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Stable modules and Wall's D(2)-problem

F. E. A. Johnson

Abstract. The D(2)-problem is to determine whether for a three-dimensional complex X , the vanishing of 3-dimensional cohomology, in all coefficients, is enough to guarantee that X is homotopically two-dimensional. We show that for finite complexes with finite fundamental group, a positive solution to the D(2)-problem is obtained precisely when all stably free algebraic 2-complexes are geometrically realizable.

The proof makes very strong use of techniques which apply to finite fundamental groups but not more generally; in particular, Yoneda's Theorem that, for finite groups, group cohomology is representable by stable modules of finite type, and also the Swan–Jacobinski Cancellation Theorem for such stable modules.

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1. Introduction

In Wall's fundamental study of Poincaré Duality [30], a finite Poincaré complex of dimension n is envisaged as a cell complex of the following *standard form*

$$(*) \quad M(\alpha) = X \cup_{\alpha} e^n$$

where X is a finite complex of dimension $\leq n-1$ and $\alpha : S^{n-1} \rightarrow X$ is a continuous map. This approach encounters a difficulty in dimension three, where it is unclear whether we can take X to be of geometrical, rather than merely cohomological, dimension two (see also [29]). This prompts the celebrated

D(2) problem. Suppose that X is a finite three dimensional cell complex such that $H^3(X; \mathcal{B}) = 0$ for all local coefficient systems \mathcal{B} on X . Is X homotopy equivalent to a finite 2-dimensional complex?

The D(2)-problem is naturally parametrized by the fundamental group of the complex involved; we say that a finitely presented group G possesses the D(2)-

property when every finite 3-complex X_G with $\pi_1(X_G) \cong G$ which is cohomologically two dimensional is indeed homotopy equivalent to a finite 2-complex. For $n \geq 3$ the corresponding D(n)-problem was solved in the affirmative by Wall [30].

Our main result, the Realization Criterion below, is an interpretation, for finite groups, of the D(2) problem in purely two-dimensional terms, or more precisely, in terms of the homotopy types of two dimensional chain complexes over $\mathbf{Z}G$. If G is a finitely presented group, by an *algebraic 2-complex* over G we mean an exact sequence of modules and homomorphisms over $\mathbf{Z}[G]$ of the following form

$$\mathbf{E} = (0 \rightarrow J \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \rightarrow \mathbf{Z} \rightarrow 0)$$

where E_r is finitely generated and *stably free* over $\mathbf{Z}[G]$ for $0 \leq r \leq 2$. We denote by \mathbf{Alg}_G the category whose objects are such algebraic 2-complexes, and whose morphisms are chain maps. The module J is an invariant of chain homotopy equivalence and plays the role of an *algebraic second homotopy group*.

To any finite 2-dimensional (*geometric*) cell complex K with $\pi_1(K) \cong G$ one associates functorially an object in \mathbf{Alg}_G , namely the ‘cellular chain complex’

$$C_*(K) = (0 \rightarrow \pi_2(K) \rightarrow C_2(\tilde{K}) \rightarrow C_1(\tilde{K}) \rightarrow C_0(\tilde{K}) \rightarrow \mathbf{Z} \rightarrow 0);$$

here \tilde{K} denotes the universal covering of K , and since \tilde{K} is simply connected, we identify $H_2(\tilde{K}; \mathbf{Z})$ with $\pi_2(\tilde{K}) = \pi_2(K)$ by means of the Hurewicz Theorem. We reduce the problem of two -dimensional geometric homotopy type to purely algebraic terms:

Algebraicity Theorem. *Let G be a finitely presented group, and let K, L be connected finite based 2-complexes with $\pi_1(K) = \pi_1(L) = G$; then there is a based homotopy equivalence $f : K \rightarrow L$ inducing the identity on π_1 if and only if $C_*(K)$ is chain homotopy equivalent to $C_*(L)$.*

Let $\mathbf{E} \in \mathbf{Alg}_G$ be an algebraic 2-complex; by a *geometric realization* of \mathbf{E} , we mean a finite connected based 2-complex K with $\pi_1(K) = G$ together with an algebraic chain equivalence $\varphi : C_*(K) \rightarrow \mathbf{E}$. A natural question is whether every algebraic 2-complex is geometrically realizable in this way; for *finite fundamental groups*, we show this is equivalent to the D(2)-problem:

Realization Criterion. If G is a finite group, the D(2)-property holds for G if and only each algebraic 2-complex $\mathbf{E} \in \mathbf{Alg}_G$ admits a geometric realization.

For finite G , the Realization Criterion thus gives a complete reduction of the D(2) -problem to one involving only two dimensional data. We shall say that a finite group is *2-tame* precisely when every algebraic 2-complex in \mathbf{Alg}_G is geometrically realizable. The question of which finite groups are 2-tame is discussed in [16], [17]. By a change of emphasis, the result of Beyl, Latiolais and Waller [1], classifying geometrical homotopy types over fundamental groups of certain 3-manifolds,

can also be interpreted in this light. In particular, cyclic groups, dihedral groups of order $4n + 2$, the binary Euclidean groups \mathbf{T}^* , \mathbf{O}^* , \mathbf{I}^* , the quaternion groups $Q(8)$, $Q(16)$ and $Q(8n + 4)$ ($n \geq 1$) are all known to be 2-tame, whereas over the quaternion groups $Q(2^n)$ ($n \geq 5$), there exist algebraic 2-complexes for which no geometrical realization is as yet known. In [15] the discussion is specialised to the problem which provided the original motivation for this work, namely the theory of Poincaré 3-complexes with finite fundamental group.

One should point out that the Algebraicity Theorem stated above is stronger than the familiar statement of ‘Whitehead’s Theorem’, namely that a based map $f : K \rightarrow L$ inducing an isomorphism on π_1 is a homotopy equivalence if and only if $C_*(f) : C_*(K) \rightarrow C_*(L)$ is a chain equivalence. Here one must construct a geometrical homotopy equivalence starting from purely algebraic data. Despite the fact that it is well known, nevertheless, it does not seem to occur, stated explicitly *in this form*, anywhere in the literature.

One can, though not without at least a little effort, retrieve the Algebraicity Theorem in the form above either from Whitehead’s general formulation of homotopy systems in [31] or the rather more specifically low-dimensional formulation given in [19]. In fact, both these formulations (‘crossed modules’, ‘k-invariants’) have slightly more structure than we need; properly speaking they refer to ‘polarised homotopy’, wherein isomorphisms to the higher homotopy groups, not merely π_1 , are specified in advance. Although we shall meet k-invariants in Section 4, to avoid the unnecessary complications of a premature encounter we begin by giving a direct proof of the Algebraicity Theorem in Section 2.

In addition, we employ two specific techniques which render algebraic homotopy theory for finite fundamental groups considerably more tractable than is the case in general. The first is Yoneda’s classification of module extensions [32]; the second is the detailed cancellation theory of $\mathbf{Z}[G]$ -modules provided by the Swan–Jacobinski Theorem [25], [12]. In Sections 3–4, we sketch both of these in the context of the ‘derived object’ formulation of homological algebra.

The results of this paper also depend heavily upon the fundamental advances of several authors. In addition to those already mentioned, there are the many other works of Swan [22], [23], [24], [26], and the pioneering contributions of Cockroft and Swan [6], and Dyer and Sieradski [7], [8].

No introduction to the subject would be complete without mention of the theory of Browning, generalizing the Swan–Jacobinski Theorem to the context of chain complexes. His work, having languished in obscurity for twenty years, is at last being accorded some of the recognition which its fundamental character deserves. Browning did not publish his results, and his work is available only in the form of his thesis [2], and some difficult-to-obtain ETH pre-prints [3], [4]. In fact, we do not employ Browning’s results directly, with the sole exception of a single use of a weak version of Browning’s Stability Theorem in Section 6. Nevertheless, the existence of his work has informed our investigations throughout. In this context, we should also mention the extremely helpful explications of Browning’s work by

Gutierrez and Latiolais [11], and from a rather different perspective, the paper of Gruenberg [10].

The preliminary work for this paper was done between September 1996 and March 1997, an equivalent, differently stated, version of the Realization Criterion first being proved in February 1997; this version was announced at the British Topology Meeting at Oxford in April 1997. The author wishes to thank Karl Gruenberg both specifically, for donating copies of Browning's ETH preprints, and generally for many enlightening conversations over the years; and particularly Charles Thomas, whose initial suggestion it was to re-examine the problem of Poincaré 3-complexes, and whose paper [27] provided an important focus at an early stage of this work.

The paper is organised as follows: the Algebraicity Theorem is proved in Section 2; in Sections 3–4 we outline the necessary results from Swan–Jacobinski cancellation theory and Yoneda's classification theory of module extensions.

For any finite group G , the second homotopy group of any 2-complex K with $\pi_1(K) = G$ determines a stable module $\Omega_3(\mathbf{Z})$. In Section 5, we show that the general question of realizing algebraic 2-complexes is reducible to the level of 'minimal' representatives of $\Omega_3(\mathbf{Z})$. Finally, in Section 6, we relate the discussion directly to the D(2) problem and prove the Realization Criterion.

2. Reduction of 2-dimensional homotopy to algebra

Let G be a finitely presented group. For $n \geq 2$, \mathbf{CW}_G^n will denote the category whose objects are connected finite connected based cell complexes of dimension $\leq n$ such that $\pi_1(K) = G$. Morphisms are based cellular maps; however, in the context in which we shall work, namely a single (based) homotopy equivalence $f : K \rightarrow L$ between given complexes K, L , no generality is lost by imposing the restriction that maps induce the isomorphism on π_1 , and without further mention, we shall always assume this condition is in effect. Without loss, we assume that basepoints are 0-cells. Also, by the usual device of collapsing a maximal tree, no generality in homotopy arguments is lost by assuming all complexes are *reduced*, that is, possess a single 0-cell, and we shall allow ourselves this latitude without further comment.

