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On the zero set of semi-invariants for tame quivers

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Abstract. Let $\mathbf{d}$ be a prehomogeneous dimension vector for a finite tame quiver Q. We show that the common zeros of all non-constant semi-invariants for the variety of representations of Q with dimension vector $N \cdot \mathbf{d}$, under the product of the general linear groups at all vertices, is a complete intersection for $N \geq 3$.

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1. Introduction

Let k be an algebraically closed field, and let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, i.e. a finite set $Q_0 = \{1, \dots, n\}$ of vertices and a finite set Q_1 of arrows $\alpha : t\alpha \to h\alpha$, where $t\alpha$ and $h\alpha$ denote the tail and the head of α, respectively.

A representation of Q over k is a collection $(X(i);\, i \in Q_0)$ of finite dimensional k-vector spaces together with a collection $(X(\alpha) : X(t\alpha) \to X(h\alpha);\, \alpha \in Q_1)$ of k-linear maps. A morphism $f : X \to Y$ between two representations is a collection $(f(i) : X(i) \to Y(i))$ of k-linear maps such that

$$f(h\alpha) \circ X(\alpha) = Y(\alpha) \circ f(t\alpha) \qquad \text{for all } \alpha \in Q_1.$$

By $\sigma(X)$ we denote the number of pairwise non-isomorphic indecomposable direct summands occurring in a decomposition of X into indecomposables. According to the theorem of Krull–Schmidt, $\sigma(X)$ is well-defined. The dimension vector of a representation X of Q is the vector

$$\mathbf{dim}\, X = (\dim X(1), \dots, \dim X(n)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of Q by $\operatorname{rep}(Q)$, and for any vector $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^{Q_0}$

$$\operatorname{rep}(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \operatorname{Mat}(d_{h\alpha} \times d_{t\alpha}, k)$$

is the vector space of representations X of Q with $X(i) = k^{d_i}$, $i \in Q_0$. The group

$$\mathrm{Gl}(\mathbf{d}) = \prod_{i=1}^{n} \mathrm{Gl}(d_i, k)$$

acts on $\mathrm{rep}(Q, \mathbf{d})$ by

$$((g_1, \dots, g_n) \star X)(\alpha) = g_{h\alpha} \cdot X(\alpha) \cdot g_{t\alpha}^{-1}.$$

Note that the $\mathrm{Gl}(\mathbf{d})$-orbit of X consists of the representations Y in $\mathrm{rep}(Q, \mathbf{d})$ which are isomorphic to X.

We call $\mathbf{d}$ a prehomogeneous dimension vector if $\mathrm{Gl}(\mathbf{d}) \star T$ is an open orbit for some T in $\mathrm{rep}(Q, \mathbf{d})$. Such a representation T is characterized by $\mathrm{Ext}^1_Q(T, T) = 0$ [9]. If Q admits only finitely many indecomposable representations, or equivalently if the underlying graph of Q is a disjoint union of Dynkin diagrams of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ [6], every vector $\mathbf{d}$ is prehomogeneous. Indeed, any representation is a direct sum of indecomposables and therefore $\mathrm{rep}(Q, \mathbf{d})$ contains finitely many orbits, one of which must be open.

Let $\mathbf{d}$ be prehomogeneous, and let $f_1, \dots, f_s \in k[\mathrm{rep}(Q, \mathbf{d})]$ be the irreducible monic polynomials whose zeros $Z(f_1), \dots, Z(f_s)$ are the irreducible components of codimension 1 of $\mathrm{rep}(Q, \mathbf{d}) \setminus \mathrm{Gl}(\mathbf{d}) \star T$, where $\mathrm{Gl}(\mathbf{d}) \star T$ is the open orbit. It is easy to see that

$$g \cdot f_i = \chi_i(g) \cdot f_i$$

for $g \in \mathrm{Gl}(\mathbf{d})$, where $\chi_i : \mathrm{Gl}(\mathbf{d}) \to k^*$ is a character. A regular function with this property is called a semi-invariant. By [11], any semi-invariant is a scalar multiple of a monomial in $f_1, \dots, f_s$, and $f_1, \dots, f_s$ are algebraically independent. We denote by

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \mathrm{rep}(Q, \mathbf{d});\ f_i(X) = 0,\ i = 1, \dots, s\}$$

the closed subscheme of $\mathrm{rep}(Q, \mathbf{d})$ of common zeros of all non-constant semi-invariants. Obviously we have $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} \leq s$, and equality means that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection.

