the classical result of Turán [20], we also have that the estimate

$$\omega(n,q) = S(n,q) + O\left(S(n,q)^{2/3}\right)$$

holds for all *n* in the interval  $x^{1/2} < n \le x$ , with at most

$$O(xS(n,q)^{-1/6}) = O(x\varepsilon(x)^{1/6}) = o(x)$$

possible exceptions, and the result now follows.

**Lemma 2.** For real numbers  $x \ge y > 1$  let

$$\Xi(x, y) = \#\{n \le x : P(\xi(n)) > y\}.$$

Then,

$$\Xi(x, y) \ll \frac{x(\log_2 x)^2}{y \log y}.$$

*Proof.* If a prime q divides  $\xi(n)$ , then clearly  $q^2 | \varphi(n)$ . The upper bound

$$\#\{n \le x : \varphi(n) \equiv 0 \pmod{q^2}\} \ll \frac{x(\log_2 x)^2}{q^2}$$

is a special partial case of Lemma 2 of [5] (see also the proof of Theorem 7.1 in [4]). In particular,

$$\#\{n \le x : P(\xi(n)) = q\} \ll \frac{x(\log_2 x)^2}{q^2}.$$
(4)

It now follows that

$$\Xi(x, y) = \sum_{y < q \le x} \sum_{\substack{n \le x \\ P(\xi(n)) = q}} 1 \ll \sum_{y < q \le x} \frac{x (\log_2 x)^2}{q^2}.$$

Using Abel summation, we estimate

$$\sum_{y < q \le x} \frac{1}{q^2} = \frac{\pi(x)}{x^2} - \frac{\pi(y)}{y^2} + 2\int_y^x \frac{\pi(t)}{t^3} dt \ll \frac{1}{x \log x} + \int_y^x \frac{1}{t^2 \log t} dt \ll \frac{1}{y \log y},$$
  
and the lemma follows.

and the lemma follows.

**Theorem 1.** If  $\varepsilon(x)$  is any admissible function, then the inequalities

$$\varepsilon(n)\log_2 n \le P(\xi(n)) \le \frac{(\log_2 n)^2}{\varepsilon(n)\log_3 n}$$

hold for almost all positive integers n.

*Proof.* By the Prime Number Theorem, for all sufficiently large real numbers *x* there exists a prime q in the interval:

$$\varepsilon(x)\log_2 x < q \le 2\varepsilon(x)\log_2 x.$$

If *n* is an integer with two prime factors  $p_1 \equiv p_2 \equiv 1 \pmod{q}$ , then  $q \mid \xi(n)$ . By Lemma 1, we derive that

$$\sum_{\substack{x^{1/2} < n \le x \\ P(\xi(n)) \ge \varepsilon(n) \log_2 n}} 1 \ge \sum_{\substack{x^{1/2} < n \le x \\ P(\xi(n)) \ge q}} 1 \ge \sum_{\substack{x^{1/2} < n \le x \\ \omega(n,q) \ge 2}} 1 = x + o(x).$$

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This proves the lower bound. The upper bound is a direct application of Lemma 2.  $\hfill \Box$ 

We remark that the upper bound of Theorem 1 improves the corollary to Theorem 2 in [9].

**Theorem 2.** As  $x \to \infty$ , we have

$$(1+o(1)) x \log_3 x \le \sum_{n \le x} \log P(\xi(n)) \le (2+o(1)) x \log_3 x.$$

*Proof.* The above lower bound follows from the lower bound from Theorem 1. For the upper bound above, we write

$$\sum_{n \le x} \log P(\xi(n)) = \sum_{q \le x} \log q \sum_{\substack{n \le x \\ P(\xi(n)) = q}} 1.$$

For  $q \leq y$ , we trivially have

$$\sum_{q \le y} \log q \sum_{\substack{n \le x \\ P(\xi(n)) = q}} 1 \le \log y \sum_{q \le y} \sum_{\substack{n \le x \\ P(\xi(n)) = q}} 1 \le \log y \sum_{n \le x} 1 \le x \log y,$$

while for q > y, we have, by (4):

$$\sum_{y < q \le x} \log q \sum_{\substack{n \le x \\ P(\xi(n)) = q}} 1 \ll x (\log_2 x)^2 \sum_{y < q \le x} \frac{\log q}{q^2} \ll x y^{-1} (\log_2 x)^2,$$

where we have used Abel summation to estimate

$$\sum_{y < q \le x} \frac{\log q}{q^2} = \pi(x) \frac{\log x}{x^2} - \pi(y) \frac{\log y}{y^2} - \int_y^x \pi(t) \left(\frac{1}{t^3} - \frac{2\log t}{t^3}\right) dt$$
$$\ll x^{-1} + \int_y^x t^{-2} dt \ll y^{-1}.$$

Setting  $y = (\log_2 x)^2$ , we obtain the desired upper bound.

**Theorem 3.** As  $x \to \infty$ , we have

$$\sum_{n \le x} P(\xi(n)) \asymp x (\log_2 x)^3.$$

*Proof.* Let  $y = (\log_2 x)^3$ ,  $z = \exp((\log x)^{1/2})$  and  $w = \exp((\log x)^{2/3})$ . We also put  $v = z^6$ . In what follows, x is taken to be arbitrarily large.

Taking A = 5/2,  $\varepsilon = 1/2$ , and  $\delta = 1/15$  in the statement of Theorem 2.1 of [1], we see that there exists an absolute constant  $D \ge 0$  and a set  $\mathcal{D}$  of cardinality  $\#\mathcal{D} \le D$ , with  $\min\{m : m \in \mathcal{D}\} \ge \log v = 6(\log x)^{1/2}$ , such that the inequality

$$\pi(t; d, 1) \ge \frac{\pi(t)}{2\varphi(d)} \tag{5}$$

holds for all positive reals t provided that  $1 \le d \le \min\{tv^{-2/3}, z^2\}$  and that d is not divisible by any element of  $\mathcal{D}$ . Note that if x is sufficiently large and  $t \ge w$ , then  $tv^{-2/3} \ge wv^{-2/3} \ge z^2$ .