By choosing once and for all a lifting of the basepoint to the universal covering, we associate to $K \in \mathbf{CW}_G^2$ an exact sequence $C_*(K)$ of $\mathbf{Z}[G]$ -modules:

$$C_*(K) = (0 \rightarrow H_2(\tilde{K}; \mathbf{Z}) \rightarrow C_2(\tilde{K}) \rightarrow C_1(\tilde{K}) \rightarrow C_0(\tilde{K}) \rightarrow \mathbf{Z} \rightarrow 0).$$

Exactness follows from the simple connectivity of \tilde{K} , and by the Hurewicz Theorem we may identify $H_2(\tilde{K}; \mathbf{Z})$ with $\pi_2(\tilde{K}) = \pi_2(K)$. Strictly speaking, we should regard $C_*(K)$ as being a chain complex which is both augmented (by $C_0(\tilde{K}) \rightarrow \mathbf{Z} \rightarrow 0$) and *coaugmented* (by $0 \rightarrow H_2(\tilde{K}; \mathbf{Z}) \rightarrow C_2(\tilde{K})$). The correspondence $K \mapsto C_*(K)$ is functorial for based cellular maps. Thus if $\alpha : K \rightarrow L$ is a based

cellular map there is a commutative diagram of $\mathbf{Z}[G]$ -homomorphisms

$$\begin{array}{ccccccc} C_*(K) & & \left(0 \rightarrow \pi_2(K) \rightarrow C_2(\tilde{K}) \rightarrow C_1(\tilde{K}) \rightarrow \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0 \right) \\ \downarrow \varphi & = & \left(\begin{array}{ccccccc} & & \downarrow \varphi_* & \downarrow \varphi_2 & \downarrow \varphi_1 & \downarrow \varphi_0 & \downarrow Id \\ 0 \rightarrow \pi_2(L) \rightarrow C_2(\tilde{L}) \rightarrow C_1(\tilde{L}) \rightarrow \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0 \end{array} \right). \end{array}$$

Let $\mathcal{G} = \langle x_1, \dots, x_g \mid W_1, \dots, W_r \rangle$ be a finite presentation for G , and let $y = y(x_1, \dots, x_g)$ be a word in the generators. Consider the presentation

$$\mathcal{G}(y) = \langle x_1, \dots, x_{g+1} \mid W_1, \dots, W_{r+1} \rangle$$

where $W_{r+1} = x_{g+1}^{-1}y$. Then $\mathcal{G}(y)$ is also a presentation of G ; the passage from \mathcal{G} to $\mathcal{G}(y)$ is effected by a so-called *Tietze transformation of the first kind*. If $K = \mathcal{K}_{\mathcal{G}}$ is the Cayley complex of \mathcal{G} , we denote by $K(y)$ the Cayley complex of $\mathcal{G}(y)$; $K(y)$ is (simply) homotopy equivalent to K .

Proposition 2.1. *Let K, L be finite reduced 2-complexes with $\pi_1(K) = \pi_1(L) = G$. Then there exist finite reduced 2-complexes K_1, L_1 such that*

- (i) $K_1 \simeq K$ and $L_1 \simeq L$;
- (ii) $K_1^{(1)} \equiv L_1^{(1)}$.

Proof. Let $\{x_1, \dots, x_m\}$ (resp. $\{y_1, \dots, y_n\}$) be the generating sets of G corresponding to the 1-cells of K (resp. L), and regard y_i as a word in $\{x_1, \dots, x_m\}$. We can regard K as the Cayley complex of some presentation. Starting with the generating set $\{x_1, \dots, x_m\}$ and adding the elements y_1, \dots, y_n successively, we obtain a sequence of presentations

$$\mathcal{G}, \mathcal{G}(y_1), \mathcal{G}(y_1, y_2), \dots, \mathcal{G}(y_1, y_2, \dots, y_n),$$

and a sequence of (simple) homotopy equivalences

$$K \subset K(y_1) \subset K(y_1, y_2) \subset \dots \subset K(y_1, y_2, \dots, y_n) = K_1.$$

Likewise, regarding L as the Cayley complex $\mathcal{L} = \mathcal{L}_{\mathcal{H}}$ of some presentation, and adding the elements x_1, \dots, x_m successively, starting with the generating set $\{y_1, \dots, y_n\}$, we obtain a sequence of presentations

$$\mathcal{H}, \mathcal{H}(x_1), \mathcal{H}(x_1, x_2), \dots, \mathcal{H}(x_1, x_2, \dots, x_m),$$

and a sequence of (simple) homotopy equivalences

$$L \subset L(x_1) \subset L(x_1, x_2) \subset \dots \subset L(x_1, x_2, \dots, x_m) = L_1.$$

Then in some ordering, the 1-cells of K_1 and those of L_1 correspond to the generating elements $(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$; that is, we may identify $K_1^{(1)}$ with $L_1^{(1)}$. \square

Proposition 2.2. *Let K, L be finite reduced 2-complexes with $\pi_1(K) = \pi_1(L) = G$, and suppose that $K^{(1)} = L^{(1)}$. If $\nu : C_*(\tilde{K}) \rightarrow C_*(\tilde{L})$ is an augmented $\mathbf{Z}[G]$ -chain map, then ν is chain homotopic to a chain map μ which restricts to the identity on $C_1(\tilde{K}) = C_1(\tilde{L})$.*

Proof. Begin with the chain map

$$\begin{array}{c} C_*(\tilde{K}) \\ \downarrow v \\ C_*(\tilde{L}) \end{array} = \begin{array}{c} \left(\begin{array}{cccc} C_2(\tilde{K}) & \xrightarrow{\partial_2} & C_1(\tilde{K}) & \xrightarrow{\partial_1} & \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0 \\ \downarrow \nu_2 & & \downarrow \nu_1 & & \downarrow \nu_0 \quad \downarrow Id \\ C_2(\tilde{L}) & \xrightarrow{\delta_2} & C_1(\tilde{L}) & \xrightarrow{\partial_1} & \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0 \end{array} \right) \end{array}$$

where, by hypothesis, $C_1(\tilde{L}) = C_1(\tilde{K})$ and $\partial_1^K = \partial_1^L = \partial_1$. Then $\text{Im}(Id - \nu_0) \subset \text{Ker}(\epsilon) = \text{Im}(\partial_1)$. Choose a $\mathbf{Z}[G]$ -homomorphism $h_0 : \mathbf{Z}[G] \rightarrow C_1(\tilde{L})$ such that $\partial_1 h_0 = Id - \nu_0$, and put $\lambda_1 = h_0 \partial_1 + \nu_1$, $\lambda_2 = \nu_2$. We obtain a chain map $\lambda : C_*(\tilde{K}) \rightarrow C_*(\tilde{L})$;

$$\begin{array}{c} C_*(\tilde{K}) \\ \downarrow \lambda \\ C_*(\tilde{L}) \end{array} = \begin{array}{c} \left(\begin{array}{cccc} C_2(\tilde{K}) & \xrightarrow{\partial_2} & C_1(\tilde{K}) & \xrightarrow{\partial_1} & \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0 \\ \downarrow \lambda_2 & & \downarrow \lambda_1 & & \downarrow Id \quad \downarrow Id \\ C_2(\tilde{L}) & \xrightarrow{\delta_2} & C_1(\tilde{L}) & \xrightarrow{\partial_1} & \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0. \end{array} \right) \end{array}$$

Clearly $\text{Im}(Id_1 - \lambda_1) \subset \text{Ker}(\partial_1) = \text{Im}(\delta_2)$. Choose a $\mathbf{Z}[G]$ -homomorphism $h_1 : C_1(\tilde{K}) \rightarrow C_1(\tilde{K})$ such that $\delta_2 h_1 = Id_1 - \lambda_1$. Put $\mu_2 = h_1 \partial_2 + \lambda_2$. Then

$$\begin{aligned} \delta_2 \mu_2 &= \delta_2(h_1 \partial_2 + \lambda_2) \\ &= \delta_2 h_1 \partial_2 + \delta_2 \lambda_2 \\ &= \partial_2 - \lambda_1 \partial_2 + \delta_2 \lambda_2 \\ &= \partial_2. \end{aligned}$$

The following commutative diagram defines the desired chain map μ

$$\begin{array}{cccc} C_2(\tilde{K}) & \xrightarrow{\partial_2} & C_1(\tilde{K}) & \xrightarrow{\partial_1} & \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0 \\ \downarrow \mu_2 & & \downarrow Id & & \downarrow Id \quad \downarrow Id \\ C_2(\tilde{L}) & \xrightarrow{\delta_2} & C_1(\tilde{L}) & \xrightarrow{\partial_1} & \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0, \end{array}$$

and $h = (h_1, h_0)$ is a chain homotopy from μ to ν . □

If $\varphi : C_*(K) \rightarrow C_*(L)$ is an algebraic chain map, by a *realization* of φ we mean a cellular map $f : K \rightarrow L$ such that $C_*(f) = \varphi$. The following Lemma, gives conditions under which some chain maps are realizable. It is modelled on the proof of result of Gutierrez and Latiolais ([11] Prop. (2.3)) in a slightly more general context, and is included here for the sake of completeness.