Let $T_1, \dots, T_r$ be pairwise non-isomorphic indecomposable representations of Q such that $\mathrm{Ext}^1_Q(T_i, T_j) = 0$ for any $i, j \leq r$. In [8] we showed that there is a positive integer N such that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection and irreducible for any dimension vector $\mathbf{d} = \sum_{i=1}^{r} \lambda_i \,\mathbf{dim}\, T_i$ with $\lambda_i \geq N$, $i = 1, 2, \dots, r$. Now our goal is to prove that N is quite small in case Q is tame; i.e., every connected component Δ of Q is either a Dynkin quiver or an extended Dynkin quiver. Our methods are completely different.

Assume that Q is tame, and set

$$N(Q) = \max N(\Delta),$$

where Δ ranges over the connected components of Q and where

$$N(\Delta) = \begin{cases} 1 & \text{if } |\Delta| = \mathbb{A}_m \text{ or } \widetilde{\mathbb{A}}_m, \\ 2 & \text{if } |\Delta| = \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7 \text{ or } \mathbb{E}_8, \\ 3 & \text{if } |\Delta| = \widetilde{\mathbb{D}}_m, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7 \text{ or } \widetilde{\mathbb{E}}_8, \end{cases}$$

and $|\Delta|$ denotes the underlying non-oriented graph of the quiver Δ. Note that $N(K) \leq N(Q)$ for any subquiver K of Q.

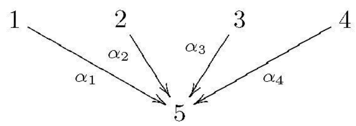
Theorem 1.1. *Suppose Q is tame. Let $T_1, \dots, T_r$ be pairwise non-isomorphic indecomposable representations of Q such that $\mathrm{Ext}^1_Q(T_i, T_j) = 0$ for any $i, j \leq r$. Choose positive integers $\lambda_1, \dots, \lambda_r$ and set $\lambda = \min \lambda_i$, $\mathbf{d} = \sum_{i=1}^r \lambda_i \, \mathbf{dim}\, T_i$. Then $\mathcal{Z}_{Q,\mathbf{d}}$ is*
(i) *a complete intersection provided $\lambda \geq N(Q)$,*
(ii) *irreducible provided $\lambda \geq N(Q) + 1$.*

Note that the case of a Dynkin quiver of type $\mathbb{A}_n$ has been treated by Chang and Weyman in [5].

In case k is the field $\mathbb{C}$ of complex numbers, the fact that $\mathcal{Z}_{Q,\mathbf{d}}$ is a complete intersection implies that $\mathrm{rep}(Q, \mathbf{d})$ is cofree as a representation of the subgroup $\mathrm{Sl}(\mathbf{d}) = \prod_{i=1}^n \mathrm{Sl}(d_i)$ of $\mathrm{Gl}(\mathbf{d})$; i.e., $\mathbb{C}[\mathrm{rep}(Q, \mathbf{d})]$ is a free module over the ring $\mathbb{C}[\mathrm{rep}(Q, \mathbf{d})]^{\mathrm{Sl}(\mathbf{d})}$ of $\mathrm{Sl}(\mathbf{d})$-invariant polynomials [13, §17], [8].

Example. Let us consider the quiver

$Q =$



with the dimension vector $\mathbf{d} = \lambda \cdot \mathbf{e}$, $\lambda \in \mathbb{N}$ and $\mathbf{e} = \begin{smallmatrix} 1 & 1 & & 1 & 1 \\ & & 3 & & \end{smallmatrix}$ as an example. There is an indecomposable representation T_1 in $\mathrm{rep}(Q, \mathbf{e})$, whose orbit is open. The complement of the open orbit of $T = T_1^\lambda$ in $\mathrm{rep}(Q, \mathbf{d})$ has 4 components of codimension 1, defined by

$$\det\Big(X(\alpha_1) \cdots \widehat{X(\alpha_j)} \cdots X(\alpha_4)\Big) = 0,$$

$j = 1, 2, 3, 4$, where the hat means "omit $X(\alpha_j)$". Using the results developed later, we know that X belongs to $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if X either contains the simple projective P_5 or else the direct sum $\bigoplus_{j=1}^4 P_j$ of the two-dimensional projectives associated to the vertices $1, \dots, 4$ as a direct summand. It is easy to check that

- $\mathcal{Z}_{Q,\mathbf{e}}$ is irreducible of codimension 2,
- $\mathcal{Z}_{Q,2\mathbf{e}}$ has two irreducible components of codimension 3 and 4, respectively,
- $\mathcal{Z}_{Q,3\mathbf{e}}$ has two irreducible components of codimension 4,
- $\mathcal{Z}_{Q,\lambda\cdot\mathbf{e}}$ is irreducible of codimension 4 for $\lambda \geq 4$.