Letting  $\mathcal{Q}$  denote the set of primes  $q \in [y, z] \setminus \mathcal{D}$ , we therefore see that the lower bound (5) holds for all  $t \in [w, x]$  and all integers  $d \in [1, z^2]$  whose prime factors all lie in  $\mathcal{Q}$ . Together with the Brun–Titchmarsh theorem (see for example Theorem 3.7 in Chapter 3 of [12]), we conclude that

$$\pi(t; d, 1) \asymp \frac{\pi(t)}{\varphi(d)}$$

holds uniformly for all  $t \in [w, x]$  and all integers d of the form d = q or  $d = q_1q_2$  composed of one or two (not necessarily distinct) primes from Q. Moreover, for any sufficiently large constant  $\gamma > 1$ , we also have

$$\pi(t; d, 1) - \pi(t/\gamma; d, 1) \asymp \frac{\pi(t)}{\varphi(d)}$$
(6)

under the same conditions.

We now let

$$k = \left\lceil \frac{\log w}{\log \gamma} \right\rceil$$
 and  $K = \left\lfloor \frac{\log x}{2 \log \gamma} \right\rfloor - 1.$ 

For any prime  $q \in Q$ , we have, by (6):

$$\sum_{\substack{w$$

On the other hand, the upper bound (3.1) in [7] (see also Lemma 1 of [5]) provides an upper bound of the same size as the above lower bound. Consequently,

$$\sum_{\substack{w 
(7)$$

We now fix a prime number q in Q. We denote by N(x, q) the number of integers  $n \le x$  for which there exists a unique representation of the form  $n = p_1 p_2 m$  for some integer m and two primes  $w < p_1 < p_2 \le x^{1/2}$  with  $p_1 \equiv p_2 \equiv 1 \pmod{q}$  and such that q is the only prime in Q dividing  $gcd(p_1 - 1, p_2 - 1)$ . We then have

$$N(x,q) \ge T_0(x,q) - T_1(x,q) - T_2(x,q) - T_3(x,q),$$

where

•  $T_0(x, q)$  is the total number of ordered triples  $(p_1, p_2, m)$  with primes  $w < p_1 < p_2 \le x^{1/2}$ ,  $p_1 \equiv p_2 \equiv 1 \pmod{q}$ , and an integer  $m \le x/p_1p_2$ . Therefore, using (7), we obtain that

$$T_{0}(x,q) \gg x \sum_{\substack{w < p_{1} < p_{2} \le x^{1/2} \\ p_{1} \equiv p_{2} \equiv 1 \pmod{q}}} \frac{1}{p_{1}p_{2}}$$

$$= \frac{x}{2} \left( \sum_{\substack{w 
$$\gg \frac{x}{2} \left( \frac{\log_{2} x}{q} \right)^{2} - \frac{x}{2q} \sum_{\substack{w 
$$= \frac{x(\log_{2} x)^{2}}{2q^{2}} + O\left( \frac{x\log_{2} x}{q^{2}} \right) \gg \frac{x(\log_{2} x)^{2}}{q^{2}}.$$$$$$

•  $T_1(x,q)$  is the number of triples  $(p_1, p_2, m)$  as above for which there exists another prime  $\ell \in \mathcal{Q}, \ell \neq q$ , such that  $p_1 \equiv p_2 \equiv 1 \pmod{\ell}$ . Then, by (7), we have that

$$\begin{split} T_1(x,q) \ll x \sum_{\substack{\ell \in \mathcal{Q} \\ \ell \neq q}} \sum_{\substack{w < p_1 < p_2 \le x^{1/2} \\ p_1 \equiv p_2 \equiv 1 \pmod{q\ell}}} \frac{1}{p_1 p_2} \le x \sum_{\ell \in \mathcal{Q}} \left( \sum_{\substack{w < p < x^{1/2} \\ p \equiv 1 \pmod{q\ell}}} \frac{1}{p} \right)^2 \\ \ll x \sum_{\ell \in \mathcal{Q}} \frac{(\log_2 x)^2}{q^2 \ell^2} \ll \frac{x (\log_2 x)^2}{q^2} \sum_{\ell > y} \frac{1}{\ell^2} \\ \ll \frac{x (\log_2 x)^2}{q^2 y \log y} = o\left(\frac{x (\log_2 x)^2}{q^2}\right). \end{split}$$

•  $T_2(x,q)$  is the number of triples  $(p_1, p_2, m)$  as above for which there exists another prime  $p_3, w < p_3 \le x^{1/2}$ , which divides *m*, and for some prime  $\ell \in \mathcal{Q}$ 