Lemma 2.3. *Let G be a finitely presented group, and let K, L be finite reduced 2-dimensional cell complexes such that $\pi_1(K) = \pi_1(L) = G$, and $K^{(1)} \cong L^{(1)}$; if $\varphi : C_*(K) \rightarrow C_*(L)$ is an algebraic chain map such that $\varphi|_{C_i(K)} = \text{Id}$ then there exists a cellular map $f : K \rightarrow L$ such that*

- (i) f extends the identity of $K^{(1)} \cong L^{(1)}$ and induces the identity on π_1 , and
- (ii) $C_*(f) = \varphi$.

Proof. By hypothesis, the following diagram commutes:

$$\begin{array}{ccc} C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) \\ \downarrow \varphi_2 & & \parallel \\ C_2(L) & \xrightarrow{\partial_2^L} & C_1(L). \end{array}$$

Begin by choosing some cellular map $\lambda : K \rightarrow L$ with the property that λ extends the identity of $K^{(1)} \equiv L^{(1)}$ and induces the identity on π_1 . For example, we may first attach cells of dimension ≥ 3 to L to kill homotopy groups in dimensions ≥ 3 , so forming a $K(G, 1)$ space with $K(G, 1)^{(2)} = L$. The identity map on $K^{(1)}$ then extends to a map $\lambda : K \rightarrow K(G, 1)$ which classifies the universal G -bundle $\tilde{K} \rightarrow K$ and which induces the identity on π_1 . Without loss of generality, the cellular approximation theorem allows us to choose λ to be cellular, extending the identity on $K^{(1)}$. In particular $\lambda : K \rightarrow \lambda(K) \subset K(G, 1)^{(2)} = L$.

The chain map $C_*(\lambda)$ induced by λ makes the following diagram commute:

$$\begin{array}{ccc} C_2(K) & \xrightarrow{\partial_2^K} & C_1(K) \\ \downarrow \lambda_2 & & \parallel \\ C_2(L) & \xrightarrow{\partial_2^L} & C_1(L). \end{array}$$

In particular, $\partial_2^L(\varphi_2 - \lambda_2) = 0$, and since the Hurewicz Theorem allows us to identify $\text{Ker}(\partial_2^L) = H_2(L; \mathbf{Z})$ can be identified with $\pi_2(L)$, then $\varphi_2 - \lambda_2$ gives a $\mathbf{Z}[G]$ -homomorphism

$$\varphi_2 - \lambda_2 : C_2(K) \rightarrow \pi_2(L).$$

Let $\{E_1, \dots, E_r\}$ denote the 2-cells and let $\{\hat{E}_1, \dots, \hat{E}_r\}$ denote the $\mathbf{Z}[G]$ -basis of $C_2(K)$ obtained by to \tilde{K} ; let $h_i : S^2 \rightarrow L$ be a map whose homotopy class $[h_i]$ satisfies

$$[h_i] = (\varphi_2 - \lambda_2)(\hat{E}_i).$$

Let $c_i : E_i \rightarrow E_i \vee S_i^2$ denote a ‘pinch map’ obtained by identifying an interior circle in E_i to a point. We may write

$$K = \left(\coprod_i E_i \right) / \sim$$

where ‘ \sim ’ makes identifications in the boundaries of the E_i . Put

$$X = \left(\coprod_i E_i \vee S_i^2 \right) / \sim$$

where ‘ \sim ’ makes the same identifications as before, observing that E_i and $E_i \vee S_i^2$ have the same boundary. One may think of X as formed from K by attaching a copy of S^2 at a single interior point within each 2-cell of K . Put $\lambda_i = \lambda|_{E_i}$ and define

$$c = \left(\coprod_i c_i \right) / \sim : K \rightarrow X \quad ; \quad \Psi = \left(\coprod_i \lambda_i \vee h_i \right) / \sim : X \rightarrow L,$$

and put $f = \Psi \circ c : K \rightarrow L$. f is a basepoint preserving map ; it is straightforward to see that f extends the identity on $K^{(1)}$ and induces the identity on π_1 . If $f_n : C_n(K) \rightarrow C_n(L)$ denotes the induced map on n -chains, one calculates easily that

$$f_2(\hat{E}_i) = \lambda_2(\hat{E}_i) + [h_i]$$

where we identify $[h_i] \in \pi_2(L)$ with its image in $C_2(L)$. However, by choice, $(\varphi_2 - \lambda_2)(\hat{E}_i) = [h_i]$. Thus

$$\begin{aligned} f_2(\hat{E}_i) &= \lambda_2(\hat{E}_i) + (\varphi_2 - \lambda_2)(\hat{E}_i) \\ &= \varphi_2(\hat{E}_i) \end{aligned}$$

and $f_2 = \varphi_2$ as required. □

Proposition 2.4. *Let K, L be finite reduced 2-complexes with $\pi_1(K) = \pi_1(L) = G$, and suppose that $\varphi : C_*(\tilde{K}) \rightarrow C_*(\tilde{L})$ is an algebraic chain homotopy equivalence; then there exists a cellular homotopy equivalence $f : K \rightarrow L$ whose induced map on the fundamental group is the identity.*

Proof. By (2.1), we may, without loss of generality, assume that $K^{(1)} = L^{(1)}$. Now $H_0(\tilde{K}; \mathbf{Z})$ and $H_0(\tilde{L}; \mathbf{Z})$ are canonically isomorphic to \mathbf{Z} , so that the isomorphism $\varphi_* : H_0(\tilde{K}; \mathbf{Z}) \rightarrow H_0(\tilde{L}; \mathbf{Z})$ is $\pm \text{Id}$. By replacing φ by $-\varphi$, if necessary, we may suppose that φ induces an isomorphism on H_0 .

Now by (2.2), we may further assume that μ restricts to the identity on $C_1(\tilde{K}) = C_1(\tilde{L})$. Finally, applying (2.3), we obtain a cellular map $f : K \rightarrow L$ inducing the identity on the fundamental group and such that $f_* = \mu : H_2(\tilde{K}; \mathbf{Z}) \rightarrow H_2(\tilde{L}; \mathbf{Z})$. In particular, $f_* : H_2(\tilde{K}; \mathbf{Z}) \rightarrow H_2(\tilde{L}; \mathbf{Z})$ is an isomorphism, so that f is a homotopy equivalence, by Whitehead's Theorem. □

As a corollary, we have:

Algebraicity Theorem. *Let G be a finitely presented group, and let K, L be connected finite based 2-complexes with $\pi_1(K) = \pi_1(L) = G$; then K is homotopy equivalent to L if and only if $C_*(K)$ is chain homotopy equivalent to $C_*(L)$.*

3. Stable modules and derived functors

Throughout rest of the paper, G will denote a *finite* group, although at the more important points we shall re-emphasize the fact; we denote by $\mathcal{F}(\mathbf{Z}[G])$ the category of $\mathbf{Z}[G]$ lattices, that is, finitely generated (right) $\mathbf{Z}[G]$ -modules whose underlying \mathbf{Z} -modules are free. We let ' \sim ' denote the stability relation on $\mathcal{F}(\mathbf{Z}[G])$; that is

$$M_1 \sim M_2 \iff M_1 \oplus \mathbf{Z}[G]^{n_1} \cong M_2 \oplus \mathbf{Z}[G]^{n_2}$$

for some $n_1, n_2 \geq 0$, and put

$$\text{Stab}(\mathbf{Z}[G]) = \mathcal{F}(\mathbf{Z}[G]) / \sim.$$

For any $\mathbf{Z}[G]$ -lattice M , we denote by $[M] \in \text{Stab}(\mathbf{Z}[G])$ the class of M under \sim .

Recall Swan's generalization of 'Schanuel's Lemma' [22]; if

$$0 \rightarrow B \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow A \rightarrow 0,$$

and

$$0 \rightarrow B' \rightarrow F'_n \rightarrow \cdots \rightarrow F'_0 \rightarrow A' \rightarrow 0$$

are exact sequences of $\mathbf{Z}[G]$ -modules in which F_i, F'_j are finitely generated and free, then $A \cong A'$ implies that $B \oplus F \cong B' \oplus F'$ for some finitely generated free $\mathbf{Z}[G]$ -modules F, F' . To each module $M \in \mathcal{F}(\mathbf{Z}[G])$, we can associate a sequence $(\Omega_r)_{r \geq 0}$ of stable modules defined by the condition that $\Omega_n(M)$ is the stable class $[D]$ of any module $D \in \mathcal{F}(\mathbf{Z}[G])$ for which there exists an exact sequence of the form

$$0 \rightarrow D \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

where each F_r is a finitely generated free module over $\mathbf{Z}[G]$. Likewise, to each module $M \in \mathcal{F}(\mathbf{Z}[G])$, we associate a sequence $(\Omega_{-r}(M))_{r \geq 0}$ of stable modules defined by the condition that $\Omega_{-n}(M)$ is the stable class $[D]$ of any module $D \in \mathcal{F}(\mathbf{Z}[G])$ for which there exists an exact sequence of the form

$$0 \rightarrow M \rightarrow F_0 \rightarrow \cdots \rightarrow F_{n-1} \rightarrow D \rightarrow 0$$

where each F_r is a finitely generated free module over $\mathbf{Z}[G]$.