2. Notations and preliminaries

The varieties considered in this paper are locally closed subsets of a k-vector space. If $\mathcal{A} \subseteq \mathcal{B}$ are two such varieties and $\mathcal{B}$ is irreducible, we denote by $\operatorname{codim}_{\mathcal{B}} \mathcal{A}$ the codimension of $\mathcal{A}$ in $\mathcal{B}$. In case $\mathcal{B} = \operatorname{rep}(Q, \mathbf{d})$, we omit the subscript $\mathcal{B}$.

We will assume throughout that the representation $T = \bigoplus_{i=1}^{r} T_i^{\lambda_i}$ is sincere, i.e., $T(l) \neq 0$ for any $l \in Q_0$. As the full subquiver K of Q which supports T is still tame with $N(K) \leq N(Q)$, this is no restriction. The assumption excludes oriented cycles as subquivers of Q. Indeed, a sincere representation of an oriented cycle cannot have an open orbit.

The Euler form of Q is the $\mathbb{Z}$-bilinear form on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha \in Q_1} d_{t\alpha} e_{h\alpha}.$$

For $X \in \operatorname{rep}(Q, \mathbf{d})$, $Y \in \operatorname{rep}(Q, \mathbf{e})$ it can be computed as

$$\langle \mathbf{d}, \mathbf{e} \rangle = [X, Y] - {}^1[X, Y],$$

where

$$[X, Y] = \dim_k \operatorname{Hom}_Q(X, Y) \qquad \text{and} \qquad {}^1[X, Y] = \dim_k \operatorname{Ext}_Q^1(X, Y).$$

The quadratic form

$$q(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d} \rangle$$

associated with the Euler form is the Tits form of Q. It is positive semi-definite as Q is tame and positive definite if Q does not contain extended Dynkin diagrams.

We follow Schofield [12] in order to describe the semi-invariants of $\operatorname{rep}(Q, \mathbf{d})$: For a representation U of Q, the right perpendicular category $U^{\perp}$ is the full subcategory of $\operatorname{rep}(Q)$ whose objects are

$$\{Y;\ [U, Y] = {}^1[U, Y] = 0\}.$$

Dually, ${}^{\perp}U$ has as objects

$$\{Z;\ [Z, U] = {}^1[Z, U] = 0\}.$$

Note that $U^{\perp} = {}^{\perp}(\tau U)$, where τ is the Auslander–Reiten translation for all non-projective indecomposable direct summands of U and $\tau(P_l) = I_l$, where P_l and I_l are the projective and injective indecomposable representations associated to the vertex $l \in Q_0$, respectively. If ${}^1[U, U] = 0$, the category $U^{\perp}$ is equivalent to the category of representations of a quiver with $n - \sigma(U)$ vertices.

Thus $T^{\perp}$ contains $n - r$ simple objects if $T = \bigoplus_{i=1}^{r} T_i^{\lambda_i}$ is a representation of Q as in the statement of the theorem. If S is one of them, the set

$$\{X \in \operatorname{rep}(Q, \mathbf{d});\ [X, S] \neq 0\}$$

is a component of codimension 1 of the complement

$$\operatorname{rep}(Q, \mathbf{d}) \setminus \operatorname{Gl}(\mathbf{d}) \star T.$$

Non-isomorphic simple objects lead to distinct components, and all components of codimension 1 are obtained in this way. Thus $\mathcal{Z}_{Q,\mathbf{d}}$ is the zero set of $n-r$ (algebraically independent) polynomials. From now on, we will denote the underlying reduced variety of $\mathcal{Z}_{Q,\mathbf{d}}$ by the same symbol. This will cause no confusion since we are only interested in the irreducibility and the dimension of $\mathcal{Z}_{Q,\mathbf{d}}$. We have the following descriptions:

$$\begin{aligned}\mathcal{Z}_{Q,\mathbf{d}} &= \{X \in \operatorname{rep}(Q,\mathbf{d});\ [X,S] \neq 0 \text{ for all simple objects } S \in T^{\perp}\}\\ &= \{X \in \operatorname{rep}(Q,\mathbf{d});\ [S',X] \neq 0 \text{ for all simple objects } S' \in {}^{\perp}T\}.\end{aligned}$$

The material presented here can be found in [12]; compare also [8]. In order to obtain part (i) of our theorem it suffices to prove $\operatorname{codim} \mathcal{Z}_{Q,\mathbf{d}} \geq n-r$.