(possibly  $\ell = q$ ) one has  $p_3 \equiv 1 \pmod{\ell}$ , and either  $p_1 \equiv 1 \pmod{\ell}$ , or  $p_2 \equiv 1 \pmod{\ell}$ . Therefore, by (7), we see that

$$T_{2}(x,q) \ll x \sum_{\ell \in \mathcal{Q}} \sum_{\substack{w < p_{1}, p_{2} \le x^{1/2} \\ w < p_{3} \le x^{1/2} \\ p_{1} \equiv p_{2} \equiv 1 \pmod{q} \\ p_{3} \equiv p_{2} \equiv 1 \pmod{q}}} \frac{1}{p_{1}p_{2}p_{3}}$$
$$\ll x \sum_{\ell \in \mathcal{Q}} \sum_{\substack{w < p_{1} \le x^{1/2} \\ p_{1} \equiv 1 \pmod{q}}} \frac{1}{p_{1}} \sum_{\substack{w < p_{2} \le x^{1/2} \\ p_{2} \equiv 1 \pmod{q\ell}}} \frac{1}{p_{2}} \sum_{\substack{w < p_{3} \le x^{1/2} \\ p_{3} \equiv 1 \pmod{\ell}}} \frac{1}{p_{3}}$$
$$\ll x (\log_{2} x)^{3} \sum_{y \le \ell \le z} \frac{1}{q^{2}\ell^{2}} \ll \frac{x (\log_{2} x)^{3}}{q^{2}y \log y} = o\left(\frac{x (\log_{2} x)^{2}}{q^{2}}\right).$$

•  $T_3(x,q)$  is the number of triples  $(p_1, p_2, m)$  as above for which there exists another triple  $(r_1, r_2, k)$  with primes  $w \le r_1 < r_2 \le x^{1/2}$  such that  $r_1 \equiv r_2 \equiv 1$  $(\text{mod } \ell)$  for some  $\ell \in Q$ , and  $p_1 p_2 m = r_1 r_2 k$ . Applying (7) once again, we obtain that

$$T_{3}(x,q) \ll x \sum_{\ell \in \mathcal{Q}} \sum_{\substack{w < p_{1} < p_{2} \le x^{1/2} \\ p_{1} \equiv p_{2} \equiv 1 \pmod{q}}} \frac{1}{p_{1}p_{2}} \sum_{\substack{w < r_{1} < r_{2} \le x^{1/2} \\ r_{1} \equiv r_{2} \equiv 1 \pmod{\ell}}} \frac{1}{r_{1}r_{2}}$$
$$\ll x (\log_{2} x)^{4} \sum_{y \le \ell \le z} \frac{1}{q^{2}\ell^{2}} \ll \frac{x (\log_{2} x)^{4}}{q^{2}y \log y} = o\left(\frac{x (\log_{2} x)^{2}}{q^{2}}\right).$$

Consequently, we have

$$N(x,q) \ge T_0(x,q) - T_1(x,q) - T_2(x,q) - T_3(x,q) \gg \frac{x(\log_2 x)^2}{q^2}.$$

We note that  $P(\xi(n)) \ge q$  for all  $n \in N(x, q)$  and that the sets N(x, q) are disjoint for different choices of  $q \in Q$ . Thus,

$$\begin{split} \sum_{n \leq x} P(\xi(n)) \gg \sum_{q \in \mathcal{Q}} q \, \# N(x,q) \gg x (\log_2 x)^2 \sum_{q \in \mathcal{Q}} \frac{1}{q} \\ \ge x (\log_2 x)^2 \Big( \sum_{y \leq q \leq z} \frac{1}{q} - \frac{D}{6(\log x)^{1/2}} \Big) \\ \gg x (\log_2 x)^2 (\log_2 z - \log_2 y + o(1)) \gg x (\log_2 x)^3. \end{split}$$

$$\sum_{n \le x} P(\xi(n)) \le \sum_{q \le x} q \sum_{\substack{n \le x \\ P(\xi(n)) = q}} 1 \ll x (\log_2 x)^2 \sum_{q \le x} \frac{1}{q} \ll x (\log_2 x)^3.$$

This completes the proof.

Concerning the minimal order of  $P(\xi(n))$ , little need be said; clearly  $P(\xi(n)) \ge 1$  for all  $n \ge 1$ , and equality holds if and only if  $n = 2, 4, p^{\nu}$  or  $2p^{\nu}$  for some odd prime p and  $\nu \ge 1$ . As for the maximal order, we have the following:

**Theorem 4.** The inequality

$$P(\xi(n)) \le \frac{(3n+1)^{1/2} - 2}{6}$$

holds for all  $n \ge 276$ , and the inequality

$$P(\xi(n)) \gg n^{0.3335}$$

holds for infinitely many n.

*Proof.* For *n* in the range  $276 \le n \le 579$ , the upper bound can be verified case by case; hence, we assume that  $n \ge 580$  in what follows. Without loss of generality, we may further assume that  $q = P(\xi(n)) > 3$ , since

$$3 \le \frac{(3n+1)^{1/2}-2}{6}$$
 holds for all  $n \ge 133$ .

If  $P(\xi(n)) = q$ , then either *n* has a prime divisor  $p \equiv 1 \pmod{q}$  and  $q^2 p \mid n$ , or *n* has two distinct prime divisors  $p_1 \equiv p_2 \equiv 1 \pmod{q}$ . In the first case, we see that

$$q < (q^2 p/2)^{1/3} \le (n/2)^{1/3} \le \frac{(3n+1)^{1/2}-2}{6},$$

the last inequality being valid for all  $n \ge 580$ . In the second case, suppose  $p_1 = aq+1$  and  $p_2 = bq + 1$ , where a < b are distinct even integers. Now if 2q + 1 is prime, then 4q + 1 is divisible by 3; thus, we must have  $a \ge 2, b \ge 6$ . Then

$$(2q+1)(6q+1) \le (aq+1)(bq+1) = p_1 p_2 \le n,$$

and we obtain the stated upper bound.