We introduce duality into $\mathcal{F}(\mathbf{Z}[G])$ as follows; when $M \in \mathcal{F}(\mathbf{Z}[G])$ we denote by M^* the right $\mathbf{Z}[G]$ -module whose underlying \mathbf{Z} -module is $\text{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ and whose G -action is given by

$$(f \bullet g)(m) = f(mg^{-1}).$$

Under our hypothesis that G is finite, M^* is a model for the $\mathbf{Z}[G]$ -dual of M . This is a special case of the 'Eckmann–Shapiro Lemma'. As an explicit model for $\text{Hom}_{\mathbf{Z}[G]}(M, \mathbf{Z}[G])$ we may take sequences $\alpha = (\alpha_x)_{x \in G}$, with each $\alpha_x \in \text{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$, satisfying the condition $\alpha_{gh}(m) = \alpha_h(mg^{-1})$. In this case, the right G -action is given by $(\alpha \bullet x)_g = \alpha_{xg}$, and the mapping

$$L : \text{Hom}_{\mathbf{Z}[G]}(M, \mathbf{Z}[G]) \rightarrow M^*; \quad L((\alpha_g)_{g \in G}) = \alpha_1$$

is an isomorphism.[†] There is now a natural equivalence $\nu : M \rightarrow M^{**}$ given by $\nu(x)(\alpha) = \alpha(x)$. Duality is introduced into $\text{Stab}(\mathbf{Z}[G])$ by writing $[M]^* = [M^*]$.

Proposition 3.1. *For any module $M \in \mathcal{F}(\mathbf{Z}[G])$ we have the following relations:*

- (i) $\Omega_m(\Omega_n(M)) = \Omega_{m+n}(M)$;
- (ii) $\Omega_n(M^*) = \Omega_{-n}(M)^*$.

[†] The referee emphasizes the point that this ceases to be true if the hypothesis that G be finite is dropped. A good example is obtained by taking $M = \mathbf{Z}$, the trivial module over the infinite cyclic group \mathbf{T} . Then $M^* \cong M$ but $\text{Hom}_{\mathbf{Z}[\mathbf{T}]}(M, \mathbf{Z}[\mathbf{T}]) = 0$.

The trivial module \mathbf{Z} is self-dual over $\mathbf{Z}[G]$, so that:

Proposition 3.2. *For any finite group G ,*

- (i) $\Omega_1(\mathbf{Z}) = [\mathbf{I}(G)];$
- (ii) $\Omega_{-1}(\mathbf{Z}) = [\mathbf{I}(G)^*].$

We wish to interpret the constructions Ω_k in terms of derived categories. This is essentially well known. In a different, though parallel, context, there is an elegant exposition in the monograph of Carlson [5]. A systematic treatment of group cohomology in the exact context required for this paper can be found in [14].

Let $f, g : M \rightarrow N$ be $\mathbf{Z}[G]$ -homomorphisms where $M, N \in \mathcal{F}(\mathbf{Z}[G])$. We write $f \approx g$ when $f - g$ factors through a projective module; that is, when $f - g$ can be written as a composite $f - g = \beta \circ \alpha$, of $\mathbf{Z}[G]$ -homomorphisms $\alpha : M \rightarrow P$ and $\beta : P \rightarrow N$, where P is projective over $\mathbf{Z}[G]$. It is straightforward to check that \approx is an equivalence relation compatible with addition and two sided composition. By the *derived category* $\mathcal{D}\text{er}(\mathcal{F}(\mathbf{Z}[G]))$ of $\mathcal{F}(\mathbf{Z}[G])$, we mean the category whose objects are modules in $\mathcal{F}(\mathbf{Z}[G])$, and in which, for any two modules M, N , the set of morphisms $\text{Hom}_{\mathcal{D}\text{er}}(M, N)$ is given by

$$\text{Hom}_{\mathcal{D}\text{er}}(M, N) = \text{Hom}_{\mathbf{Z}[G]}(M, N) / \approx .$$

Let $M_1, M_2 \in \mathcal{F}(\mathbf{Z}[G])$; it is clear that if $M_1 \oplus \mathbf{Z}[G]^{m_1} \cong M_2 \oplus \mathbf{Z}[G]^{m_2}$ then M_1 and M_2 are isomorphic in the derived category. Thus each stable module $\Omega_k(M)$ gives rise to an isomorphism class, denoted by the same symbol, in the derived category. The traditional interpretation of $H^k(-, -)$ as the k^{th} derived functor of $\text{Hom}(-, -)$ can be made explicit in this context (see [13]).

Proposition 3.3. *On the category $\mathcal{F}(\mathbf{Z}[G])$, for all $k \geq 1$ there is a natural equivalence of functors $\nu : H^k(M, N) \rightarrow \text{Hom}_{\mathcal{D}\text{er}}(\Omega_k(M), N)$.*

In particular, we see that, in the derived category, $\Omega_k(M)$ is a *corepresenting object* for the functor $X \mapsto H^k(M, X)$. Moreover, this statement remains true for $k = 0$ provided $H^0(M, N)$ is 'correctly interpreted' as Tate cohomology. We note that, $\Omega_1(\mathbf{Z})$ is the stable class of the augmentation ideal, and $\Omega_2(\mathbf{Z})$ is the stable class of the so-called 'second Fox ideal' of a presentation. Since the category $\mathcal{F}(\mathbf{Z}[G])$ has good duality properties, one sees also that:

Proposition 3.4. *On the category $\mathcal{F}(\mathbf{Z}[G])$ there are natural equivalences*

$$\nu : H^k(M, N) \rightarrow \text{Hom}_{\mathcal{D}\text{er}}(M, \Omega_{-k}(N)).$$

all $k \geq 1$; that is, in the derived category, $\Omega_{-k}(N)$ is a representing object for the functor $X \mapsto H^k(X, N)$.

minimal rank has the cancellation property. It follows easily that:

Proposition 3.6. $\Omega_{2n+1}(\mathbf{Z})$ is a fork for each $n \geq 1$.

4. Module extensions

We begin by summarizing Yoneda's cohomological classification of module extensions [32]; in fact, we shall need a relative version of Yoneda's extension theory, where all exact sequences are taken over the category $\mathcal{F}(\mathbf{Z}[G])$. This is also well known in principle, but it is difficult to find an account in the literature which covers our requirements exactly. In fact, the precise details required here may be found in [14]. Thus let M, N be modules in $\mathcal{F}(\mathbf{Z}[G])$; then $\mathbf{Ext}_{\mathcal{F}(\mathbf{Z}[G])}^n(M, N)$ will denote the category whose objects are exact sequences of $\mathbf{Z}[G]$ -modules and homomorphisms

$$\mathbf{E} = (0 \rightarrow N \rightarrow E_{n-1} \rightarrow \cdots \cdots \rightarrow E_0 \rightarrow M \rightarrow 0)$$

in which all modules lie in $\mathcal{F}(\mathbf{Z}[G])$, and whose morphisms are commutative diagrams of $\mathbf{Z}[G]$ -homomorphisms thus:

$$\begin{array}{c} \mathbf{E} \\ \downarrow h \\ \mathbf{F} \end{array} = \begin{pmatrix} 0 \rightarrow N \rightarrow E_{n-1} \rightarrow \cdots \cdots \rightarrow E_0 \rightarrow M \rightarrow 0 \\ \downarrow h_+ \quad \downarrow h_{n-1} \qquad \qquad \qquad \downarrow h_0 \quad \downarrow h_- \\ 0 \rightarrow N \rightarrow F_{n-1} \rightarrow \cdots \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \end{pmatrix}.$$

When G is understood, we abbreviate $\mathbf{Ext}_{\mathcal{F}(\mathbf{Z}[G])}^n(M, N)$ to $\mathbf{Ext}^n(M, N)$.

If $\mathbf{E}, \mathbf{F} \in \mathbf{Ext}^n(A, B)$, a morphism $\varphi : E \rightarrow F$ is said to be an *elementary congruence* when it induces the identity at both ends thus:

$$\begin{array}{c} \mathbf{E} \\ \downarrow \varphi \\ \mathbf{F} \end{array} = \begin{pmatrix} 0 \rightarrow B \rightarrow E_{n-1} \rightarrow \cdots \cdots \rightarrow E_0 \rightarrow A \rightarrow 0 \\ \downarrow \text{Id} \quad \downarrow \varphi_{n-1} \qquad \qquad \qquad \downarrow \varphi_0 \quad \downarrow \text{Id} \\ 0 \rightarrow B \rightarrow F_{n-1} \rightarrow \cdots \cdots \rightarrow F_0 \rightarrow A \rightarrow 0 \end{pmatrix}.$$

We write $\mathbf{E} \rightsquigarrow \mathbf{F}$ when there exists an elementary congruence $\varphi : \mathbf{E} \rightarrow \mathbf{F}$. In the case $n = 1$, it is a consequence of the Five Lemma that elementary congruence is an equivalence relation on $\mathbf{Ext}^1(M, N)$, and we write:

$$\text{Ext}^1(M, N) = \mathbf{Ext}^1(M, N) / \rightsquigarrow .$$

By contrast, \rightsquigarrow is not an equivalence relation when $n \geq 2$; in this case, we denote by ' \equiv ' the equivalence relation generated by ' \rightsquigarrow '; that is $\mathbf{E}, \mathbf{F} \in \mathbf{Ext}^n(M, N)$ are said to be *congruent*, written $\mathbf{E} \equiv \mathbf{F}$, when there exists a sequence $\mathbf{E}_r, (0 \leq r \leq m) \in \mathbf{Ext}^n(M, N)$ such that $\mathbf{E} = \mathbf{E}_0, \mathbf{F} = \mathbf{E}_m$, and for each $r \leq m - 1$, either $\mathbf{E}_r \rightsquigarrow \mathbf{E}_{r+1}$ or $\mathbf{E}_{r+1} \rightsquigarrow \mathbf{E}_r$. We denote by $\text{Ext}^n(M, N)$ the quotient set

$$\text{Ext}^n(M, N) = \mathbf{Ext}^n(M, N) / \equiv .$$

Suppose that $K \in \Omega_n(M)$, and fix an extension

$$\mathbf{F} = (0 \rightarrow K \rightarrow F_{n-1} \rightarrow \cdots \cdots \rightarrow F_0 \rightarrow M \rightarrow 0)$$

where each F_i is finitely generated free $\mathbf{Z}[G]$ -module. Suppose that

$$\mathbf{E} = (0 \rightarrow N \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_0 \rightarrow M \rightarrow 0) \in \mathbf{Ext}^n(M, N)$$