Fix a sink $z \in Q_0$; i.e., a vertex z which is the head of some arrows $\alpha_j : y_j \to z$, $j = 1,\dots,s$, but the tail of none. The vertices $y_1,\dots,y_s$ need not be distinct. Let E be the simple projective supported at z. By $\overline{Q}$ we denote the full subquiver of Q with $\overline{Q}_0 = Q_0 \setminus \{z\}$ and by $\overline{\mathbf{d}}$ the restriction of $\mathbf{d}$ to $\overline{Q}_0$. Note that the orbit of the restriction $\overline{T} = \bigoplus_{i=1}^{r} \overline{T}_i^{\lambda_i}$ to $\overline{Q}$ is open in $\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})$. As E is the simple projective supported at z, we have

$$E^{\perp} = \{X \in \operatorname{rep}(Q);\ X(z) = 0\},$$

which we identify with $\operatorname{rep}(\overline{Q})$. There is a short exact sequence

$$0 \to E^{d_z} \to T \to \overline{T} \to 0.$$

Considering the long exact sequence of Hom's and Ext^1's from it, we find that $E^{\perp} \cap T^{\perp} = E^{\perp} \cap \overline{T}^{\perp} = \overline{T}^{\perp_{\overline{Q}}}$.

We decompose $\mathcal{Z}_{Q,\mathbf{d}}$ as a disjoint union

$$\mathcal{Z}_{Q,\mathbf{d}} = \mathcal{Z}'_{Q,\mathbf{d}} \cup \mathcal{Z}''_{Q,\mathbf{d}},$$

where

$$\mathcal{Z}'_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}};\ [X,E] = 0\} \qquad \text{and} \qquad \mathcal{Z}''_{Q,\mathbf{d}} = \{X \in \mathcal{Z}_{Q,\mathbf{d}};\ [X,E] \neq 0\}.$$

We will estimate the codimensions of $\mathcal{Z}'_{Q,\mathbf{d}}$ and $\mathcal{Z}''_{Q,\mathbf{d}}$ in $\operatorname{rep}(Q,\mathbf{d})$ separately.

Throughout the article, $T = \bigoplus_{i=1}^{r} T_i^{\lambda_i}$ will denote a sincere representation of a tame quiver Q, and we set $\lambda = \min \lambda_i \geq 1$ and $\mathbf{dim}\, T = \mathbf{d}$.

3. The variety $\mathcal{Z}''_{Q,\mathbf{d}}$

Proposition 3.1. *A representation X in $\mathcal{Z}_{Q,\mathbf{d}}$ belongs to $\mathcal{Z}''_{Q,\mathbf{d}}$ if and only if*

(i) *the restriction $\overline{X}$ to $\overline{Q}$ lies in $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$*

and

(ii) $\operatorname{rank}(X(\alpha_1) \cdots X(\alpha_s)) < d_z$.

In particular,

$$\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} = \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \max\left(0, \left(\sum_{j=1}^{s} d_{y_j}\right) - d_z + 1\right).$$

Proof. The second condition just says that E is a direct summand of X, or equivalently that $[X,E] \neq 0$. A representation $X = X' \oplus E$ belongs to $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if

$$[X,S] = [X',S] + [E,S] > 0$$

for any simple object $S \in T^{\perp}$. Equivalently,

$$[X',S] > 0$$

holds for any simple representation $S \in T^{\perp}$ with $[E,S] = \dim S(z) = 0$. These are precisely the simple objects of $\overline{T}^{\perp_{\overline{Q}}}$, and moreover we have

$$[X',S] = [\overline{X'},S] = [\overline{X},S] > 0$$

since $S(z) = 0$.

As for the statement about $\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}}$, observe that, in case $d_z > \sum_{j=1}^{s} d_{y_j}$, any $d_z \times \sum_{j=1}^{s} d_{y_j}$-matrix has rank less than d_z, whereas for $d_z \leq \sum_{j=1}^{s} d_{y_j}$, the subvariety

$$\mathcal{N}_{\mathbf{d}} = \left\{A \in \operatorname{Mat}\left(d_z \times \sum_{j=1}^{s} d_{y_j}\right);\ \operatorname{rank} A < d_z\right\}$$

is of codimension $\left(\sum_{j=1}^{s} d_{y_j}\right) - d_z + 1$. □

Corollary 3.2. *Suppose that $\lambda \geq N(Q)$ and that E is not a direct summand of T.*

(i) *We have*

$$\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) \geq \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n-1) + \sigma(\overline{T}).$$

(ii) *If moreover $d_z < \sum_{j=1}^{s} d_{y_j}$, we have*

$$\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) \geq \operatorname{codim} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n-1) + \sigma(\overline{T}) + \lambda - N(Q).$$

In order to prove this result, we need some information about the number $\sigma(\overline{T})$ of pairwise non-isomorphic indecomposables occurring as direct summands of $\overline{T}$. We start by estimating $\sigma(\overline{U})$ for an indecomposable representation U:

