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Delzant's T-invariant, Kolmogorov complexity and one-relator groups

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Abstract. We prove that for "random" one-relator groups the Delzant T-invariant (which measures the smallest size of a finite presentation of a group) is comparable in magnitude with the length of the defining relator. The proof relies on our previous results regarding isomorphism rigidity of generic one-relator groups and on the methods of the theory of Kolmogorov–Chaitin complexity.

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1. Introduction

Delzant [14] introduced an extremely interesting but still rather enigmatic invariant for finitely presentable groups. For any finite presentation $\Pi = \langle X \mid R \rangle$ define *the length* $\ell(\Pi)$ as

$$\ell(\Pi) := \sum_{r \in R} \max\{|r| - 2, 0\}.$$

If G is a finitely presentable group, the *T-invariant* $T(G)$ of G, which we also call the *presentation rank of* G, is defined [14] as

$$T(G) := \min\{\ell(\Pi) \mid \Pi \text{ is a finite presentation of the group } G\}.$$

The T-invariant plays a central role in Delzant and Potyagailo's proof of the strong accessibility (or "hierarchical decomposition") theorem for finitely presented groups [16]. This theorem is the strongest and most difficult of numerous accessibility results ([17], [18], [5], [6], [40], [15], [42], [31]). One can also define a closely related notion, the *non-reduced* T*-invariant* $T_1(G)$, as the minimum of sums of lengths of the

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defining relators, taken over all finite presentations of G. As we observe in Lemma 5.2 below, if G is a finitely presentable group without elements of order two then

$$T(G) \leq T_1(G) \leq 3T(G).$$

The non-reduced T-invariant has been studied in the context of 3-manifolds, where it turns out to be related to the notion of Matveev complexity. We refer the reader to a recent paper of Pervova and Petronio [37] for a discussion on this subject.

If G is a finitely generated group then the ordinary *rank*, $\mathrm{rk}(G)$, of G is the smallest cardinality of a finite generating set for G. The first (and already quite nontrivial) accessibility result is Grushko's theorem [26] which asserts that for finitely generated groups G_1 and G_2 we have $\mathrm{rk}(G_1 * G_2) = \mathrm{rk}(G_1) + \mathrm{rk}(G_2)$. In [14] Delzant proved a similar theorem for the presentation rank, namely that

$$T(G_1 * G_2) = T(G_1) + T(G_2)$$

if G_1, G_2 are finitely presentable groups.

The hierarchical decomposition theorem proved in [16] implies, for example, that an iterated process of JSJ-decomposition (in any sense of the word), see [41], [38], [19], [21], [7], applied to a finitely presented group, then to the factors of its JSJ-decomposition, and so on, always terminates. The T-invariant is also crucial in Delzant's generalization [15] of Sela's acylindrical accessibility result [40] for finitely presented groups.

If Π is a finite presentation, let $G(\Pi)$ be the group defined by Π. We can regard T as a function defined over finite presentations by setting $T(\Pi) = T(G(\Pi))$. If G is given by a particular finite presentation Π then $\ell(\Pi)$ gives an obvious upper bound for $T(G(\Pi))$. However, it is very unclear in general how to estimate $T(G)$ from below. For example, if $\Pi = \langle X \,|\, R\rangle$ and $\alpha \in \mathrm{Aut}(F(X))$ then the presentations Π and $\Pi' = \langle X \,|\, \alpha(R)\rangle$ define isomorphic groups but it is easy to produce examples where $\ell(\Pi')$ is arbitrarily smaller than $\ell(\Pi)$.

We prove however that for "most" one-relator presentations this does not happen and that the value of Delzant's T-invariant is comparable in magnitude with the length of the defining relator. If $r \in F(a_1, \ldots, a_k)$, let $G_r := \langle a_1, \ldots, a_k \,|\, r\rangle$ be the one-relator group whose defining relator is r. Our main result is:

Theorem A. *Fix an integer $k > 1$ and let $F = F(a_1, \ldots, a_k)$. For any number $0 < \varepsilon < 1$ there is an integer $n_1 > 0$ and a constant $M = M(k, \varepsilon) > 0$ with the following property.*

Let J be the set of all nontrivial cyclically reduced words r such that

$$T(G_r) \log_2 T(G_r) \geq M|r|.$$

Then for any $n \geq n_1$

$$\frac{\#\{r \in J : |r| = n\}}{\#\{r \in F : r \text{ is cyclically reduced and } |r| = n\}} \geq 1 - \varepsilon.$$

Thus for any fixed $0 < \varepsilon, \delta < 1$ we asymptotically have $T(G_r) \geq c|r|^{1-\delta}$, where c is a constant, for at least the fraction $(1 - \varepsilon)$ of all cyclically reduced words r of a given length. This says that the description of a one-relator group by a generic relator r is "essentially incompressible". In view of the above remarks about the connection between $T(G)$ and $T_1(G)$, the same conclusion as in Theorem A also holds for $T_1(G_r)$.

This is a good place to observe that the function T is not computable.

Observation 1.1. *The function T, as a function over finite presentations, is not a computable function.*

Proof. We say that a finitely generated group G is *essentially free* if G is the free product of a finitely generated free group and finitely many cyclic groups of order two. The only defining relators in the "standard presentation" Π_0 of such a group are the squares of those generators which have order two and so $\ell(\Pi_0) = 0$.

It is easy to use Tietze transformations to show that any group G having a finite presentation in which all relators have length at most two is essentially free. Hence, by the definition of $T(G)$, a finitely presentable group G has $T(G) = 0$ if and only if G is essentially free.

Recall that a property $\mathcal{P}$ of finitely presented groups is a *Markov property* if $\mathcal{P}$ is independent of presentation, there are finitely presented groups with $\mathcal{P}$ and there is a finitely presented group G_* such that G_* cannot be embedded in any finitely presented group with $\mathcal{P}$. Being essentially free is clearly a Markov property. We can take G_* to be the cyclic group of order three. The classic Adian–Rabin Theorem [32] says that if $\mathcal{P}$ is any Markov property then there is no algorithm over all finite presentations which, when given a finite presentation Π, decides whether or not the group $G(\Pi)$ has $\mathcal{P}$.