To establish the lower bound, we recall the result of Fouvry [10], which asserts that for all large x, the set Q of primes p in the interval  $x^{1/2} \le p \le x$  and satisfying  $P(p-1) \gg p^{0.667}$  is of cardinality  $\#Q \gg x/\log x$ . We also recall that, by Brun's

method (see Theorem 2.2 in [12]), for any integer *m*, the number of primes of the form  $p = mq + 1 \le x$  for some other prime *q* is

$$O\left(\frac{x}{\varphi(m)(\log(x/m))^2}\right) = O\left(\frac{x}{\varphi(m)(\log x)^2}\right)$$

provided that  $m < x^{1/2}$ . Summing up the above inequalities over all positive integers  $m \le \log_2 x$ , we see that

$$\#\{p \le x : P(p-1) \ge x/\log_2 x\} \ll \frac{x}{\log^2 x} \sum_{m < \log x} \frac{1}{\varphi(m)} \ll \frac{x\log_2 x}{\log^2 x} = o(\mathcal{Q}).$$

Thus, most of the primes p in Q in the interval have  $q = P(p-1) < x/\log_2 x$ , and therefore there exist two primes  $p_1, p_2 \in Q$  with the same value of  $P(p_1 - 1) = P(p_2 - 1) = q$ . With  $n = p_1 p_2$ , we see that  $P(\xi(n)) \ge q \gg \max \{p_1^{0.667}, p_2^{0.667}\} \gg n^{0.3335}$ .

As is clear from the proof, the upper bound of Theorem 4 is tight under the prime *k*-tuplet conjecture of Hardy and Littlewood (see, for example, [3]). We also remark that the trivial upper bound  $P(\xi(n)) \leq n^{1/2}$  holds for all  $n \geq 1$ .

Unfortunately, our method of proof for the lower bound of Theorem 4 can not be combined with the more recent results of [2], since the set of primes considered there is too thin.

**Theorem 5.** The inequalities

$$\Omega(\xi(n)) = (1 + o(1)) \log_2 n \log_4 n \quad and \quad \frac{\log_2 n}{(\log_3 n)^2} \ll \omega(\xi(n)) \ll \log_2 n$$

hold for almost all positive integers n.

*Proof.* We start with  $\Omega(\xi(n))$  and first turn our attention to the upper bound. Let x be a large positive real number, and let  $A_1$  be the set of all positive integers n in the interval  $[x/\log x, x]$ . Clearly,  $A_1$  contains all but o(x) positive integers  $n \le x$ . Let  $A_2$  be the set of those integers  $n \in A_1$  for which  $P(\xi(n)) \le (\log_2 x)^2$ ; by Theorem 1,  $A_2$  contains all but o(x) positive integers  $n \le x$ . Let  $y = (\log_2 x)^2$ . For any positive integer m, we write

$$\omega_{\mathbf{y}}(m) = \sum_{\substack{p < y \\ p \mid m}} 1 \quad \text{and} \quad \Omega_{\mathbf{y}}(m) = \sum_{\substack{p < y \\ p^{\nu} \parallel m}} \nu.$$

Thus, the inequality  $\Omega(\xi(n)) \leq \Omega_y(\varphi(n))$  holds for all  $n \in A_2$ . The argument on page 349 in [8] shows that

$$\sum_{n \le x} \left| \Omega_y(\varphi(n)) - \log_2 x \log_2 y \right|^2 \ll x \log_2 x (\log_2 y)^2.$$
(8)

Now let  $\varepsilon_1(x) = (\log_2 x)^{-1/3}$ , and let  $\mathcal{B}$  be the set of those  $n \le x$  such that

$$\Omega_{\mathbf{y}}(\varphi(n)) > (1 + \varepsilon_1(x)) \log_2 x \log_2 y.$$

Using (8), it follows that

$$#\mathcal{B} \ll \frac{x}{\varepsilon_1(x)^2 \log_2 x} = o(x).$$

The set  $A_3 = A_2 \setminus B$  contains all but o(x) positive integers  $n \leq x$ , and for each  $n \in A_3$  we have

$$\Omega(\xi(n)) \le \Omega_{y}(\varphi(n)) \le (1 + \varepsilon_{1}(x)) \log_{2} x \log_{2} y = (1 + o(1)) \log_{2} x \log_{4} x.$$
(9)

Since  $n \ge x/\log x$  for all  $n \in A_3$ , this shows that

$$\Omega(\xi(n)) \le (1 + o(1)) \log_2 n \log_4 n$$

for almost all positive integers n.

Next we turn to the lower bound for  $\Omega(\xi(n))$ . As before, let x be a large real number, and put  $\varepsilon_2(x) = (\log_3 x)^{-1/3}$  and  $Q = (\log_2 x)^{1/2}$ . For natural numbers n and q, we again write  $\omega(n, q)$  for the number of prime factors p of n that are congruent to 1 modulo q. For a prime  $q \le Q$  we define the sets

$$\mathbb{C}_q = \left\{ n \le x : \omega(n,q) \le (1-\varepsilon_2(x)) \frac{\log_2 x}{\varphi(q)} \right\},\,$$

and

$$\mathfrak{C} = \bigcup_{q \le Q} \mathfrak{C}_q.$$

We claim that #C = o(x) as  $x \to \infty$ . Indeed, for a fixed prime  $q \le Q$ , by a result of Turán [20] (see also (1.2) of [17]), we have

$$\#\mathcal{C}_q \ll \frac{xq}{\varepsilon_2^2(x)\log_2 x} \ll \frac{x(\log_3 x)^{2/3}}{\log_2 x} q.$$

Therefore,

$$\#\mathcal{C} \leq \sum_{q \leq Q} \#\mathcal{C}_q \ll \frac{x(\log_3 x)^{2/3}}{\log_2 x} \sum_{q \leq (\log_2 x)^{1/2}} q \ll \frac{x}{(\log_3 x)^{1/3}} = o(x).$$

Now let  $\mathcal{D}$  be the set of those positive integers  $n \leq x$  not lying in  $\mathcal{C}$ . Then for each

 $n \in \mathcal{D}$ , one has

$$\begin{split} \Omega(\xi(n)) &\geq \sum_{q \leq Q} \left( \omega(n,q) - 1 \right) = \sum_{q \leq Q} \omega(n,q) - \pi(Q) \\ &\geq \left( 1 - \varepsilon_2(x) \right) \log_2 x \sum_{q \leq Q} \frac{1}{\varphi(q)} - \pi(Q) \\ &\geq \left( 1 - \varepsilon_2(x) \right) \log_2 x \sum_{q \leq Q} \frac{1}{q} - \pi(Q) \\ &\geq \left( 1 + o(1) \right) \log_2 x \log_4 x \geq \left( 1 + o(1) \right) \log_2 n \log_4 n. \end{split}$$

This completes the proof of the normal order of  $\Omega(\xi(n))$ .