Lemma 3.3. *For an indecomposable representation $U \neq E$ of Q, we have*

$$\sigma(\overline{U}) \leq 1 + N(Q) \cdot \left(\left(\sum_{j=1}^{s} \dim U(y_j)\right) - \dim U(z)\right).$$

Proof. As U is indecomposable, we may assume Q to be connected. We use the following abbreviations:

$$\dim U(z) = u, \quad \dim U(y_j) = u_j, j = 1, \dots, s, \quad u' = \left(\sum_{j=1}^{s} u_j\right) - u.$$

Note that $u' \geq 0$ since U is indecomposable and $U \neq E$. If $u = 0$, $\overline{U} = U$ is indecomposable and $\sigma(\overline{U}) = 1$. In case $u' = 0$, the map

$$[U(\alpha_1), \cdots, U(\alpha_s)] : \bigoplus_{j=1}^{s} U(y_j) \to U(z)$$

is an isomorphism, and again $\overline{U}$ is indecomposable. Thus we may suppose $u > 0$ and $u' > 0$.

Recall that the value of the Tits form $q(\mathbf{dim}\, U)$ equals 0 or 1, as Q is tame. We compute:

$$\begin{aligned} q(\mathbf{dim}\, U - \mathbf{dim}\, E) &= q(\mathbf{dim}\, U) + q(\mathbf{dim}\, E) - \langle \mathbf{dim}\, U, \mathbf{dim}\, E\rangle - \langle \mathbf{dim}\, E, \mathbf{dim}\, U\rangle \\ &= q(\mathbf{dim}\, U) + q(\mathbf{dim}\, E) + u' - u \leq 2 + u' - u. \end{aligned}$$

As q is positive definite or positive semi-definite in case Q is a Dynkin quiver or an extended Dynkin quiver, respectively, we obtain:

$$u \leq \begin{cases} u' + 2 \leq 2u' + 1 & \text{if } Q \text{ is an extended Dynkin quiver,} \\ u' + 1 & \text{if } Q \text{ is a Dynkin quiver.} \end{cases}$$

Now clearly $\overline{U}$ has at most $\sum_{j=1}^{s} u_j$ indecomposable direct summands, and thus

$$\sigma(\overline{U}) \leq \sum_{j=1}^{s} u_j = u + u' \leq \begin{cases} 1 + 3u' & \text{if } Q \text{ is an extended Dynkin quiver,} \\ 1 + 2u' & \text{if } Q \text{ is a Dynkin quiver,} \end{cases}$$

which proves the lemma except in case $|Q| = \mathbb{A}_n$ or $|Q| = \widetilde{\mathbb{A}}_{n-1}$.

If $|Q| = \mathbb{A}_n$, we have $u \leq 1$ and hence $\sigma(\overline{U}) \leq 1 + u'$. In case $|Q| = \widetilde{\mathbb{A}}_{n-1}$, the number of indecomposable (possible isomorphic) direct summands in a decomposition of $\overline{U}$ is at most $1 + u'$. This can be seen by inspecting the list of indecomposable representations of Q. Such representations are string or band representations, and they are described by words (non-oriented paths) in Q (see [4] for details). □

Proof of Corollary 3.2. We set

$$t'_i = \left(\sum_{j=1}^{s} \dim T_i(y_j)\right) - \dim T_i(z), \ i = 1, \dots, r$$

and

$$t' = \sum_{i=1}^{r} t'_i.$$

Note that, by definition,

$$\sum_{i=1}^{r} \lambda_i t'_i = \left(\sum_{j=1}^{s} d_{y_j} \right) - d_z.$$

Our lemma implies:

$$\begin{aligned}\sigma(\overline{T}) &\leq \sum_{i=1}^{r} \sigma(\overline{T_i}) \leq r + N(Q) \cdot t' \leq r + \left(\sum_{i=1}^{r} \lambda_i t'_i \right) - (\lambda - N(Q)) \cdot t' \\ &= \sigma(T) + \left(\sum_{j=1}^{s} d_{y_j} \right) - d_z - (\lambda - N(Q)) \cdot t'.\end{aligned}$$

Combining this with Proposition 3.1 we find that

$$\begin{aligned}\operatorname{codim} \mathcal{Z}''_{Q,\mathbf{d}} - n + \sigma(T) &= \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} + \left(\sum_{j=1}^{s} d_{y_j} \right) - d_z + 1 - n + \sigma(T) \\ &\geq \operatorname{codim}_{\operatorname{rep}(\overline{Q},\overline{\mathbf{d}})} \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} - (n-1) + \sigma(\overline{T}) + (\lambda - N(Q)) \cdot t'.\end{aligned}$$