If the function T were computable then, for any finite presentation Π, we could decide the essential freeness of $G(\Pi)$ by computing $T(\Pi)$. Hence T cannot be computable. □

The proof of Theorem A involves several different probabilistic tools. The idea introduced in this paper is the use of Kolmogorov complexity, a concept that plays an important role in coding theory, algorithmic probability and complexity theory. This notion is also sometimes known as "Kolmogorov–Chaitin complexity" because of the contributions of Chaitin to the subject. Roughly speaking, the Kolmogorov complexity of a word is the size of the smallest computer program (in a fixed programming language) that can compute the given word. Surprisingly, the only previous use of Kolmogorov complexity in group theory known to us is a 1985 paper of Grigorchuk [22], giving an interesting application of Kolmogorov complexity to algorithmic problems in group theory.

Our results here also depend on [29] and [30] where we obtained a number of results regarding a very strong Mostow-type "isomorphism rigidity" for generic one-relator groups. These results use a combination of the Arzhantseva–Ol'shanskii minimization technique and their ingenious "non-readability" small cancellation condition [29] and Large Deviation Theory [30] to study the behavior of random words under an arbitrary automorphism of the ambient free group. The isomorphism rigidity theorems proved in [30] allow us, given any finite presentation $\Pi = \langle X \mid R \rangle$ defining a group isomorphic to $G_r = \langle a_1, \ldots, a_k \mid r \rangle$ (where $k > 1$ is fixed) for a generic relator r plus a small initial segment u of r, to algorithmically recover the word r. This implies that r is uniquely algorithmically determined by an amount $O(\ell(\Pi) \log \ell(\Pi))$ of information. (The logarithmic term comes from the fact that the subscripts in the enumeration of letters in X also need to be encoded.) From here one can deduce that the Kolmogorov complexity of r is $\leq O(\ell(\Pi) \log \ell(\Pi))$. On the other hand, using the methods of algorithmic probability, in particular the notion of *prefix complexity*, we can deduce that a cyclically reduced word r of a given length has Kolmogorov complexity $\geq c|r|$ asymptotically with probability $\geq 1 - \varepsilon$. These inequalities taken together yield the conclusion of Theorem A.

We believe that the general analogue of Theorem A is true. This would say that if we fix a number $k \geq 2$ of generators and any number $m \geq 1$ of defining relators, then a generic k-generator m-relator presentation should essentially be the shortest description of the group defined. We have seen that the proof of Theorem A relies on two components: the Kolmogorov complexity arguments used in this paper and the isomorphism rigidity results for random one-relator groups established in [29], [30]. Now most of the arguments and statements of [29], [30] needed to prove isomorphism rigidity actually go through for generic groups with an arbitrary fixed number of relators and we believe that "generic groups are rigid" in general. However, to actually infer rigidity, at the end of the proof we use a crucial fact about one-relator groups. Namely, we need the classical theorem of Magnus (see, for example, [32]) which says that if two elements r and s have the same normal closures in a free group F then r is conjugate in F to s or s^{-1}. This statement does not hold in general for tuples consisting of more than one element of F. However, we believe that the desired analogue does hold generically.

If $\tau = (u_1, \ldots, u_m)$ is an m-tuple of elements of the free group F_k, the *symmetrized set* $R(\tau)$ *generated by* τ consists of all the cyclic permutations of cyclically reduced forms of $u_i^{\pm 1}$.

Conjecture 1.2 (Stability Conjecture). *Fix $k \geq 2$ and $m \geq 1$. Let $F = F(a_1, \ldots, a_k)$. Then there exists an algorithmically recognizable generic class $\mathcal{C}$ of m-tuples of elements of F with the following property. If $\sigma, \tau \in \mathcal{C}$ and $\alpha \in \mathrm{Aut}(F)$ are such that $R(\sigma)$ and $R(\alpha(\tau))$ have the same normal closure in F then $R(\sigma) = R(\alpha(\tau))$.*

Magnus' theorem implies that the Stability Conjecture holds for $m = 1$ with $\mathcal{C} = F_k$. If one could establish the Stability Conjecture, then both the isomorphism rigidity results of [30] and the results of this paper would then follow for finitely presented groups with any fixed numbers of generators and relators exactly as in the one-relator case.

In [30] we showed that for a fixed $k \geq 2$ the number I_n of *isomorphism types* of k-generator one-relator groups with cyclically reduced defining relators of length n satisfies

$$\frac{c_1(2k-1)^n}{n} \leq I_n \leq \frac{c_2(2k-1)^n}{n},$$

where $c_1 = c_1(k) > 0, c_2 = c_2(k) > 0$ are some constants independent of n. Using auxiliary results from the proof of Theorem A we obtain an improvement of this estimate in the present paper and compute the precise asymptotics of I_n:

Theorem B. *Let $k \geq 2$ be a fixed integer. Then the number I_n of isomorphism types of k-generator one-relator groups with cyclically reduced defining relators of length n satisfies:*

$$I_n \sim \frac{(2k-1)^n}{nk!2^{k+1}}.$$

Here $f(n) \sim g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

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2. Kolmogorov complexity

The T-invariant is a measure of "smallest descriptive complexity" in the framework of finite presentations of groups while Kolmogorov complexity is a general theory of "minimal descriptive complexity". We provide here only a brief discussion of the relevant facts regarding Kolmogorov complexity and refer the reader to the survey of Fortnow [20] for an overview and to the excellent and comprehensive book of Li and Vitanyi [34] for detailed background information.

Intuitively speaking, the Kolmogorov complexity $C(x)$ of a finite binary string x is the size of the smallest computer program M that can compute x. In order for this notion to make sense one needs to first fix a "programming language" but it turns out that all reasonable choices yield measures which are equivalent up to an additive constant.

Note that $C(x)$, as a measure of *descriptive* complexity of x, totally disregards how long the particular program M will have to run in order to compute x. Some

strings clearly admit much shorter descriptions then their length. For example, if x_0 is the binary representation of the number $2^{2^{2^{10}}}$ then the length of x_0 is huge, namely $1 + 2^{2^{10}}$. Yet we were just able to give a very short unambiguous description of x_0. Thus x_0 has small Kolmogorov complexity and $C(x_0) \ll |x_0|$. On the other hand it is intuitively clear that for a "random" string x of large length, the shortest description of x is essentially x itself. In this case $C(x) \approx |x|$. This phenomenon is called "incompressibility" and plays an important role in complexity theory for establishing lower complexity bounds of various algorithms.