We now turn our attention to  $\omega(\xi(n))$  and start with the lower bound. Again, let x be a large positive real number, and let  $\varepsilon_3(x)$  be any admissible function. Let q be a prime number and let  $v_q(m)$  denote the largest power of q dividing a natural number m. It suffices to show that there exists a constant  $c_1$  such that for all but o(x) positive integers  $n \leq x$ , the estimate

$$\nu_q(\xi(n)) \ge \varepsilon_3(x) \log_2 x,\tag{10}$$

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holds simultaneously for all primes  $q \le c_1 \log_2 x / \log_3 x$ .

Let us define

$$\mathcal{W}_q = \left\{ n \le x : \omega(n,q) < \frac{\log_2 x}{2\varphi(q)} \right\}.$$

By the result of Turán mentioned above, we have  $\#W_q \ll xq/\log_2 x$ ; summing up these estimates for all  $q \leq (\log_3 x)^{1/2}$ , we see that

$$\sum_{q \le (\log_3 x)^{1/2}} \# W_q \ll \frac{x}{\log_2 x} \sum_{q \le (\log_3 x)^{1/2}} q \ll \frac{x \log_3 x}{\log_2 x \log_4 x} = o(x).$$

We also note that for  $q \leq (\log_3 x)^{1/2}$ , we have

$$\frac{\log_2 x}{2\varphi(q)} \gg \frac{\log_2 x}{(\log_3 x)^{1/2}}$$

which establishes (10) for q in this small range if  $\varepsilon_3(x) \leq (\log_3 x)^{-1/2}$ , which we now assume.

Next we consider the case in which  $q > (\log_3 x)^{1/2}$ . Let us denote by  $\omega_y(n)$  the number of prime factors p of n with  $p \le y$ . Let  $\mathcal{N}$  be the set of integers  $x^{1/2} \le n \le x$  for which

$$\omega_y(n) = \log_2 y + O((\log_2 y)^{2/3})$$

holds simultaneously for  $y = \exp((\log x)^{1/2})$  and for y = x. By [20], we have that  $\#\mathcal{N} = x + o(x)$ .

Let  $\mathscr{E}_q$  be the set of  $n \in \mathcal{N}$  such that  $p^2 \mid n$  for some  $p \equiv 1 \pmod{q}$  and let  $\mathscr{E}$  be the union of all  $\mathscr{E}_q$  for  $q > (\log_3 x)^{1/2}$ . Clearly,

$$\#\mathcal{E}_q \ll \sum_{p \equiv 1 \pmod{q}} \frac{x}{p^2} \le \frac{x}{q^2} \sum_{t \ge 1} \frac{1}{t^2} \ll \frac{x}{q^2},$$

and therefore

$$\#\mathcal{E} \leq \sum_{q > (\log_3 x)^{1/2}} \#\mathcal{E}_q \ll x \sum_{q > (\log_3 x)^{1/2}} \frac{1}{q^2} = o\left(\frac{x}{(\log_3 x)^{1/2}}\right) = o(x).$$

For a fixed positive integer k and primes  $p_1 \equiv \cdots \equiv p_k \equiv 1 \pmod{q}$ , let  $\mathcal{N}_{k,q}(p_1, \ldots, p_k)$  be the set of integers  $n \in \mathcal{N} \setminus \mathcal{E}$  such that  $n = p_1 \ldots p_k m$  holds with some integer m with  $\omega(m, q) = 0$ .

We first show that if  $k \le 0.5 \log_2 x$ , then  $\mathcal{N}_{k,q}(p_1, \ldots, p_k)$  is empty unless

$$\frac{x}{p_1 \dots p_k} \ge z,\tag{11}$$

where  $z = \exp((\log x)^{1/2})$ . Indeed, in the opposite case, we see that for  $n \in \mathcal{N}_{k,q}(p_1, \ldots, p_k)$ ,

$$\omega(n) \le k + \omega(m) \le k + \omega_z(n) \le 0.5 \log_2 x + O\left((\log_2 x)^{1/2}\right),$$

which is impossible because  $\omega(n) \sim \log_2 n \sim \log_2 x$  for  $n \in \mathcal{N}$ .

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We now have

$$\#\mathcal{N}_{k,q}(p_1,\ldots,p_k) \le \sum_{\substack{m \le x/(p_1\ldots p_k) \\ q \not \downarrow \varphi(m)}} 1.$$
(12)

It has been shown in the proof of Theorem 4.1 of [7] that there exists an absolute constant  $c_2 > 0$  such that the upper bound

$$\sum_{\substack{m \leq t \\ \not \mid \varphi(m)}} 1 \ll t \exp\left(-c_2 S(t,q)\right)$$

holds uniformly when  $\log t > q$ , where S(t, q) is given by (2). By Theorem 3.4 of [7], we know that the lower bound

$$S(t,q) \gg \frac{\log_2 t}{q}$$

holds provided that  $q < \log t$ . Thus, assuming (11), and remarking that  $\log z = (\log x)^{1/2} > q$ , we derive from (12) that the estimate

$$\#\mathcal{N}_{k,q}(p_1,\ldots,p_k) \ll \frac{x}{p_1\ldots p_k} \exp\left(-c_3 \frac{\log_2 x}{q}\right)$$

holds with some absolute constant  $c_3 > 0$ .