As $t'_i \geq 0$ for all i, this yields part (i) of Corollary 3.2. Part (ii) follows from the fact that $\sum_{i=1}^{r} \lambda_i t'_i = \left(\sum_{j=1}^{s} d_{y_j} \right) - d_z > 0$ implies $t'_i > 0$ for some i and hence $t' > 0$. □

4. Reflection functors

We define two new quivers $\widetilde{Q}$ and Q': $\widetilde{Q}$ is obtained from Q by adding a vertex z' and arrows $\beta_j : z' \to y_j$, $j = 1, \dots, s$. Deleting z and $\alpha_1, \dots, \alpha_s$ in $\widetilde{Q}$ yields Q'. Note that Q' is tame as well. We denote by E' the simple injective representation of Q' supported at z'.

We consider the reflection functor

$$\mathcal{F} : \operatorname{rep}(Q) \to \operatorname{rep}(Q')$$

associated with z. Recall that

$$(\mathcal{F}X)(i) = \begin{cases} X(i) & i \neq z' \\ \ker \left(\bigoplus X(y_j) \xrightarrow{[X(\alpha_1),\dots,X(\alpha_s)]} X(z) \right) & i = z', \end{cases}$$

and that

$$(\mathcal{F}X)(\beta_l) : (\mathcal{F}X)(z') \to (\mathcal{F}X)(y_l) = X(y_l)$$

is the inclusion of $(\mathcal{F}X)(z')$ into $\bigoplus_{j=1}^{s} X(y_j)$ followed by the projection to $X(y_l)$ (see [1], [6]). The functor $\mathcal{F}$ restricts to an equivalence

$$\mathcal{F} : (\operatorname{rep}(Q))' \to (\operatorname{rep}(Q'))'$$

from the full subcategory $(\mathrm{rep}(Q))'$ of $\mathrm{rep}(Q)$ whose objects do not contain E as a direct summand, or equivalently have no non-trivial morphisms to E, to the full subcategory $(\mathrm{rep}(Q'))'$ of $\mathrm{rep}(Q')$ whose objects do not contain E' as a direct summand.

Suppose that E is neither a direct summand of T nor an element of $T^{\perp}$. This implies that $[T,E]=0$ and ${}^1[T,E]>0$ and thus the vector $\mathbf{d}'\in\mathbb{Z}^{Q_0'}$, where Q_0' denotes the set of vertices of Q', defined by

$$d'_x=\begin{cases} d_x, & x\neq z' \\ \left(\sum_{j=1}^s d_{y_j}\right)-d_z, & x=z'\end{cases}$$

has positive entries. Indeed, we have

$$d'_{z'}=\left(\sum_{j=1}^s d_{y_j}\right)-d_z=-\langle\mathbf{d},\mathbf{dim}\,E\rangle=-[T,E]+{}^1[T,E]>0. \tag{4.1}$$

Note that in fact we have $d'_{z'}\geq\lambda$ as ${}^1[T_i,E]>0$ for some i implies ${}^1[T,E]\geq\lambda_i\geq\lambda$. We let $\widetilde{\mathbf{d}}$ be the dimension vector for $\widetilde{Q}$ which coincides with $\mathbf{d}$ on Q_0 and with $\mathbf{d}'$ on Q_0'.

As E is not a direct summand of T, the latter belongs to $(\mathrm{rep}\,Q)'$. Therefore $\mathcal{F}T$ lies in $(\mathrm{rep}\,Q')'$, and we have $\mathbf{dim}\,\mathcal{F}T=\mathbf{d}'$, ${}^1[\mathcal{F}T,\mathcal{F}T]={}^1[T,T]=0$, and thus $\mathbf{d}'$ is prehomogeneous. Choose T' in the open orbit of $\mathrm{rep}(Q',\mathbf{d}')$. As T' is isomorphic to $\mathcal{F}T$, we have $T'=\bigoplus_{i=1}^r(T_i')^{\lambda_i}$ with T_i' indecomposable, pairwise non-isomorphic and ${}^1[T_i',T_j']=0$ for all i,j. Moreover, we know $T^{\perp}\subsetneq(\mathrm{rep}\,Q)'$, as E does not belong to $T^{\perp}$, and $(T')^{\perp}\subsetneq(\mathrm{rep}\,Q')'$, as $d'_{z'}=[T',E']>0$. We conclude that $(T')^{\perp}$ is equivalent to $\mathcal{F}(T^{\perp})$, the category of representations of a quiver with $n-r$ vertices. Hence $\mathcal{Z}_{Q',\mathbf{d}'}$ is given by $n-r$ equations as well. We decompose $\mathcal{Z}_{Q',\mathbf{d}'}$ as a disjoint union $\mathcal{Z}_{Q',\mathbf{d}'}=\mathcal{W}'_{Q',\mathbf{d}'}\cup\mathcal{W}''_{Q',\mathbf{d}'}$, where