Recall that any Turing machine M on the set of finite binary strings $\{0, 1\}^*$ computes a partial recursive function $\{0, 1\}^* \to \{0, 1\}^*$ and, moreover, every partial recursive function $\{0, 1\}^* \to \{0, 1\}^*$ arises in this fashion.

Once one has fixed the formalism of Turing machines, one can identify a Turing machine with its sequence of instructions and think of Turing machines as programs. A Turing machine M can then itself be coded as a binary string according to some fixed effective method and we write $\langle M \rangle$ for the code of the machine M. The pair consisting of a Turing machine M and an input w can then be given the code $\langle M \rangle w$. A basic feature of the theory of computability is the existence of a *universal* Turing machine U, which, if its input is a code $\langle M \rangle w$, simulates M on input w. To be more precise, a Turing machine U is *universal* if for any Turing machine M there is a binary string $\langle M \rangle$ such that for any string $w \in \{0, 1\}^*$ the machine U produces the same result on input $\langle M \rangle w$ as M does on w.

Definition 2.1. Fix a universal Turing machine U with the alphabet $\Sigma := \{0, 1\}$. Then U computes a universal partial recursive function ϕ from Σ^* to Σ^*. That is, for any partial recursive function ψ there is a string $z \in \Sigma^*$ such that for all $x \in \Sigma^*$, $\phi(zx) = \psi(x)$.

For a finite binary string $x \in \Sigma^*$ we define the *Kolmogorov complexity* $C(x)$ as

$$C(x) := \min\{|p| : p \in \Sigma^*, \phi(p) = x\}.$$

Kolmogorov complexity is traditionally defined for finite binary strings. However, if $s > 1$ is a fixed integer, then all of the standard definitions and theorems go through essentially unchanged if one considers finite strings x in a fixed s-letter alphabet A. This can be done in either of two essentially equivalent ways. First, one can modify Definition 2.1 by choosing U to be a universal Turing machine with the alphabet A_s computing a universal partial recursive function from A_s^* to A_s^*. Alternatively, one can fix a recursive bijection $h\colon A_s^* \to \Sigma^*$ and define $C_s(x)$, where $x \in A_s^*$ to be $C(h(x))$. We choose the latter option since most theorems in [34] are stated for binary strings and we want to be able to cite the results of [34] verbatim.

Definition 2.2. Let $s > 1$ be an integer and let A_s be an alphabet with s letters. Fix a recursive bijection $h\colon A_s^* \to \{0, 1\}^*$.

For any string $x \in A_s^*$ define its *Kolmogorov complexity* $C_s(x)$ as

$$C_s(x) := C(h(x)).$$

Kolmogorov complexity lacks some mathematical properties which are essential for certain arguments. Fortunately, this difficulty can be overcome by using the closely related notion of *prefix complexity*. For a detailed discussion of this notion we refer the reader to Chapters 2 and 3 of [34]. In the present paper we need only cite a few basic facts regarding prefix complexity from [34]. A partial recursive function ϕ on Σ^* is called a *prefix function* if whenever $\phi(x)$ is defined and x is a proper initial segment of y, then $\phi(y)$ is undefined. There is a corresponding notion of a *prefix machine*. Informally speaking, a prefix machine does not require an "end-of-tape" symbol for the input word and decides whether or not to halt only based on its current state and before scanning the next letter of the input. The machine starts working on an infinite input word and, after performing a computational step on the working and output tapes, the machine either moves one letter to the right on the input tape or halts and terminates its work.

Just as with ordinary Turing machines, there exist universal prefix machines computing universal prefix partial recursive functions (see Theorem 3.1.1 in [34]).

Definition 2.3. Fix a universal prefix Turing machine U' with the alphabet $\Sigma = \{0, 1\}$. Then U' computes a universal prefix partial recursive function ψ from Σ^* to Σ^*.

For a finite binary string $x \in \Sigma^*$ we define the *prefix complexity* $K(x)$ as

$$K(x) := \min\{|p| : p \in \Sigma^*, \psi(p) = x\}.$$

Similarly to the case of Kolmogorov complexity, prefix complexity can be defined not only for binary but also for s-ary strings.

Definition 2.4. Let $s > 1$ be an integer and let A_s be an alphabet with s letters. Fix the same recursive bijection $h\colon A_s^* \to \{0, 1\}^*$ as in Definition 2.2.

For any string $x \in A_s^*$ define its *prefix complexity* $K_s(x)$ as

$$K_s(x) := K(h(x)).$$

For our purposes, the crucial way in which prefix complexity is better than Kolmogorov complexity is that $\sum_{x\in\{0,1\}^*} 2^{-K(x)} \leq 1$ while $\sum_{x\in\{0,1\}^*} 2^{-C(x)}$ diverges.

We list here some relevant properties of Kolmogorov and prefix complexity.

Proposition 2.5. *Let $s > 1$ be a fixed integer and let A_s be an s-letter alphabet. Then the following holds.*

(1) *We have*

$$\sum_{x\in\{0,1\}^*} 2^{-K(x)} \leq 1.$$

(2) *Up to additive constants, for any* $x \in \{0,1\}^*$ *we have*

$$C(x) \leq K(x) \leq C(x) + \log_2 C(x).$$

(3) *We have*

$$\sum_{x\in A_s^*} 2^{-K_s(x)} \leq 1.$$

(4) *Up to additive constants for any* $x \in A_s^*$ *we have*

$$C_s(x) \leq K_s(x) \leq C_s(x) + \log_2 C_s(x).$$

Proof. Part (1), as observed by Levin [33], is a direct corollary of Kraft's Inequality, which is ubiquitous in information theory (see also 4.2.2(b) in [34]). Part (2) is statement 3.1.3 in [34]. Clearly, (1) implies (3) and, also, (2) implies (4). Since part (1) is quite important for our purposes, we provide a proof here.