Therefore, the set  $\mathcal{N}_{k,q}$  consisting of all integers n in  $\mathcal{N} \setminus \mathcal{E}$  that belong to at least one of the sets  $\mathcal{N}_{k,q}(p_1, \ldots, p_k)$ , for fixed k and q, has cardinality at most

$$\#\mathcal{N}_{k,q} = \frac{1}{k!} \sum_{\substack{p_1 < x \\ p_1 \equiv 1 \pmod{q}}} \cdots \sum_{\substack{p_k < x \\ p_k \equiv 1 \pmod{q}}} \#\mathcal{N}_{k,q}(p_1, \dots, p_k)$$

$$\leq \frac{1}{k!} \sum_{\substack{p_1 < x \\ p_1 \equiv 1 \pmod{q}}} \cdots \sum_{\substack{p_k < x \\ p_k \equiv 1 \pmod{q}}} \frac{x}{p_1 \dots p_k} \exp\left(-c_3 \frac{\log_2 x}{q}\right)$$

$$\leq \frac{x}{k!} \exp\left(-c_3 \frac{\log_2 x}{q}\right) S(x, q)^k.$$

Put  $K_q = \varepsilon_3(x)(\log_2 x)/q$ . Recalling the bound (3) and using the Stirling formula, we obtain

$$\sum_{k \le K_q} \#\mathcal{N}_{k,q} \ll x \exp\left(-c_3 \frac{\log_2 x}{q}\right) \sum_{k \le K_q} \frac{\left(2\log_2 x\right)^k}{q^k k!} \\ \ll x \exp\left(-c_3 \frac{\log_2 x}{q}\right) \sum_{k \le K_q} \left(\frac{6\log_2 x}{qk}\right)^k.$$

Furthermore, we derive

$$\sum_{k \le K_q} \left( \frac{6 \log_2 x}{qk} \right)^k \ll \sum_{0 \le i \le \log K_q} \sum_{K_q K_q e^{-i-1} \le k \le K_q e^{-i}} \left( \frac{6e^{i+1} \log_2 x}{qK_q} \right)^k$$
$$= \sum_{0 \le i \le \log K_q} \sum_{K_q K_q e^{-i-1} \le k \le K_q e^{-i}} \left( 6\varepsilon_3^{-1}(x)e^{i+1} \right)^k$$
$$\ll \sum_{0 \le i \le \log K_q} \left( 6\varepsilon_3^{-1}(x)e^{i+1} \right)^{K_q e^{-i}}$$
$$\ll \exp\left( c_4 K_q \log\left(\varepsilon_3^{-1}(x)\right) \right)$$

for some constant  $c_4$ . Therefore, for an appropriate constant  $c_1$ ,

$$\sum_{q \le c_1 \log_2 x/\log_3 x} \sum_{k \le K_q} \# \mathcal{N}_{k,q}$$

$$\ll x \sum_{q \le c_1 \log_2 x/\log_3 x} \exp\left(-c_3 \frac{\log_2 x}{q} + c_4 K_q \log\left(\varepsilon_3^{-1}(x)\right)\right)$$

$$\ll x \sum_{q \le c_1 \log_2 x/\log_3 x} \exp\left(-0.5c_3 \frac{\log_2 x}{q}\right) = o(x)$$

provided that x is large enough. Clearly, the inequality (10) implies the desired lower bound on  $\omega(\xi(n))$ .

We now prove the upper bound on  $\omega(\xi(n))$ . By (1), we know that the inequality

$$\log(\xi(n)) \ll \log_2 n \log_3 n \tag{13}$$

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holds on a set of positive integers 1 of asymptotic density 1. The upper bound on  $\omega(\xi(n))$  claimed by our Theorem 5 follows now from inequality (13) above combined with the classical estimate

$$\omega(\xi(n)) \ll \frac{\log \xi(n)}{\log_2 \xi(n)},$$

which concludes the proof.

It is easy to see that Theorem 5 implies that for some constant  $c_5 > 0$ , the bound

$$\tau(\xi(n)) \ge 2^{\omega(\xi(n))} \gg \exp\left(c_5 \frac{\log_2 n}{(\log_3 n)^2}\right)$$

holds for almost all positive integers *n*, where, as usual,  $\tau(k)$  denotes the number of divisors of an integer  $k \ge 1$ .

It is also clear that for any positive integer n

$$\omega(\xi(n)) \le \omega(\varphi(n)) \ll \frac{\log \varphi(n)}{\log_2 \varphi(n)} \ll \frac{\log n}{\log_2 n}$$

and

$$\Omega(\xi(n)) \ll \Omega(\varphi(n)) \ll \log \varphi(n) \ll \log n.$$

**Theorem 6.** The inequalities

$$\Omega(\xi(n)) \gg \log n \quad and \quad \omega(\xi(n)) \gg \frac{\log n}{\log_2 n}$$

hold for infinitely many positive integers n.