The universal property of free modules gives a commutative diagram of $\mathbf{Z}[G]$ -modules thus:

$$\begin{array}{c} \mathbf{F} \\ \downarrow h \\ \mathbf{E} \end{array} = \begin{pmatrix} 0 \rightarrow K \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \\ \downarrow h_{\mathbf{E}}^{\mathbf{F}} \quad \downarrow h_{n-1} \quad \quad \quad \downarrow h_0 \quad \downarrow \text{Id} \\ 0 \rightarrow N \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_0 \rightarrow M \rightarrow 0 \end{pmatrix}.$$

The essence of Yoneda's Theorem [32], [14] is that, for fixed \mathbf{F} , the correspondence $\mathbf{E} \mapsto h_{\mathbf{E}}^{\mathbf{F}} \in \text{Hom}_{\mathbf{Z}[G]}(K, N)$ gives rise to a bijection

$$\text{Ext}^n(M, N) \xrightarrow{\cong} \text{Hom}_{\mathcal{D}_{\text{er}}}(K, N) \cong \text{Hom}_{\mathcal{D}_{\text{er}}}(\Omega_n(M), N).$$

Using the corepresentability result of (3.3), we obtain the following version of Yoneda's Theorem:

Proposition 4.1. *For each $M \in \mathcal{F}(\mathbf{Z}[G])$, the functors $N \mapsto \text{Ext}^n(M, N)$ and $N \mapsto H^n(M, N)$ are naturally equivalent for all $n \geq 1$.*

This bijection imposes a (slightly unnatural) group structure on $\text{Ext}^n(M, N)$. We note however, that in the case $n = 1$, the group structure is entirely natural, being given by Baer sum ([18] Chap. 3), and in that case the Yoneda map is an equivalence of group valued functors.

For any $M, N \in \mathcal{F}(\mathbf{Z}[G])$ we denote by $\mathbf{Free}^n(M, N)$ the subclass of $\mathbf{Ext}^n(M, N)$ consisting of extensions of the form

$$0 \rightarrow N \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$$

with $A_i \in \mathcal{F}(\mathbf{Z}[G])$ in which each A_i is free. Likewise $\mathbf{Proj}^n(M, N)$ (resp. $\mathbf{SF}^n(M, N)$) will denote the subclass of $\mathbf{Ext}^n(M, N)$ which each A_i is projective (resp. stably free). We have inclusions

$$\mathbf{Free}^n(M, N) \subset \mathbf{SF}^n(M, N) \subset \mathbf{Proj}^n(M, N) \subset \mathbf{Ext}^n(M, N).$$

Proposition 4.2. *Let $M, N \in \mathcal{F}(\mathbf{Z}[G])$; then for each $n \geq 1$, the following conditions are equivalent:*

- (i) $\mathbf{SF}^n(M, N) \neq \emptyset$;
- (ii) $N \in \Omega_n(M)$;
- (iii) $M \in \Omega_{-n}(N)$.

Moreover, when either $n \geq 2$ or $n = 1$ and G has the cancellation property for free modules, these conditions are equivalent to

- (iv) $\mathbf{Free}^n(M, N) \neq \emptyset$.

Proof. We first assume that $n \geq 2$ and prove the equivalence of (i)–(iv). The equivalence of (ii) and (iii) is obvious, as are the implications (iv) \implies (i) and (iv) \implies (ii). Thus it suffices to show that (ii) \implies (i), and (i) \implies (iv).

(ii) \implies (i). First note that if F is a finitely generated free module over $\mathbf{Z}[G]$, and $F_1 \subset F$ is a free $\mathbf{Z}[G]$ submodule, then F/F_1 is stably free over $\mathbf{Z}[G]$ if and only if F/F_1 is torsion free over \mathbf{Z} ; this is an easy argument using the injectivity of F_1 relative to $\mathcal{F}(\mathbf{Z}[G])$ or, what is the same, by double dualization using the universal property of free modules. Now suppose that $N \in \Omega_n(M)$; by definition, there exists a module D which occurs in an exact sequence

$$0 \rightarrow D \rightarrow \mathbf{Z}[G]^\gamma \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0.$$

with $\gamma > 0$, each F_r finitely generated free over $\mathbf{Z}[G]$, such that

$$N \oplus \mathbf{Z}[G]^a \cong D \oplus \mathbf{Z}[G]^b$$

for some $a, b \geq 0$. Now $D \oplus \mathbf{Z}[G]^b$ occurs in the exact sequence

$$0 \rightarrow D \oplus \mathbf{Z}[G]^b \rightarrow \mathbf{Z}[G]^{\gamma+b} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0,$$

so that $N \oplus \mathbf{Z}[G]^a$ occurs in the exact sequence

$$0 \rightarrow N \oplus \mathbf{Z}[G]^a \rightarrow \mathbf{Z}[G]^{\gamma+b} \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0,$$

We split this last sequence as a pair thus:

$$(I) \quad 0 \rightarrow N \oplus \mathbf{Z}[G]^a \rightarrow \mathbf{Z}[G]^{\gamma+b} \rightarrow K \rightarrow 0$$

$$(II) \quad 0 \rightarrow K \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Dividing through (I) by $\mathbf{Z}[G]^m$ gives an extension

$$(III) \quad 0 \rightarrow N \rightarrow \mathbf{Z}[G]^{\gamma+b}/\mathbf{Z}[G]^a \rightarrow K \rightarrow 0$$

in which $N, K \in \mathcal{F}(\mathbf{Z}[G])$. Thus, $\mathbf{Z}[G]^{\gamma+b}/\mathbf{Z}[G]^a$ is torsion free over \mathbf{Z} , so that, by the remark above, $\mathbf{Z}[G]^{\gamma+b}/\mathbf{Z}[G]^a = S$ is stably free over $\mathbf{Z}[G]$. Splicing (II) and (III) gives an element $0 \rightarrow N \rightarrow S \rightarrow F_{n-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$, of $\mathbf{SF}^n(M, N)$ as required. This completes the proof that (ii) \implies (i).

(i) \implies (iv). Suppose that $0 \rightarrow N \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S_0 \rightarrow M \rightarrow 0$ is an element of $\mathbf{SF}^n(M, N)$ where each S_i is finitely generated stably free. If $n = 2$ (this is only place where we need the hypothesis that $n \geq 2$) then there exists $a > 0$ so that $S_i \oplus \mathbf{Z}[G]^a$ is free for $i = 1, 2$. Adding a summand of $\mathbf{Z}[G]^a$ to the middle terms gives an extension

$$0 \rightarrow N \rightarrow S_1 \oplus \mathbf{Z}[G]^a \rightarrow S_0 \oplus \mathbf{Z}[G]^a \rightarrow M \rightarrow 0$$

which defines an element of $\mathbf{Free}^2(M, N)$. If $n > 2$, we may first add a suitable free summand to S_1 and S_0 to make S_0 free; the result follows by induction after an easy splicing argument. This completes the proof that (i) \implies (iv).

Now suppose that $n = 1$; again the equivalence of (ii) and (iii) is obvious, and the proof that (ii) \implies (i) given above goes through with a a minor change of indexing, so that it suffices to show that (i) \implies (ii). Thus suppose that $0 \rightarrow N \rightarrow S \rightarrow M \rightarrow 0$ is an extension in which S is a finitely generated stably free module. Then for some $a \geq 0$, $S \oplus \mathbf{Z}[G]^a$ is free, and from the exact sequence

$$0 \rightarrow N \oplus \mathbf{Z}[G]^a \rightarrow S \oplus \mathbf{Z}[G]^a \rightarrow M \rightarrow 0$$

we see that $N \oplus \mathbf{Z}[G]^a \in \Omega_1(M)$. Thus $N \in \Omega_1(M)$, showing that (i) \implies (ii), and hence the equivalence of (i), (ii) and (iii) in the case $n = 1$.

Finally, if G has the cancellation property for free modules then (i) is equivalent to (iv). \square

We denote by $\text{Proj}^n(M, N)$ (resp. $\text{SF}^n(M, N)$, $\text{Free}^n(M, N)$) the image of $\mathbf{Proj}^n(M, N)$ (resp. $\mathbf{SF}^n(M, N)$, $\mathbf{Free}^n(M, N)$) under the canonical map

$$\mathbf{Ext}^n(M, N) \rightarrow \text{Ext}^n(M, N).$$

Evidently $\text{Free}^n(M, N) \subset \text{SF}^n(M, N)$; one may extend the argument of (4.2) to show that for $n \geq 2$, $\mathbf{SF}^n(M, N)$ and $\mathbf{Free}^n(M, N)$ represent the same congruence classes; that is:

(4.3) For all $n \geq 2$, $\text{Free}^n(M, N) = \text{SF}^n(M, N)$.

For any finite group G , we proceed to give a description of $\text{Ext}^n(\mathbf{Z}, \Omega_n(\mathbf{Z})) \cong H^n(\mathbf{Z}, \Omega_n(\mathbf{Z}))$ when \mathbf{Z} denotes the trivial module over $\mathbf{Z}[G]$ and $N \in \Omega_n(\mathbf{Z})$.