A subset $S \subseteq \{0,1\}^*$ is *prefix-free* if whenever $p, q \in S$, $p \neq q$ then p is not an initial segment of q. Recall that by definition $K(x)$ is the shortest length of a prefix program p with $\psi(p) = x$. Thus the set S of such all such p corresponding to $x \in \{0,1\}^*$ is prefix-free. If p is a binary string, then $2^{-|p|}$ is the Lebesgue measure of the subset S_p of the unit interval $I = [0,1]$ consisting of those numbers whose binary expansion begins with p. Since S is prefix-free, subsets S_p and S_q are disjoint for $p \neq q$. The inequality thus follows from the countable additivity of Lebesgue measure. □

We also recall the classical Markov inequality from probability theory which can be found in most probability textbooks (see, for example, Lemma 1.7.1 in [39]):

Lemma 2.6 (Markov Inequality). *Let* $X : \Omega \to \mathbb{R}$ *be a nonnegative random variable on a sample probability space* Ω *with the expected value* $E(X) > 0$. *Then for any* $\delta > 0$ *we have*

$$P\big(X \geq \delta\big) \leq \frac{E(X)}{\delta}.$$

Lemma 2.7. *Let* $s > 1$ *be a fixed integer and let* A_s *be an* s*-letter alphabet. Let* $\Omega \subset A_s^*$ *be a nonempty subset equipped with a discrete non-vanishing probability measure* Π, *so that* $\sum_{x\in\Omega} P(\{x\}) = 1$. *Denote* $\mu(x) := P(\{x\})$ *for any* $x \in \Omega$.

Then for any $\delta > 0$ *we have*

$$P\big(K_s(x) \geq -\log_2 \mu(x) - \log_2 \delta\big) = P\big(2^{-K_s(x)} \leq \delta\mu(x)\big) \geq 1 - \frac{1}{\delta}.$$

Proof. Consider the function $X\colon \Omega \to \mathbb{R}$ defined by $X(x) = \frac{2^{-K_s(x)}}{\mu(x)}$.

The Π-expected value of X is

$$E(X) = \sum_{x\in\Omega} \mu(x)\frac{2^{-K_s(x)}}{\mu(x)} \leq \sum_{x\in\Omega} 2^{-K_s(x)} \leq \sum_{x\in A_s^*} 2^{-K_s(x)} \leq 1,$$

where the last inequality holds by Proposition 2.5.

Therefore by Markov's inequality

$$P\left(\frac{2^{-K_s(x)}}{\mu(x)} \geq \delta\right) \leq \frac{E(X)}{\delta} \leq \frac{1}{\delta},$$

and so

$$P\left(\frac{2^{-K_s(x)}}{\mu(x)} \leq \delta\right) \geq P\left(\frac{2^{-K_s(x)}}{\mu(x)} < \delta\right) \geq 1 - \frac{1}{\delta},$$

as required. □

3. Kolmogorov complexity and freely reduced words

Convention 3.1. Let $k > 1$ and let $F = F(a_1, \ldots, a_k)$. Put

$$A_{2k} := \{a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}\}.$$

As usual we identify F with the set of all freely reduced words in A_{2k}^*. Thus if $g \in F$ then $|g|$ is the length of the unique freely reduced word representing g. For a subset $S \subseteq F$ denote by $\gamma(n, S)$ the number of all $x \in S$ with $|x| = n$. Similarly, denote by $\rho(n, S)$ the number of all $x \in S$ such that $|x| \leq n$. Note that $\gamma(n, F) = 2k(2k-1)^{n-1}$ for $n \geq 1$. Denote by CR the set of all cyclically reduced words in A_{2k}^*. Thus CR $\subseteq F$. These notations will be fixed for the remainder of the paper, unless specified otherwise.

The next result is is easy to see.

Lemma 3.2 ([30]). *For any $n \geq 1$ we have*

$$(2k-1)^n \leq \gamma(n, \mathrm{CR}) \leq 2k(2k-1)^n.$$

Moreover, in Proposition 5.8 below we will see an explicit formula for $\gamma(n, \mathrm{CR})$, which we do not need for the moment.

Proposition 3.3. *Let $c \geq 1$. Denote by Z the set of all cyclically reduced words x such that*

$$C_{2k}(x) \geq -\frac{c}{2} + |x|\frac{\log_2(2k-1)}{2}.$$

Then there is $n_0 > 1$ such that for any $n \geq n_0$ we have

$$\frac{\gamma(n, Z)}{\gamma(n, \mathrm{CR})} \geq 1 - \frac{1}{2^c}.$$

Proof. Let $n > 0$ be an integer and let $\mathcal{W}_n$ be the set of all cyclically reduced words of length n with the uniform discrete probability measure P. As in Lemma 2.7 denote $\mu(x) := P(\{x\})$ for any $x \in \mathcal{W}_n$. Then by Lemma 3.2 for any $x \in \mathcal{W}_n$ we have

$$\frac{1}{2k}(2k-1)^{-n} \leq P(\{x\}) = \mu(x) = \frac{1}{\gamma(n, \mathrm{CR})} \leq (2k-1)^{-n}.$$

We apply Lemma 2.7 with $\delta = 2^c$. Hence

$$\begin{aligned} 1 - \frac{1}{2^c} &\leq P\big(2^{-K_{2k}(x)} \leq 2^c \mu(x)\big) \\ &\leq P\big(2^{-K_{2k}(x)} \leq 2^c (2k-1)^{-n}\big) \\ &= P\big(-K_{2k}(x) \leq c - n\log_2(2k-1)\big) \\ &= P\big(K_{2k}(x) \geq -c + n\log_2(2k-1)\big). \end{aligned}$$

Recall that by Proposition 2.5

$$K_{2k}(x) \leq C_{2k}(x) + \log_2 C_{2k}(x) + c_0$$

where c_0 is some fixed constant. There is $n_0 > 1$ such that for any word $x \in A_{2k}^*$ of length $n \geq n_0$ we have

$$K_{2k}(x) \leq 2C_{2k}(x).$$

Therefore if $n \geq n_0$ then

$$\begin{aligned} 1 - \frac{1}{2^c} &\leq P\big(K_{2k}(x) \geq -c + n\log_2(2k-1)\big) \\ &\leq P\big(C_{2k}(x) + \log_2 C_{2k}(x) + c_0 \geq -c + n\log_2(2k-1)\big) \\ &\leq P\big(2C_{2k}(x) \geq -c + n\log_2(2k-1)\big), \end{aligned}$$

as required. □

4. Genericity in free groups

If $b_n, b \in \mathbb{R}$ and $\lim_{n\to\infty} b_n = b$, we say that the convergence is *exponentially fast* if there exist $C > 0$ and σ with $0 < \sigma < 1$ such that for all n we have

$$|b_n - b| \leq C\sigma^n.$$

Definition 4.1. Let $S \subseteq Q \subseteq F$. We say that S is *Q-generic* if

$$\lim_{n\to\infty} \frac{\rho(n, S)}{\rho(n, Q)} = 1.$$

If in addition the convergence in the above limit is exponentially fast, we say that S is *exponentially Q-generic*.