$$\mathcal{W}'_{Q',\mathbf{d}'}=\{X'\in\mathcal{Z}_{Q',\mathbf{d}'};\ [E',X']=0\}$$

and

$$\mathcal{W}''_{Q',\mathbf{d}'}=\{X'\in\mathcal{Z}_{Q',\mathbf{d}'};\ [E',X']\neq 0\}.$$

Proposition 4.1. *Suppose E is neither a direct summand of T nor an element of $T^{\perp}$. Then we have*

(i) $\operatorname{codim}\mathcal{Z}'_{Q,\mathbf{d}}=\operatorname{codim}_{\mathrm{rep}(Q',\mathbf{d}')}\mathcal{W}'_{Q',\mathbf{d}'}$

and

(ii) *$\mathcal{Z}'_{Q,\mathbf{d}}$ is irreducible if $\mathcal{W}'_{Q',\mathbf{d}'}$ has this property.*

Proof. By construction, X belongs to $\mathcal{Z}'_{Q,\mathbf{d}}$ if and only if $\mathcal{F}X$ is isomorphic to some $X'\in\mathcal{W}'_{Q',\mathbf{d}'}$, but unfortunately the functor $\mathcal{F}$ cannot be made into a regular map from

$$\mathrm{rep}(Q,\mathbf{d})'=\{X\in\mathrm{rep}(Q,\mathbf{d});\ [X,E]=0\}$$

to

$$\operatorname{rep}(Q',\mathbf{d}')' = \{X' \in \operatorname{rep}(Q',\mathbf{d}');\ [E',X'] = 0\}.$$

We use the following détour (compare [7] and Section 4.2 in [3]): The set

$$\Big\{X \in \operatorname{rep}(\widetilde{Q},\widetilde{\mathbf{d}});\ \sum_{j=1}^{s} X(\alpha_j)X(\beta_j) = 0,\ [X(\beta_1),\dots,X(\beta_s)]^t \text{ injective},$$
$$[X(\alpha_1),\dots,X(\alpha_s)] \text{ surjective}\Big\}$$

is a principal $\mathrm{Gl}(d'_{z'})$-bundle over $\operatorname{rep}(Q,\mathbf{d})'$ and a principal $\mathrm{Gl}(d_z)$-bundle over $\operatorname{rep}(Q',\mathbf{d}')'$ via the projections π and π' deleting z' and z, respectively. Hence the claim follows from $\pi^{-1}(\mathcal{Z}'_{Q,\mathbf{d}}) = (\pi')^{-1}(\mathcal{W}'_{Q',\mathbf{d}'})$. □

5. Proof of Theorem 1.1

We proceed by induction on the number n of vertices of Q. We may assume the theorem to be true for $\mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}}$. First we treat the cases that

(i) E is a direct summand of T

and

(ii) E belongs to $T^{\perp}$.

In both cases, we have that E is a direct summand of X for all $X \in \mathcal{Z}_{Q,\mathbf{d}}$; i.e., $\mathcal{Z}''_{Q,\mathbf{d}} = \mathcal{Z}_{Q,\mathbf{d}}$. Indeed, in case (i) this follows from the fact that $\operatorname{Hom}_Q(E,T) \neq 0$, which is a closed condition. In case (ii), E is a simple object in $T^{\perp}$.

As any direct summand $T_i \not\simeq E$ of T belongs to ${}^{\perp}E$, we have

$$\dim T_i(z) - \sum_{j=1}^{s} \dim T_i(y_j) = \langle \mathbf{dim}\, T_i, \mathbf{dim}\, E\rangle = [T_i,E] - {}^1[T_i,E] = 0.$$

By Lemma 3.3, $\overline{T_i}$ is indecomposable, and therefore

$$\sigma(\overline{T}) = \begin{cases} r-1 & \text{in case (i)}, \\ r & \text{in case (ii)}. \end{cases}$$

The induction hypothesis together with Corollary 3.2 implies the first part of our theorem. We conclude from Proposition 3.1 that $\mathcal{Z}_{Q,\mathbf{d}} \simeq \mathcal{Z}_{\overline{Q},\overline{\mathbf{d}}} \times \mathcal{N}_{\mathbf{d}}$, where

$$\mathcal{N}_{\mathbf{d}} = \{A \in \operatorname{Mat}\left(d_z \times \sum_{j=1}^{s} d_{y_j}\right);\ \operatorname{rank} A < d_z\}.$$

The second part follows from the fact that the set $\mathcal{N}_{\mathbf{d}}$ is irreducible in case $d_z \geq \sum_{j=1}^{s} d_{y_j}$.