*Proof.* Let *k* be a sufficiently large integer, and then let  $p_1$  and  $p_2$  be the first two primes in the arithmetic progression 1 (mod  $2^k$ ). By Linnik's Theorem, in the form given by Heath-Brown [13], we know that  $\max\{p_1, p_2\} \ll 2^{11k/2}$ , With  $n = p_1 p_2$ , we have that  $2^k | \xi(n)$ ; therefore  $\Omega(\xi(n)) \ge k \gg \log n$ . Finally, let *y* be large and let  $M = \prod_{p < y} p$ . By the Prime Number Theorem, we have  $\log M = (1 + o(1))y$ . Let  $p_1$  and  $p_2$  be the first two primes in the arithmetic progression 1 (mod *M*). We again have that  $\max\{p_1, p_2\} \ll M^{11/2}$ , and with  $n = p_1 p_2$  we have that  $M | \xi(n)$ . Thus,

$$\omega(\xi(n)) \gg \omega(M) = \pi(y) \gg \frac{\log M}{\log_2 M} \gg \frac{\log n}{\log_2 n},$$

which finishes the proof.

## 3. Average q-adic norm and order of $\varphi(n)$

Let q be a prime, and let  $|m|_q$  be the q-adic norm of m, that is,  $|m|_q = q^{-\nu_q(m)}$  where, as before,  $\nu_q(m)$  is the largest power of q dividing m. In this section, we address the average value of  $|\varphi(n)|_q$  and  $\nu_q(\varphi(n))$ .

Recall that an arithmetic function f(n) is said to be *multiplicative* if f(nm) = f(n)f(m) for any integers *n* and *m* with gcd(n, m) = 1. Accordingly, if f(nm) = f(n) + f(m) for any integers *n* and *m* with gcd(n, m) = 1 then f(n) is called *additive*.

In particular,  $v_q(\varphi(n))$  is an additive function. Thus,  $|\varphi(n)|_q$  is a bounded multiplicative function, and therefore it is natural that our principal tool is the following theorem of Wirsing [21].

**Lemma 3.** Assume that a real-valued multiplicative function f(n) satisfies the following conditions:

- $f(n) \ge 0, n = 1, 2, ...;$
- $f(p^{\nu}) \leq ab^{\nu}, \nu = 2, 3, \dots$ , for some constants a, b > 0 with b < 2;
- there exists a constant  $\tau > 0$  such that

$$\sum_{p \le x} f(p) = (\tau + o(1)) \frac{x}{\log x}.$$

Then, for any  $x \ge 0$ ,

$$\sum_{n \le x} f(n) = \left(\frac{1}{e^{\gamma \tau} \Gamma(\tau)} + o(1)\right) \frac{x}{\log x} \prod_{p \le x} \left(\sum_{\nu=0}^{\infty} \frac{f(p^{\nu})}{p^{\nu}}\right),$$

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$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

is the  $\Gamma$ -function.

**Lemma 4.** For any fixed prime q,

$$\prod_{p \le x} \left( 1 + \frac{|p-1|_q}{p-1} \right) = \left( \eta_q + o(1) \right) (\log_2 x)^{\alpha_q},$$

where  $\alpha_q = (q^2 - q - 1)/(q^2 - 1)$ , and  $\eta_q$  is a constant depending only on q.

Proof. We have

$$\log\left(1 + \frac{|p-1|_q}{p-1}\right) = \frac{|p-1|_q}{p} + O\left(\frac{|p-1|_q}{p^2}\right),$$

therefore the series

$$\zeta_q = \sum_p \left| \log \left( 1 + \frac{|p-1|_q}{p-1} \right) - \frac{|p-1|_q}{p} \right|$$

converges absolutely. Hence, it is enough to show that

$$\sum_{p \le x} \frac{|p-1|_q}{p} = \alpha_q \log_2 x + \beta_q + o(1)$$
(14)

holds with some constant  $\beta_q$ .

We have:

$$\sum_{p \le x} \frac{|p-1|_q}{p} = \sum_{k=0}^{\infty} \left( \sum_{\substack{p \le x \\ p \equiv 1 \pmod{q^k}}} \frac{q^{-k}}{p} - \sum_{\substack{p \le x \\ p \equiv 1 \pmod{q^{k+1}}}} \frac{q^{-k}}{p} \right)$$
  
=  $S(x, 1) - (q-1) \sum_{k=1}^{\infty} q^{-k} S(x, q^k),$  (15)

where, as before,  $S(x, q^k)$  is given by (2). We write K for the largest positive integer such that  $q^K \le \log_2 x$ ; thus,  $K \asymp$ log<sub>3</sub> x. Using the classical Page bound (see Chapter 20 of [6]) and partial summation (see a remark in Chapter 22 of [6]), we have

$$\pi(t; q^k, 1) = \frac{t}{(q-1)q^{k-1}\log t} + O\left(\frac{t}{q^k(\log t)^2}\right)$$
(16)

for all positive integers  $k \leq K$  and real  $t \geq e^{K}$ .

Therefore, using the same partial summation arguments as in the proof of Theorem 1 of [18] (see also Lemma 6.3 of [17]), and using (16) in the appropriate place (starting with the value of  $t \ge e^K$ ), we derive that for every  $k \le K$ ,

$$S(x, q^k) = \frac{\log_2 x}{(q-1)q^{k-1}} + A_{k,q} + O\left(\frac{1}{(\log x)^{1/2}}\right),\tag{17}$$

for some constants  $A_{k,q}$  depending only on k and q. Moreover, by Theorem 1 of [18] or Lemma 6.3 of [17],  $A_{k,q} = O(1)$  uniformly for q and k = 0, 1, ... (see (3)).

For  $k \ge K$ , we use the fact that

$$S(x, q^k) \ll \frac{\log_2 x}{(q-1)q^{k-1}}$$
 (18)

(see the bound (3.1) in [7] and also Lemma 1 of [5]). Define

$$\beta_q = A_{k,0} - (q-1) \sum_{k \ge 1} \frac{A_{k,q}}{q^k}.$$

Using (17) and (18) in (15), and taking into account that

$$1-(q-1)\sum_{k\geq 1}\frac{1}{(q-1)q^{2k-1}}=\frac{q^2-q-1}{q^2-1}=\alpha_q,$$

we get (14) and thus finish the proof.