Similarly, S is called *(exponentially) Q-negligible* if $Q - S$ is (exponentially) Q-generic.

Note that the union of two (exponentially) Q-negligible sets is (exponentially) Q-negligible and the intersection of two (exponentially) Q-generic sets is (exponentially) Q-generic.

Proposition 4.2 ([30]). *The following hold:*

(1) *A subset $S \subseteq F$ is exponentially F-negligible if and only if*

$$\lim_{n\to\infty} \frac{\gamma(n, S)}{(2k-1)^n} = 0$$

with exponentially fast convergence.

(2) *A subset $S \subseteq \mathrm{CR}$ is exponentially* CR*-negligible if and only if*

$$\lim_{n\to\infty} \frac{\gamma(n, S)}{(2k-1)^n} = 0$$

with exponentially fast convergence.

(3) *A subset $Q \subseteq \mathrm{CR}$ is exponentially* CR*-generic if and only if*

$$\lim_{n\to\infty} \frac{\gamma(n, Q)}{\gamma(n, \mathrm{CR})} = 1$$

with exponentially fast convergence.

Definition 4.3. An automorphism $\tau : F \to F$ is called a *relabeling* automorphism if the restriction $\tau|_{A_{2k}}$ is a permutation of A_{2k}.

Convention 4.4. For the remainder of the paper we adopt the following convention. If $r \geq 0$ is a real number, by saying that w is a word of length r we will mean that w is a word of length $\lfloor r \rfloor$.

Lemma 4.5. *Let $0 < \lambda < 1/3$. Let τ be a nontrivial relabeling automorphism of F.*

Define $S(\lambda, \tau)$ as the set of all cyclically reduced words x such that x and some cyclic permutation of $\tau(x)$ have a common initial segment of length $\geq \lambda|x|$.

Then $S(\lambda, \tau)$ is exponentially CR*-negligible.*

Proof. Suppose $x \in S(\lambda, \tau)$ and $|x| = n > 1$. Then there exist an initial segment u of x with $|u| = \lambda n$ and a cyclic permutation v taking $\tau(x)$ to x' such that u is also an initial segment of x'.

Case 1. Suppose first that v is a trivial cyclic permutation. Then u is an initial segment of $\tau(x)$ and $u = \tau(u)$. Since τ is a relabeling automorphism, this implies that there is some letter $a \in A_{2k}$ such that $a^{\pm 1}$ does not occur in u. Then the number of possibilities for u is at most $2k(2k-3)^{\lambda n - 1}$ and the number of possibilities for v is at most $2k(2k-1)^{(1-\lambda)n-1}$. Hence the number of all such u is at most

$$\frac{4k^2}{(2k-1)(2k-3)}(2k-3)^{\lambda n}(2k-1)^{(1-\lambda)n},$$

which is exponentially smaller than $(2k-1)^n$.

Case 2. Suppose now that v is a nontrivial cyclic permutation, so that v has "translation length" $l \neq 0 (\mod n)$, $1 \leq l \leq n-1$. Thus $\tau(x) = y_1 y_2$, $x' = y_2 y_1$ and $|y_2| = l$ and $|y_1| = n - l$.

The idea is that there are at least $\lambda n/6$ letters of x for which there is no choice and which are predetermined by the rest of x. Hence the number of possibilities for x is exponentially smaller than $(2k-1)^n$. There are basically two cases: when the overlap between the positions of u in x and in $\tau(x)$ is small (that is both l and $-l$ are large $\mod n$) and when the overlap is large (that is one of l, $-l$ is small $\mod n$).

Subcase 2.A. Assume first that $l, n - l \geq |u|/6 = \lambda n/6$, so that the overlap between the positions of u in x and $\tau(x)$ has length at most $|u|/6$.

Then $y_2 = u y_2'$ and $\tau(x) = y_1 u y_2'$ where $|y_1| \geq |u|$. Hence $x = uv = uv_1\tau^{-1}(u)v_2$ where $|uv_1| = |y_1|$. We see that in this case the segment $u' = \tau(u)$ of x of length $\lambda n/6$ occurring in the same position in x as u does in $\tau(x)$ is uniquely determined (for a fixed l) by the rest of the word x. The number of choices for l is at most n. Given l the number of choices for (uv_1, v_2) is at most $\frac{(2k)^2}{(2k-1)^2}(2k-1)^{n-\lambda n/6}$. Hence the number of possibilities for such u is at most

$$n\frac{(2k)^2}{(2k-1)^2}(2k-1)^{n(1-\lambda/6)},$$

which is exponentially smaller than $(2k-1)^n$.

Subcase 2.B. Suppose now that $0 < l < |u|/6$ or $0 < n - l < |u|/6$.

We will assume that $0 < l < |u|$ as the other case is similar. Thus $x = uv$ and $\tau(x) = y_1 u y_2$ with $|y_1| = l$. So the positions in which u occurs in x and in $\tau(x)$ have an overlap of length $|u| - l$. That is we can write $u = z_1 u_1$ with $|u_1| = l$

Represent $|u| = m_0 l + d_0$ with $0 \leq d_0 < l$. Note that $m_0 \geq 5$ and $d_0 < l \leq |u|/6 = \lambda n/6$.

Now write u as

$$u = z' u_{m_0} u_{m_0-1} \dots u_1$$

where $|u_i| = l$ for $i = 1, \dots, m_0$ and $|z'| = d_0$. Since

$$x = uv = z' u_{m_0} u_{m_0-1} \dots u_2 u_1 v$$

and

$$\tau(x) = y_1 u y_2 = y_1 z' u_{m_0} u_{m_0-1} \dots u_2 u_1 y_2,$$

and $|y_1| = l$, we see that

$$u_2 = \tau(u_1), u_3 = \tau(u_2) = \tau^2(u_1), \dots u_{m_0} = \tau(u_{m_0-1}) = \tau^{m_0-1}(u_1).$$

Thus, given l, the words u_1 and z' determine uniquely the rest of the word u, namely the word $w = u_{m_0} \dots u_2$. Recall that $|z'| \leq l$, $|u_1| = l$ and hence

$$|w| \geq |u| - 2l \geq |u| - 2|u|/6 = 2|u|/3 = 2\lambda n/3.$$

Recall that $|z'| = d_0$ is determined by l. So, given l (for which there are at most n choices), the word w is uniquely determined by the rest of the word x.