We begin by computing the ring $\text{End}_{\mathcal{D}\text{er}}(\mathbf{Z})$. Note that $\text{End}_{\mathbf{Z}[G]}(\mathbf{Z}) \cong \mathbf{Z}$ under the isomorphism $\alpha \mapsto \alpha(1)$; also there are $\mathbf{Z}[G]$ -homomorphisms $\Sigma : \mathbf{Z} \rightarrow \mathbf{Z}[G]$ and $\epsilon : \mathbf{Z}[G] \rightarrow \mathbf{Z}$ given by

$$\Sigma(n) = n \left(\sum_{g \in G} g \right) ; \quad \epsilon \left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g.$$

Moreover,

(4.4) If $\varphi : \mathbf{Z} \rightarrow \mathbf{Z}[G]$ is a $\mathbf{Z}[G]$ -homomorphism then $\varphi = n\Sigma$ for some $n \in \mathbf{Z}$;

(4.5) If $\psi : \mathbf{Z}[G] \rightarrow \mathbf{Z}$ is a $\mathbf{Z}[G]$ -homomorphism then $\psi = n\epsilon$ for some $n \in \mathbf{Z}$.

Suppose $\alpha : \mathbf{Z} \rightarrow \mathbf{Z}$ factors through $\mathbf{Z}[G]^m$ for some $m \geq 1$ thus:

$$\begin{array}{ccc} & \mathbf{Z}[G]^m & \\ \varphi \nearrow & & \searrow \psi \\ \mathbf{Z} & \xrightarrow{\alpha} & \mathbf{Z} \end{array}$$

We may write $\varphi = (a_1 \Sigma, \dots, a_m \Sigma)$ and $\psi = \begin{pmatrix} b_1 \epsilon \\ \vdots \\ b_m \epsilon \end{pmatrix}$ so that

$$\alpha = \left(\sum_i a_i b_i \right) \epsilon \Sigma = \left(\sum_i a_i b_i \right) |G|.$$

Conversely, if $\alpha = a|G|$ then α factors through $\mathbf{Z}[G]$ since $\alpha = a\epsilon \circ \Sigma$. Thus we see that

(4.6) α factors through some $\mathbf{Z}[G]^m \iff \alpha = n|G|$ for some n .

Clearly α factors through a projective if and only if it factors through some $\mathbf{Z}[G]^n$, so that

Proposition 4.7. *Let \mathbf{Z} denote the trivial module over $\mathbf{Z}[G]$ where G is a finite group; then there is an isomorphism of rings*

$$\text{End}_{\mathcal{D}\text{er}}(\mathbf{Z}) \cong \mathbf{Z}/|G|.$$

However, each Ω_k is a self-equivalence of the derived category ([13]), so that more generally we have:

Proposition 4.8. *If $N \in \Omega_k(\mathbf{Z})$ then there is an isomorphism of rings*

$$\text{End}_{\mathcal{D}\text{er}}(N) \cong \mathbf{Z}/|G|.$$

If $M \in \mathcal{F}(\mathbf{Z}[G])$, if $N \in \Omega_n(M)$ then $H^n(M, N) \cong H^n(M, \Omega_n(M))$, and by the corepresentation formula (3.3), $H^n(M, \Omega_n(M)) \cong \text{Hom}_{\mathcal{D}\text{er}}(\Omega_n(M), \Omega_n(M))$. Since Ω_n is a self-equivalence of $\mathcal{D}\text{er}(\mathcal{F}(\mathbf{Z}[G]))$,

$$\text{Hom}_{\mathcal{D}\text{er}}(\Omega_n(M), \Omega_n(M)) \cong \text{Hom}_{\mathcal{D}\text{er}}(M, M) \cong \text{End}_{\mathcal{D}\text{er}}(M).$$

Thus in the case where $M = \mathbf{Z}$ we obtain

$$(4.9) \quad \text{Ext}^n(\mathbf{Z}, N) \cong \mathbf{Z}/|G|$$

when $N \in \Omega_n(\mathbf{Z})$, and more generally, by dimension shifting,

$$(4.10) \quad \text{Ext}^n(M, N) \cong \mathbf{Z}/|G|$$

for $M \in \Omega_m(\mathbf{Z})$ and $N \in \Omega_n(M)$.

The explicit parametrization of the congruence classes in (4.10) is effected by the method of 'k-invariants' thus; let $M \in \Omega_m(\mathbf{Z})$ and $N \in \Omega_n(M)$. Since $N \in \Omega_{m+n}(\mathbf{Z})$, there is a ring isomorphism $\kappa : \text{End}_{\mathcal{D}\text{er}}(N) \rightarrow \mathbf{Z}/|G|$. If $\mathbf{P} \in \mathbf{Proj}^n(M, N)$ and \mathbf{E} is an arbitrary element of $\mathbf{Ext}^n(M, N)$, the universal property of projective modules leads to a commutative diagram of $\mathbf{Z}[G]$ -modules

thus:

$$\begin{array}{c} \mathbf{P} \\ \downarrow \alpha \\ \mathbf{E} \end{array} = \begin{pmatrix} 0 \rightarrow N \rightarrow P_{n-1} \rightarrow \cdots \cdots \rightarrow P_0 \rightarrow M \rightarrow 0 \\ \downarrow \alpha_+ \quad \downarrow \alpha_{n-1} \qquad \qquad \qquad \downarrow \alpha_0 \quad \downarrow \text{Id} \\ 0 \rightarrow N \rightarrow E_{n-1} \rightarrow \cdots \cdots \rightarrow E_0 \rightarrow M \rightarrow 0 \end{pmatrix}.$$

We define the k -invariant of the transition, $k(\mathbf{P} \rightarrow \mathbf{E}) \in \mathbf{Z}/|G|$, by

$$k(\mathbf{P} \rightarrow \mathbf{E}) = \kappa(\alpha_+).$$

The k -invariant construction has the following easily verified properties:

Proposition 4.11. *Let $\mathbf{P}, \mathbf{Q} \in \mathbf{Proj}^n(M, N)$, and let $\mathbf{E} \in \mathbf{Ext}^n(M, N)$; then*

- (i) $k(\mathbf{P} \rightarrow \mathbf{E}) = k(\mathbf{P} \rightarrow \mathbf{Q})k(\mathbf{Q} \rightarrow \mathbf{E})$;
- (ii) $k(\mathbf{P} \rightarrow \mathbf{P}) = 1$;
- (iii) $k(\mathbf{P} \rightarrow \mathbf{Q})k(\mathbf{Q} \rightarrow \mathbf{P}) = 1$;
- (iv) $\mathbf{E} \in \mathbf{Proj}^n(M, N) \iff k(\mathbf{P} \rightarrow \mathbf{E}) \in (\mathbf{Z}/|G|)^*$.

Making a specific choice of projective extension $\mathbf{P} \in \mathbf{Proj}^n(M, N)$ we see that:

Proposition 4.12. *The correspondence $\mathbf{E} \mapsto k(\mathbf{P} \rightarrow \mathbf{E})$ defines a bijective mapping $\mathbf{Proj}^n(M, N) \rightarrow (\mathbf{Z}/|G|)^*$ for $M \in \Omega_m(\mathbf{Z})$ and $N \in \Omega_n(M)$.*

5. Algebraic 2-complexes

If K is a finite connected 2-complex with $\pi_1(K) = G$, then the universal covering \tilde{K} is simply connected, so that the cellular chain complex $C_*(\tilde{K})$ can be written as an exact sequence of $\mathbf{Z}[G]$ modules thus:

$$C_*(\tilde{K}) = (0 \rightarrow \pi_2(K) \rightarrow C_2(\tilde{K}) \xrightarrow{\partial_2} C_1(\tilde{K}) \xrightarrow{\partial_1} C_0(\tilde{K}) \rightarrow \mathbf{Z} \rightarrow 0).$$

In particular, the second homotopy group $\pi_2(K)$ is naturally a $\mathbf{Z}[G]$ -module belonging to the stable module $\Omega_3(\mathbf{Z})$, and the cell complex itself determines an element of $\mathbf{Free}^3(\mathbf{Z}, \pi_2(K))$. We have seen already in (4.3) that the set $\mathbf{Free}^3(\mathbf{Z}, \pi_2(K))$, of congruence classes of free extensions coincides with the corresponding set $\mathbf{SF}^3(\mathbf{Z}, \pi_2(K))$ formed from stably free extensions. By an *algebraic 2-complex* we shall mean an extension of the form $\mathbf{E} = (0 \rightarrow J \xrightarrow{j} E_2 \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \rightarrow \mathbf{Z} \rightarrow 0)$ where each E_i is finitely generated stably free; in particular, $J \in \Omega_3(\mathbf{Z})$ and $\mathbf{E} \in \mathbf{SF}^3(\mathbf{Z}, J)$.

For any $n \geq 1$, the n -fold stabilization of \mathbf{E} is the complex

$$\mathcal{S}^n(\mathbf{E}) = (0 \rightarrow J \oplus \mathbf{Z}[G]^n \xrightarrow{j \oplus \text{Id}} E_2 \oplus \mathbf{Z}[G]^n \xrightarrow{\partial_2 \circ \pi} E_1 \xrightarrow{\partial_1} E_0 \rightarrow \mathbf{Z} \rightarrow 0)$$

where $\pi : E_2 \oplus \mathbf{Z}[G]^n \rightarrow E_2$ is the projection.