**Theorem 7.** For any prime q,

$$\sum_{n \le x} |\varphi(n)|_q = (\gamma_q + o(1)) x (\log x)^{-q/(q^2 - 1)}$$

where  $\gamma_q$  is a constant depending only on q.

*Proof.* For  $p \neq q$ , we have

$$\sum_{\nu=0}^{\infty} \frac{|\varphi(p^{\nu})|_q}{p^{\nu}} = 1 + \sum_{\nu=1}^{\infty} \frac{|p-1|_q}{p^{\nu}} = \frac{|p-1|_q}{p-1},$$

and certainly

$$\sum_{\nu=0}^{\infty} \frac{|\varphi(q^{\nu})|_q}{q^{\nu}} = 1 + \sum_{\nu=1}^{\infty} \frac{1}{q^{2\nu-1}} = 1 + \frac{q}{q^2-1} = \frac{q^2+q-1}{q^2-1}.$$

Combining Lemma 3 and Lemma 4, we obtain the desired result.

We now show that the classical *Turán–Kubilius* inequality can be used to study the normal order of  $v_q(\varphi(n))$ .

**Theorem 8.** For any prime q, the estimate

$$v_q(\varphi(n)) = \left(\frac{q}{(q-1)^2} + o(1)\right)\log_2 n$$

holds for almost all positive integers n.

*Proof.* Because  $v_q(\varphi(n))$  is an additive function, by the Turán–Kubilius inequality (see [14], [19]), we have

$$\frac{1}{x}\sum_{n\leq x}\left|\nu_q(\varphi(n))-A_q(x)\right|^2\ll D_q(x)$$

where

$$A_q(x) = \sum_{p^r \le x} \frac{\nu_q(\varphi(p^r))}{p^r} \quad \text{and} \quad D_q(x) = \sum_{p^r \le x} \frac{\nu_q^2(\varphi(p^r))}{p^r}$$

and in both sums the summation is extended over all prime powers  $p^r \le x$ . Thus, it is enough to show that

$$A_q(x) = \left(\frac{q}{(q-1)^2} + o(1)\right)\log_2 x \quad \text{and} \quad D(x) = o((\log_2 x)^2).$$
(19)

Because  $v_q(\varphi(p)) \ll \log p$ , using the Prime Number Theorem, we derive that

$$\sum_{\substack{p^r \le x \\ r \ge 2}} \frac{\nu_q(\varphi(p))}{p^r} \ll \sum_{r=2}^x \sum_{k=2}^\infty \frac{\log k}{(0.5k \log k)^r} \ll \sum_{r=2}^x \sum_{k=2}^\infty \frac{1}{k^r} \ll \sum_{r=2}^x 2^{-r} \ll 1.$$

Thus

$$A_{q}(x) = \sum_{p \le x} \frac{v_{q}(\varphi(p))}{p} + O(1) = \sum_{\substack{p \le x \\ p \ne q}} \frac{v_{q}(\varphi(p))}{p} + O(1).$$

Furthermore, as in the proof of Lemma 4, we derive that

$$\sum_{\substack{p \le x \\ p \ne q}} \frac{\nu_q(\varphi(p))}{p} = \sum_{k=1}^{\infty} \left( \sum_{\substack{p \le x \\ p \equiv 1 \pmod{q^k}}} \frac{k}{p} - \sum_{\substack{p \le x \\ (\text{mod } q^{k+1})}} \frac{k}{p} \right)$$
$$= \sum_{k=1}^{\infty} S(x, q^k) = \left( \frac{q}{(q-1)^2} + o(1) \right) \log_2 x.$$

Similar arguments show that  $D_q(x) = O(\log_2 x)$  (in fact, our arguments give an asymptotic formula for  $D_q(x)$ ). Therefore, we obtain (19), which finishes the proof.

#### 4. Distribution of $\Omega(\varphi(n)) - \omega(\varphi(n))$

It has been shown in [8] that for almost all positive integers *n*, both  $\Omega(\varphi(n))$  and  $\omega(\varphi(n))$  are close to  $0.5(\log_2 n)^2$ . Here, we study the behavior of the difference  $\Omega(\varphi(n)) - \omega(\varphi(n))$ .

Theorem 9. The estimate

 $\Omega(\varphi(n)) - \omega(\varphi(n)) = (1 + o(1)) \log_2 n \log_4 n$ 

holds for almost all positive integers n.

*Proof.* By Theorem 5, we know that

$$\Omega(\xi(n)) = (1 + o(1)) \log_2 n \log_4 n$$

holds for almost all positive integers n. Since

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = \Omega(\varphi(n)) - \omega(\lambda(n)) \ge \Omega(\varphi(n)) - \Omega(\lambda(n)) \ge \Omega(\xi(n)),$$

we see that

$$\Omega(\varphi(n)) - \omega(\varphi(n)) \ge (1 + o(1)) \log_2 n \log_4 n$$

holds for almost all positive integers n.

To obtain the upper bound, let x be a large positive real number, and let  $y = (\log_2 x)^2$ . The argument on page 404 of [16] shows that the set of all positive integers  $n \le x$  such that  $\varphi(n)$  is not divisible by the square of any prime q > y has cardinality x + o(x) (see the bound on  $\#\mathcal{E}_2$  in Theorem 9 of [16]). Thus, for all but o(x) positive integers  $n \le x$ , we have that

$$\Omega(\varphi(n)) - \omega(\varphi(n)) = \Omega_{\nu}(\varphi(n)) - \omega_{\nu}(\varphi(n)) \le \Omega_{\nu}(\varphi(n)).$$

Now using (9) (which is established with the same value of y), we finish the proof.

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