Hence the number of possibilities for x is at most

$$n \frac{(2k)^2}{(2k-1)^2} (2k-1)^{n-2\lambda n/3},$$

which is exponentially smaller than $(2k-1)^n$.

By summing up the numbers of possibilities for x in the above cases we see that

$$\lim_{n\to\infty} \frac{\gamma(n, S(\lambda, \tau))}{(2k-1)^n} = 0$$

with exponentially fast convergence.

Hence $S(\lambda, \tau)$ is exponentially CR-generic by Proposition 4.2. □

The same type of an argument as in the proof of Lemma 4.5 yields the following result.

Lemma 4.6. *Let $0 < \lambda < 1/3$. Let τ be a nontrivial relabeling automorphism of F.*

Define $S'(\lambda, \tau)$ as the set of all cyclically reduced words x such that x and some cyclic permutation of $\tau(x^{-1})$ have a common initial segment of length $\geq \lambda|x|$. Then $S'(\lambda, \tau)$ is exponentially CR*-negligible.*

Definition 4.7. Let $0 < \lambda < 1/3$. For a non-proper power cyclically reduced word x let $\mathcal{Y}(x, \lambda)$ be the set of all y satisfying one of the following:

(1) the word y is a cyclic permutation of $\tau(x)$ for some nontrivial relabeling automorphism τ;

(2) the word y is a cyclic permutation of $\tau(x^{-1})$ for some (possibly trivial) trivial relabeling automorphism τ;

(3) the word y is obtained by a nontrivial cyclic permutation of x.

Lemma 4.8. *Let $0 < \lambda < 1/3$. Define $E(\lambda)$ as the set of all non-proper power cyclically reduced words x such that for every $y \in \mathcal{Y}(x, \lambda)$ the lengths of the maximal common initial segment of x and y is $< \lambda|x|$. Then $E(\lambda)$ is exponentially* CR*-generic.*

Proof. As proved by Arzhantseva and Ol'shanskii [1] (and easy to see directly by arguments similar to those used in Lemma 4.5 and Lemma 4.6), the set of non-proper power cyclically reduced words x whose symmetrized closures satisfy the $C'(\lambda)$ small cancellation condition (see [32] for definitions) is exponentially CR-generic. Since there are only finitely many relabeling automorphisms, the result now follows from Lemma 4.5 and Lemma 4.6 by intersecting a finite number of exponentially CR-generic sets. □

Remark 4.9. Note that by definition the set $E(\lambda)$ is closed under taking inverses, cyclic permutations and applying relabeling automorphisms. Let M be the number of all (including the trivial one) relabeling automorphisms. Then for any $x \in E(\lambda)$ the set $\mathcal{Y}(x, \lambda)$ contains exactly $2M|x| - 1$ distinct elements.

5. Delzant's T-invariant for one-relator groups

Definition 5.1 (Non-reduced T-invariant). For a finite group presentation $\Pi = \langle X \mid R \rangle$ denote $\ell_1(\Pi) := \sum_{r \in R} |r|$.

If G is a finitely presentable group, define

$$T_1(G) := \min\{\ell_1(\Pi) : \Pi \text{ is a finite presentation of } G\}.$$

We call $T_1(G)$ the *non-reduced T-invariant of G.*

Obviously, for any Π we have $\ell(\Pi) \leq \ell_1(\Pi)$ and hence for every finitely presentable group G we have $T(G) \leq T_1(G)$. It turns out that under some mild assumptions there is a similar inequality in the other direction:

Lemma 5.2. *Let G be a finitely presentable group with no elements of order two. Then there exists a finite presentation Π of G such that $\ell(\Pi) = T(G)$ and such that every relation in Π has length at least three, and therefore $T_1(G) \leq \ell_1(\Pi) \leq 3\ell(\Pi) = 3T(G)$.*

Consequently,

$$T(G) \leq T_1(G) \leq 3T(G).$$

Proof. Among all finite presentations Π of G with $\ell(\Pi) = T(G)$, choose a presentation $\Pi = \langle X \,|\, R\rangle$ of minimal ℓ_1-length.

We claim that every relation in Π has length at least three. Clearly, the minimality assumptions on Π imply that Π has no relations of length one. Suppose Π has a relation r of length two. Thus $r = xy$ where $x, y \in X^{\pm 1}$. We may assume that $y \in X$.

If $x \neq y$ in $F(X)$, let Π' be the presentation obtained from Π by the Tietze transformation consisting of replacing every occurrence of y in the relators of R different from r by x^{-1}, freely reducing the resulting relators if needed, then removing the relator xy and removing the generator y from X. Then $\ell(\Pi') \leq \ell(\Pi) = T(G)$ and hence $\ell(\Pi') = T(G)$. By construction, $\ell_1(\Pi') < \ell_1(\Pi)$ contradicting the minimality of Π.

If $r = x^2$, the assumption that G has no elements of order two implies that $x = 1$ in G. Let Π'' be the presentation obtained from Π by removing the generator x from X, removing the relation $r = x^2$ and deleting all the occurrences of x from the other relations of R and freely reducing the results if necessary. We again have $\ell(\Pi'') = \ell(\Pi) = T(G)$ and hence $\ell(\Pi'') = T(G)$. By construction $\ell_1(\Pi'') < \ell_1(\Pi)$, contradicting the choice of Π.

Thus every relation in Π has length at least three, as claimed. □

Recall that, as specified in Convention 3.1, $k > 1$ is a fixed integer and $F = F(a_1, \ldots, a_k)$. As before we identify F with the set of all freely reduced words in the alphabet $A_{2k} = \{a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}\}$. For $u \in F$ we denote by G_u the one-relator group $G_u := \langle a_1, \ldots, a_k \,|\, u = 1\rangle$. If Π is a presentation, $G(\Pi)$ denotes the group presented by Π.

We now recall an important result about isomorphism rigidity of generic one-relator groups that we obtained in [30].