This corresponds to the following geometrical construction; if K is a finite 2-complex, we denote by $\Sigma^n(K)$ the complex obtained from the one point union of

K with a collection of n copies of the 2-sphere;

$$\Sigma^n(K) = K \vee \underbrace{S^2 \vee \dots \vee S^2}_n.$$

It is straightforward to check that $C_*(\widetilde{\Sigma^n(K)}) \cong S^n(C_*(\widetilde{K}))$.

The discussion can be carried out in the context of group presentations: if $\mathcal{G} = \langle x_1, \dots, x_g \mid W_1, \dots, W_r \rangle$ is a finite presentation of G , we denote by $C_*(\mathcal{G})$ the cellular chain complex of $\widetilde{\mathcal{K}_G}$ where \mathcal{K}_G is the Cayley complex of \mathcal{G} :

$$C_*(\mathcal{G}) = (0 \rightarrow \pi_2(\mathcal{G}) \rightarrow C_2(\widetilde{\mathcal{K}_G}) \xrightarrow{\partial_2} C_1(\widetilde{\mathcal{K}_G}) \xrightarrow{\partial_1} \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0).$$

It is clear from Schanuel's Lemma that the module $\pi_2(\mathcal{G}) = \pi_2(K_G)$ is stably unique and belongs to $\Omega_3(\mathbf{Z})$. If we wish to emphasise the group presentational aspects, as against the module theoretic aspects, of the situation we could also, as was essentially done by Fox [9], observe that the stable uniqueness of $\pi_2(\mathcal{G})$ already follows from Tietze's Theorem [28]. In any case, denoting by $\pi_2(G)$ the stable module $[\pi_2(\mathcal{G})]$, we have:

Proposition 5.1. *For any finite group G , $\pi_2(G) = \Omega_3(\mathbf{Z})$.*

The process of stabilization also has an interpretation in terms of group presentations; we denote by $\Sigma^n \mathcal{G}$ any presentation of G obtained from \mathcal{G} by adding n trivial relations. It is straightforward to see that $C_*(\Sigma^n \mathcal{G})$ is congruent to $S^n(C_*(\mathcal{G}))$.

A morphism $h : \mathbf{E} \rightarrow \mathbf{F}$ in $\mathbf{Ext}^3(M, N)$ is said to be a *homotopy equivalence* (strictly speaking, a *weak homotopy equivalence*) when $h_M = Id$ and h_N is an isomorphism; that is, when it has the form

$$\begin{array}{c} \mathbf{E} \\ \downarrow h \\ \mathbf{F} \end{array} = \begin{pmatrix} 0 \rightarrow N \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0 \\ \downarrow h_N \quad \downarrow h_2 \quad \downarrow h_1 \quad \downarrow h_0 \quad \downarrow Id \\ 0 \rightarrow N \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \end{pmatrix}.$$

We write $\mathbf{E} \simeq \mathbf{F}$ when there exists a weak homotopy equivalence $h : \mathbf{E} \rightarrow \mathbf{F}$. Weak homotopy equivalence is always reflexive and transitive, but is not, in general, symmetric. If $h : \mathbf{E} \rightarrow \mathbf{F}$ is a weak homotopy equivalence, the mapping cone $M(h)$ is necessarily acyclic; if, in addition, \mathbf{E}, \mathbf{F} are projective then $M(h)$ is chain contractible. By the usual argument ([21] p. 167), h then has an inverse up to homotopy. Thus 'weak homotopy equivalence' is indeed an equivalence relation when restricted to $\mathbf{Proj}^n(\mathbf{Z}, J)$.

A module $J \in \Omega_3(\mathbf{Z})$ is said to be *minimal* when $\text{rk}_{\mathbf{Z}}(J) \leq \text{rk}_{\mathbf{Z}}(J')$ for all $J' \in \Omega_3(\mathbf{Z})$. If $J \in \Omega_3(\mathbf{Z})$ and $\mathbf{E} = (0 \rightarrow J \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbf{Z} \rightarrow 0)$ is an algebraic 2-complex then \mathbf{E} is said to be *minimal* when J is minimal. G is said to be *2-tame* (resp. *minimally 2-tame*) when for every algebraic 2-complex

\mathbf{E} (resp. every minimal algebraic 2-complex \mathbf{E}) there exists a presentation \mathcal{G} such that $C_*(\mathcal{G}) \simeq \mathbf{E}$.

Proposition 5.2. *Let G be a finite group; then*

$$G \text{ is 2-tame} \iff G \text{ is minimally 2-tame.}$$

Proof. The implication (\implies) is clear. Thus suppose that G is minimally 2-tame, let $J \in \Omega_3(\mathbf{Z})$ be a specific choice of minimal module, and let

$$\mathbf{F} = (0 \rightarrow J \xrightarrow{j} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \rightarrow \mathbf{Z} \rightarrow 0)$$

be a minimal algebraic 2-complex whose left-hand end is J but whose choice is otherwise arbitrary. Now let

$$\mathbf{E} = (0 \rightarrow \Lambda \rightarrow E_2 \rightarrow E_1 \rightarrow E_0 \rightarrow \mathbf{Z} \rightarrow 0)$$

be a non-minimal free algebraic 2-complex. By the Swan–Jacobinski Theorem, we may suppose, without any loss of generality, that $\Lambda = J \oplus \mathbf{Z}[G]^n$. Put

$$r = k(\mathcal{S}^n(\mathbf{F}) \rightarrow \mathbf{E}) \in (\mathbf{Z}/|G|)^*.$$

Let $\mathbf{D} = (0 \rightarrow J \rightarrow D_2 \rightarrow D_1 \rightarrow D_0 \rightarrow \mathbf{Z} \rightarrow 0)$ be an element of $\mathbf{Ext}^3(\mathbf{Z}, J)$ such that

$$k(\mathbf{F} \rightarrow \mathbf{D}) = r.$$

Since $r \in (\mathbf{Z}/|G|)^*$ it follows that $\mathbf{D} \in \mathbf{Proj}^3(\mathbf{Z}, J)$. Furthermore, we may make the identification

$$\mathbf{Ext}^3(\mathbf{Z}, J \oplus \mathbf{Z}[G]^n) \cong \mathbf{Ext}^3(\mathbf{Z}, J \oplus \mathbf{Z}[G]^n) \cong (\mathbf{Z}/|G|)^*.$$

Since $k(\mathcal{S}^n(\mathbf{F}) \rightarrow \mathbf{E}) = k(\mathbf{F} \rightarrow \mathbf{D}) = r$, and \mathbf{E} is free, it follows that \mathbf{D} is stably free, and hence a minimal algebraic 2-complex. By hypothesis, there exists a presentation \mathcal{G} such that $C_*(\mathcal{G}) \simeq \mathbf{D}$. However $k(\mathcal{S}^n \mathbf{F} \rightarrow \mathcal{S}^n \mathbf{D}) = r$, so that $\mathcal{S}^n \mathbf{D}$ is congruent to \mathbf{E} and hence $\mathcal{S}^n \mathbf{D} \simeq \mathbf{E}$. Let $\Sigma^n \mathcal{G}$ be a presentation obtained from \mathcal{G} by adding n trivial relations. Then $C_*(\Sigma^n \mathcal{G}) \simeq \mathcal{S}^n(\mathbf{D}) \simeq \mathbf{E}$. \square

For algebraic 2-complexes, and more generally for projective complexes, the relation ‘weak homotopy equivalence’ is symmetrical; that is, if \mathbf{E}, \mathbf{E}' are algebraic 2-complexes, and $f : \mathbf{E} \rightarrow \mathbf{E}'$ is a homotopy equivalence, then there exists an inverse homotopy equivalence $g : \mathbf{E}' \rightarrow \mathbf{E}$. If \mathbf{E} is an algebraic 2-complex over $\mathbf{Z}[G]$, say that \mathbf{E} is *geometrically realized* when there is a finite 2-complex L with $\pi_1(L) = G$, and a homotopy equivalence $\varphi : C_*(L) \xrightarrow{\sim} \mathbf{E}$; equivalently, when there is a finite presentation \mathcal{G} of G and a homotopy equivalence $\varphi : C_*(\mathcal{G}) \xrightarrow{\sim} \mathbf{E}$. A finite group G is said to be *2-trivial* when there exists a single presentation \mathcal{G} of G such that $\mathbf{E} \simeq C_*(\mathcal{G})$ for each minimal algebraic 2-complex \mathbf{E} . It is straightforward to see that

Proposition 5.3. *If G is 2-trivial then G is 2-tame.*

At present, it is still unknown whether every finite group G is necessarily 2-tame. However, we point out that the condition of 2-triviality imposes strong requirements on G . In particular, the stable module $\Omega_3(\mathbf{Z})$ must then contain a unique module at the minimal level, so that the tree structure has no branching at all. Browning [4] classified the minimal 2-complexes of finite abelian groups, and showed that they arise from suitable presentations; thus finite abelian groups are 2-tame, but, by results of Metzler [20], *definitely not 2-trivial*.

6. Virtual homotopy type and the D(2)-problem

We now proceed to reduce the D(2)-problem, for finite fundamental groups, to a question involving only 2-dimensional algebraic homotopy types.