(iii) Finally, suppose that E is neither a direct summand of T nor does it belong to $T^{\perp}$, or equivalently that $d_z < \sum_{j=1}^{s} d_{y_j}$. Using Corollary 3.2 and its dual, Proposition 4.1 and remembering that the codimension of any irreducible component of $\mathcal{Z}_{Q,\mathbf{d}}$ is at most $n-r$, we see that the theorem is true for $\mathcal{Z}_{Q,\mathbf{d}}$ if and only if it holds for $\mathcal{Z}_{Q',\mathbf{d}'}$.

In case either T contains a preprojective direct summand or $T^{\perp}$ a preprojective representation, we may apply a series of reflection functors until we reach the situation that a simple projective either is a direct summand of T or else belongs to $T^{\perp}$, and we can reduce by (i) or (ii). This finishes the proof in case Q is of finite representation type as any indecomposable representation is preprojective.

If Q is not representation finite, we are left with the situation that no preprojective representation is a direct summand of T nor an element of $T^{\perp}$. Dually, we may assume T does not contain a preinjective direct summand either. Indeed, suppose a simple injective representation E' is a direct summand of T or belongs to ${}^{\perp}T$, a situation we will reach after a series of (inverse) reflection functors. Then apply the dual of the first or the second reduction step above; recall that $\mathcal{Z}_{Q,\mathbf{d}}$ has a dual description as

$$\mathcal{Z}_{Q,\mathbf{d}} = \{X \in \operatorname{rep}(Q,\mathbf{d});\ [S',X] \neq 0 \text{ for all simple objects } S' \in {}^{\perp}T\}.$$

The following lemma finishes the proof of Theorem 1.1.

Lemma 5.1. *Let Q be an extended Dynkin quiver. Suppose T is a regular representation with an open orbit. Then $T^{\perp}$ contains a non-zero preprojective representation.*

Proof. Consider a Bongartz completion $\widetilde{T}$ for T [2]; i.e., an exact sequence

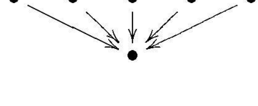
$$0 \to kQ \to \widetilde{T} \to \bigoplus_{i=1}^{r} T_i^{\nu_i} \to 0$$

for which the induced map

$$\operatorname{Hom}_Q\Big(T_l, \bigoplus_{i=1}^{r} T_i^{\nu_i}\Big) \to \operatorname{Ext}^1_Q(T_l, kQ)$$

is surjective for $l = 1,\dots,r$. There is a $\mathbb{Z}$-linear map $\partial : \mathbb{Z}^{Q_0} \to \mathbb{Z}$, called defect, such that any indecomposable representation Y of Q is preprojective, regular and preinjective if and only if $\partial(\mathbf{dim}\, Y)$ is negative, zero and positive, respectively (see for instance [10]). As T is regular, $\partial\widetilde{T} = \partial kQ < 0$ and therefore $\widetilde{T}$ contains an indecomposable preprojective direct summand Y, and $Y \in T^{\perp}$. Indeed, we have ${}^{1}[T,Y] = 0$ for all direct summands of $\widetilde{T}$ and $[T,Y] = 0$ since Y is preprojective and T is regular [10, Theorem 3.6.5]. □

Example. Working out the following example, one can see that if Q is not tame, it may happen that both T and $T^{\perp}$ belong to the set of regular representations:

$Q =$  , $\mathbf{d} =$

1	6	6	6	6
		16		

.

As $q(\mathbf{d}) = 1$, there exists an irreducible $T \in \operatorname{rep}(Q, \mathbf{d})$ having an open orbit. The simple objects in $T^{\perp}$ have dimension vectors

$$\begin{smallmatrix} 3 & 1 & 1 & 1 & 1 \\ & & 3 & & \end{smallmatrix}, \quad \begin{smallmatrix} 0 & 1 & 1 & 1 & 0 \\ & & 2 & & \end{smallmatrix}, \quad \begin{smallmatrix} 0 & 1 & 1 & 0 & 1 \\ & & 2 & & \end{smallmatrix}, \quad \begin{smallmatrix} 0 & 1 & 0 & 1 & 1 \\ & & 2 & & \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} 0 & 0 & 1 & 1 & 1 \\ & & 2 & & \end{smallmatrix}.$$

It is easy to check that these simple objects are regular representations of Q.

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