Theorem 5.3 ([30]). *Let $k > 1$ be a fixed integer and $F = F(a_1, \ldots, a_k)$. There exists an exponentially* CR*-generic set $Q_k \subseteq$ CR with the following properties:*

(1) *There is an exponential time algorithm which, given $w \in F$, decides whether or not $w \in Q_k$.*

(2) *The set Q_k is closed under taking cyclic permutations, inverses and applying relabeling automorphisms.*

(3) *Each $u \in Q_k$ is minimal in its $\mathrm{Aut}(F)$-orbit, that is $|u| \leq |\alpha(u)|$ for any $\alpha \in \mathrm{Aut}(F)$.*

(4) *If $r \in Q_k$ then G_r is torsion-free freely indecomposable non-elementary word-hyperbolic group.*

(5) *If $u \in Q_k$ and $v \in F$ are such that $|u| = |v|$ and $\mathrm{Aut}(F)u = \mathrm{Aut}(F)v$ then $v \in Q_k$ and there is a relabeling automorphism τ of F such that v is a cyclic permutation of $\tau(u)$.*

(6) *Let $u \in Q_k$ and $v \in F$ be such that $|u| = |v|$. Then $G_u \cong G_v$ if and only if $v \in \mathrm{CR}$ and there is a relabeling automorphism τ of F such that v is a cyclic permutation of $\tau(u)$ or $\tau(u)^{-1}$.*

(7) *If $u \in Q_k$ and $v \in F$ then $G_u \cong G_v$ if and only if there is $\alpha \in \mathrm{Aut}(F)$ such that $\alpha(v) = u$ or $\alpha(v) = u^{-1}$.*

The following lemma is just the "general enumeration argument".

Lemma 5.4. *Let $\mathcal{C}$ be a recursively enumerable class of finite presentations of groups. There is a partial algorithm $\Omega(\mathcal{C})$ which, when given a finite presentation $\Pi = \langle X \mid R \rangle$, finds a finite presentation $\Pi' \in \mathcal{C}$ such that $G(\Pi')$ is isomorphic to $G(\Pi)$ if such a presentation Π' exists.*

Proof. We assume that the generating sets for Π and for all presentations in $\mathcal{C}$ are initial segments of a fixed recursive set $\{x_1, x_2, \dots\}$ of generators. We enumerate *all* tuples (Π', b, h, h') where

$$\Pi' = \langle X' \mid R' \rangle \in \mathcal{C}, \quad d \in \mathbb{N}^+, \quad h\colon X \longrightarrow F(X'), \quad h'\colon X' \longrightarrow F(X).$$

When such a tuple is enumerated, we then enumerate the first d elements of $N' = \mathrm{ncl}(R') \subset F(X')$ and of $N = \mathrm{ncl}(R) \subset F(X)$. We then check if all of the following hold using only the elements of N and N' which have just been enumerated:

$$\begin{aligned} h(r) &\in N' \quad \text{for all } r \in R, \\ h'(r) &\in N \quad \text{for all } r \in R', \\ h'h(x)x^{-1} &\in N \quad \text{for all } x \in X, \\ hh'(x)x^{-1} &\in N' \quad \text{for all } x \in X'. \end{aligned}$$

If all of these memberships are witnessed by the elements of N and N' just enumerated, then h and h' define mutually inverse isomorphisms between $G(\Pi)$ and $G(\Pi')$ and we output Π'. If not, we go on to the next tuple. □

Convention 5.5. For $0 < \lambda < 1/3$ denote $Q_k(\lambda) = Q_k \cap E(\lambda)$ where $E(\lambda)$ is as in Lemma 4.8 and Q_k is from Theorem 5.3. By Lemma 4.8 and Theorem 5.3 the set $Q_k(\lambda)$ is exponentially CR-generic.

Lemma 5.6. *There exists a constant $N = N(k) > 0$ with the following property. Let $0 < \lambda < 1/3$ be a rational number and let $r \in Q_k(\lambda)$ be a nontrivial cyclically reduced word and $G_r := \langle a_1, \ldots, a_k \,|\, r = 1\rangle$. Thus r is not a proper power and it satisfies the $C'(\lambda)$ small cancellation condition.*

Suppose G_r can be presented by a finite presentation

$$\Pi = \langle b_1, \ldots b_m \,|\, r_1, \ldots, r_t\rangle \qquad (\dagger)$$

where $t \geq 1$.

Then $C_{2k}(r) \leq N\ell_1(\Pi)\log_2 \ell_1(\Pi) + |r|N\lambda + N$.

Proof. We describe an algorithm $\mathcal{A}$, which, given a presentation $(\dagger)$ for G_r and an initial segment u of r of length $\lambda|r|$, will recover the word r.

First, note that we are assuming that $(\dagger)$ defines a group isomorphic to the k-generator one-relator group G_r with defining relator in Q_k. We first apply the algorithm $\Omega(\mathcal{C})$ from Lemma 5.4 with $\mathcal{C}$ the class of all k-generator one-relator presentations with defining relators from Q_k. (Note that $\mathcal{C}$ is recursive by part (1) of Theorem 5.3.) This procedure finds some cyclically reduced word $v \in Q_k$ such that $(\dagger)$ defines a group isomorphic to G_v.

Thus $G_r \cong G_v$ and both r and v (as well as v^{-1}) are minimal cyclically reduced words from Q_k. By Theorem 5.3 $|v| = |r|$ and there is a relabeling automorphism τ of F such that r is a cyclic permutation of $\tau(v)$ or $\tau(v)^{-1}$.

Construct the set B consisting of all words x with the property that there is a relabeling automorphism τ of F such that x is a cyclic permutation of $\tau(v)$ or $\tau(v)^{-1}$. Thus $r \in B$. By Lemma 4.8 there is a unique element of B having the same initial segment of length $\lambda|r|$ as does r, namely r itself. Recall that the initial segment u of r of length $\lambda|r|$ is part of the input for algorithm $\mathcal{A}$. Then we list all elements of B and check which one of them has initial segment u. That element is r.

The algorithm $\Omega(\mathcal{C})$ is fixed. The further input of $\mathcal{A}$, required to compute r, consists of the presentation $(\dagger)$ and the initial segment u of r with $|u| = \lambda|r|$. We need to estimate the length of this input when expressed as a binary sequence. Put $T = \ell_1(\Pi)$. First note that in $(\dagger)$ every b_i must occur in some $r_j^{\pm 1}$ since G_r is a one-ended group by Theorem 5.3 and therefore $m \leq T$.

We can now encode the presentation $(\dagger)$ by writing each subscript $i = 1, \ldots, m$ for each occurrence of b_i in $(\dagger)$ as a binary integer. Using $\bar{i}$ to denote the binary expression for i, we replace each occurrence of b_i in $(\dagger)$ by $b\bar{i}$ and each occurrence of b_i^{-1} by $-b\bar{i}$. Note that the bit-length of the binary expression $\bar{i}$ of i is at most

$\log_2 i$. This produces an unambiguous encoding of (†) as a string W of length at most $O(T \log_2 T)$ over the six letter alphabet

$$b \quad 0 \quad 1 \quad - \quad , \quad |$$

and this alphabet can then be block-coded into binary in the standard way.

Since the number k of generators is fixed, describing u requires at most $O(|u|)$ number of bits.

Hence there exist a constant $N = N(k) > 0$ such that

$$C_{2k}(r) \leq NT \log_2 T + |r|N\lambda + N. \qquad \square$$

Theorem 5.7. *Let $k > 1$ be a fixed integer and $F = F(a_1, \ldots, a_k)$. For any ε, $0 < \varepsilon < 1$ there is an integer $n_1 > 0$ and a constant $M = M(k, \varepsilon) > 0$ with the following property.*

Let J be the set of all nontrivial cyclically reduced words r such that

$$T(G_r) \log_2 T(G_r) \geq M|r|.$$

Then for any $n \geq n_1$

$$\frac{\gamma(n, J)}{\gamma(n, \mathrm{CR})} \geq 1 - \varepsilon.$$

Proof. Let $N > 0$ be the constant provided by Lemma 5.6. Choose a rational number λ, $0 < \lambda < 1/3$, so that $\frac{\log_2(2k-1)}{2} - N\lambda > 0$.

Let $c > 0$ be an arbitrary integer. Let $n_0 > 1$ be the integer provided by Proposition 3.3. As in Proposition 3.3 let Z be the set of all cyclically reduced words x of length $\geq n_0$ such that

$$C_{2k}(x) \geq -\frac{c}{2} + |x| \frac{\log_2(2k-1)}{2}.$$

Then by Proposition 3.3 for any $n \geq n_0$ we have

$$\frac{\gamma(n, Z)}{\gamma(n, \mathrm{CR})} \geq 1 - \frac{1}{2^c}.$$

Since $Q_k(\lambda)$ is exponentially generic, Proposition 4.2 implies that there is $n_1 \geq n_0$ such that for any $n \geq n_1$

$$\frac{\gamma(n, Z \cap Q_k(\lambda))}{\gamma(n, \mathrm{CR})} \geq 1 - 2\frac{1}{2^c}.$$

Now suppose $r \in Z \cap Q_k(\lambda)$ and $|r| \geq n_1$.

Then by Lemma 5.6

$$-\frac{c}{2}+|r|\frac{\log_2(2k-1)}{2}\le C_{2k}(r)\le NT_1(G_r)\log_2 T_1(G_r)+|r|N\lambda+N,$$

and hence by Lemma 5.2

$$\begin{aligned}|r|\left(\frac{\log_2(2k-1)}{2}-N\lambda\right)-N-\frac{c}{2}&\le NT_1(G_r)\log_2 T_1(G_r)\\&\le 3NT(G_r)\log_2 3T(G_r)\\&=3NT(G_r)(\log_2 T(G_r)+\log_2 3)\\&\le 30NT(G_r)\log_2 T(G_r),\end{aligned}$$

yielding the conclusion of the theorem. □

We need the following result of Rivin on the precise number of cyclically reduced words of a given length.

Proposition 5.8. *For any $n\ge 1$ we have*

$$\gamma(n,\mathrm{CR})=(2k-1)^n+1+(k-1)[1+(-1)^n].$$

Thus for a fixed $k\ge 2$ we have $\gamma(n,\mathrm{CR})\sim(2k-1)^n$.
The next statement is obvious.

Lemma 5.9. *The number of relabeling automorphisms is $k!2^k$.*

Theorem 5.10. *Fix an integer $k\ge 2$. Let I_n be the number of isomorphism types of groups admitting a k-generator one-relator presentation where the defining relator is cyclically reduced and has length n. Then*

$$I_n\sim\frac{(2k-1)^n}{nk!2^{k+1}}.$$

Proof. Choose $0<\lambda<1/3$ so that $Q_k(\lambda)\subseteq\mathrm{CR}$ is exponentially CR-generic. Recall that $Q_k(\lambda)$ is closed under applying inverses, cyclic permutations and relabeling automorphisms.

Denote by $M=k!2^k$ the number of all relabeling automorphisms of $F=F(a_1,\dots,a_k)$.

By Remark 4.9 for any $u\in Q_k(\lambda)$ we have $\#\mathcal{Y}(u,\lambda)=2M|u|-1$. Hence by Theorem 5.3 the number of all $v\in Q_k(\lambda)$ with $G_u\cong G_v$ is equal to $2M|u|$. Therefore the set of words of length n in $Q_k(\lambda)$ defines precisely $\frac{\gamma(n,Q_k(\lambda))}{2Mn}$ isomorphism types of one-relator groups. Denote $b_n=\gamma(n,\mathrm{CR})-\gamma(n,Q_k(\lambda))$. Thus $\frac{b_n}{(2k-1)^n}\to 0$ exponentially fast as $n\to\infty$.

Hence

$$\left| I_n - \frac{\gamma(n, Q_k(\lambda))}{2Mn} \right| \leq b_n,$$

and so

$$\left| \frac{2nMI_n}{(2k-1)^n} - \frac{\gamma(n, Q_k(\lambda))}{(2k-1)^n} \right| \leq \frac{2Mnb_n}{(2k-1)^n}.$$

By CR-genericity of $Q_k(\lambda)$ and by Rivin's formula we have

$$\lim_{n\to\infty} \frac{\gamma(n, Q_k(\lambda))}{(2k-1)^n} = \lim_{n\to\infty} \frac{\gamma(n, Q_k(\lambda))}{\gamma(n, \mathrm{CR})} \frac{\gamma(n, \mathrm{CR})}{(2k-1)^n} = 1 \cdot 1 = 1.$$

Since $\lim_{n\to\infty} \frac{2Mnb_n}{(2k-1)^n} = 0$, this implies

$$\lim_{n\to\infty} \frac{2nMI_n}{(2k-1)^n} = 1,$$

and hence

$$I_n \sim \frac{(2k-1)^n}{2Mn} = \frac{(2k-1)^n}{nk!2^{k+1}}$$

as required. □

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