Proposition 6.1. *Let X be a finite 3-complex with $\pi_1(X) \cong G$, and put $K = X^{(2)}$; then there is a canonical exact sequence of $\mathbf{Z}[G]$ -modules*

$$0 \rightarrow H_3(\tilde{X}; \mathbf{Z}) \rightarrow C_3(\tilde{X}) \xrightarrow{\partial_3} \pi_2(K) \rightarrow \pi_2(X) \rightarrow 0.$$

Proof. We have a pair of exact sequences

$$0 \rightarrow Z_3(\tilde{X}) \rightarrow C_3(\tilde{X}) \xrightarrow{\partial_3} \text{Im}(\partial_3) \rightarrow 0$$

$$0 \rightarrow \text{Im}(\partial_3) \rightarrow Z_2(\tilde{X}) \rightarrow H_2(\tilde{X}; \mathbf{Z}) \rightarrow 0,$$

which, spliced together give:

$$0 \rightarrow Z_3(\tilde{X}) \rightarrow C_3(\tilde{X}) \xrightarrow{\partial_3} Z_2(\tilde{X}) \rightarrow H_2(\tilde{X}; \mathbf{Z}) \rightarrow 0.$$

Since $K = X^{(2)}$ we have $H_2(\tilde{K}; \mathbf{Z}) = Z_2(\tilde{K}) = Z_2(\tilde{X})$, whilst $Z_3(\tilde{X}) = H_3(\tilde{X}; \mathbf{Z})$, so that we have an exact sequence

$$0 \rightarrow H_3(\tilde{X}; \mathbf{Z}) \rightarrow C_3(\tilde{X}) \xrightarrow{\partial_3} H_2(\tilde{K}; \mathbf{Z}) \rightarrow H_2(\tilde{X}; \mathbf{Z}) \rightarrow 0.$$

The result now follows from the Hurewicz Theorem, since $\pi_2(K) \cong \pi_2(\tilde{K}) = H_2(\tilde{K}; \mathbf{Z})$ and $\pi_2(X) \cong \pi_2(\tilde{X}) = H_2(\tilde{X}; \mathbf{Z})$. \square

As a consequence, we have:

Proposition 6.2. *Let X be a finite 3-complex with $\pi_1(X) \cong G$; if $H^3(X; \mathcal{B}) = 0$ for any local coefficient system \mathcal{B} on X then*

- (i) $\text{Im}(\partial_3)$ is isomorphic to the free $\mathbf{Z}[G]$ -module $C_3(\tilde{X})$;
- (ii) $\text{Im}(\partial_3)$ is a direct summand of $\pi_2(K)$, and
- (iii) $\pi_2(X) \cong \pi_2(K)/\text{Im}(\partial_3)$.

Proof. Since X is three dimensional, $H_3(\tilde{X}; \mathbf{Z})$ is torsion free, since by the Universal Coefficient Theorem, $\text{Tor}(H_3(\tilde{X}; \mathbf{Z})) = \text{Tor}(H^4(\tilde{X}; \mathbf{Z})) = 0$. Moreover, by the cohomology hypothesis on X , we have $H_3(\tilde{X}; \mathbf{Q}) = H^3(\tilde{X}; \mathbf{Q}) = 0$. It follows that $H_3(\tilde{X}; \mathbf{Z}) = 0$. Thus $\partial_3 : C_3(\tilde{X}) \rightarrow \text{Im}(\partial_3)$ is an isomorphism. This proves (i).

Furthermore, the exact sequence of (6.1) reduces to

$$0 \rightarrow C_3(\tilde{X}) \xrightarrow{\partial_3} \pi_2(K) \rightarrow \pi_2(X) \rightarrow 0.$$

However,

$$\text{Tor}(\pi_2(X)) = \text{Tor}(H_2(\tilde{X}; \mathbf{Z})) = \text{Tor}(H^3(\tilde{X}; \mathbf{Z}))$$

so that, again by the cohomology assumption on X , $\pi_2(X)$ is torsion free. Thus we have a short exact sequence in $\mathcal{F}(\mathbf{Z}[G])$

$$0 \rightarrow C_3(\tilde{X}) \xrightarrow{\partial_3} \pi_2(K) \rightarrow \pi_2(X) \rightarrow 0,$$

in which $C_3(\tilde{X}) \cong \text{Im}(\partial_3)$ is $\mathbf{Z}[G]$ -free (of rank n say) and hence relatively injective. Thus

$$\pi_2(K) \cong \pi_2(X) \oplus \text{Im}(\partial_3) \cong \pi_2(X) \oplus \mathbf{Z}[G]^n.$$

This proves (ii) and (iii) simultaneously. \square

It follows from the above that:

Corollary 6.3. *Let X be a finite 3-complex with $\pi_1(X) \cong G$; if $H^3(X; \mathcal{B}) = 0$ for any local coefficient system \mathcal{B} on X then $\pi_2(X) \in \Omega_3(\mathbf{Z})$.*

Throughout the rest of this section, until further notice, X will denote a finite 3-complex with $\pi_1(X) \cong G$ which is cohomologically 2-dimensional in the sense that $H^3(X; \mathcal{B}) = 0$ for any local coefficient system \mathcal{B} on X . Up to homotopy type, there is no loss of generality in assuming that X is also reduced, so we assume that as well. K will denote the 2-skeleton of X . We may refine the above analysis slightly. Firstly, since $H_3(\tilde{X}; \mathbf{Z}) = 0$ the boundary map $\partial_3 : C_3(\tilde{X}) \rightarrow C_2(\tilde{X})$ is injective. Moreover, we have:

Proposition 6.4. $\pi_2(X) \cong \text{Ker}(\partial_2 : C_2(\tilde{X})/\text{Im}(\partial_3) \rightarrow C_1(\tilde{X}))$.

Proof. This follows directly from the exactness of the following sequence of $\mathbf{Z}[G]$ -modules $0 \rightarrow \pi_2(K)/\text{Im}(\partial_3) \rightarrow C_2(\tilde{X})/\text{Im}(\partial_3) \xrightarrow{\partial_2} C_1(\tilde{X})$. \square

Proposition 6.5. $\text{Im}(\partial_3)$ is a direct summand of $C_2(\tilde{X})$ and $C_2(\tilde{X})/\text{Im}(\partial_3)$ is stably free over $\mathbf{Z}[G]$.

Proof. We have an exact sequence $0 \rightarrow \pi_2(X) \rightarrow C_2(\tilde{X})/\text{Im}(\partial_3) \xrightarrow{\partial_2} C_1(\tilde{X})$ in which $\pi_2(X)$, being a representative of $\Omega_3(\mathbf{Z})$, is torsion free over \mathbf{Z} . However,

$C_1(\tilde{X})$ is free over $\mathbf{Z}[G]$ and hence free over \mathbf{Z} . Thus $C_2(\tilde{X})/\text{Im}(\partial_3)$ is also torsion free over \mathbf{Z} . The result now follows from the injectivity of the free module $\text{Im}(\partial_3)$ relative to $\mathcal{F}(\mathbf{Z}[G])$, given that $C_2(\tilde{X})$ is also free. \square

Let $j : \pi_2(X) \rightarrow C_2(\tilde{X})/\text{Im}(\partial_3)$ be the inclusion obtained by making the identifications

$$\pi_2(X) = \pi_2(K)/\text{Im}(\partial_3) ; \quad C_2(\tilde{X}) = C_2(\tilde{K}).$$

We let $\langle X \rangle$ denote the following algebraic 2-complex;

$$0 \rightarrow \pi_2(X) \xrightarrow{j} C_2(\tilde{X})/\text{Im}(\partial_3) \rightarrow C_1(\tilde{X}) \rightarrow C_0(\tilde{X}) \rightarrow \mathbf{Z} \rightarrow 0.$$

Observe that $\langle X \rangle$ is functorial in the cell structure of X . Clearly $\langle X \rangle$ belongs to $\mathbf{SF}^3(\mathbf{Z}, \pi_2(X))$, and we regard it as a ‘virtual’ 2-dimensional homotopy type representing X . There is a natural $\mathbf{Z}[G]$ -chain map $\varphi : C_*(X) \rightarrow \langle X \rangle$ from the cellular chain complex of X obtained by collapsing $C_3(\tilde{X}) \cong \text{Im}(\partial_3)$:

$$\begin{array}{ccccccccc} 0 & \rightarrow & C_3(\tilde{X}) & \rightarrow & C_2(\tilde{X}) & \rightarrow & C_1(\tilde{X}) & \rightarrow & C_0(\tilde{X}) & \rightarrow & \mathbf{Z} & \rightarrow & 0 \\ & & \downarrow & & \downarrow \varphi_2 & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \\ & & 0 & \rightarrow & C_2(\tilde{X})/\text{Im}(\partial_3) & \rightarrow & C_1(\tilde{X}) & \rightarrow & C_0(\tilde{X}) & \rightarrow & \mathbf{Z} & \rightarrow & 0 \end{array}$$

where, since X is reduced, we may identify the epimorphism $C_0(\tilde{X}) \rightarrow \mathbf{Z}$ with the augmentation map.

By the Hurewicz Theorem, we may identify $\pi_2(X)$ with $H_2(\tilde{X}; \mathbf{Z})$. The ‘algebraic second homotopy group’ of $\langle X \rangle$ is given by

$$\pi_2(\langle X \rangle) = \pi_2(K)/\text{Im}(\partial_3).$$

It is straightforward to check that

Proposition 6.6. $\varphi : C_*(X) \rightarrow \langle X \rangle$ induces an isomorphism

$$\varphi : \pi_2(X) \xrightarrow{\cong} \pi_2(\langle X \rangle).$$

We are tempted to restate (6.6) as saying that $\varphi : C_*(X) \rightarrow \langle X \rangle$ is a weak homotopy equivalence. The difficulty is that $C_*(X)$ and $\langle X \rangle$ are slightly different sorts of objects; $\langle X \rangle$ is both augmented (by $C_0(\tilde{X}) \rightarrow \mathbf{Z}$) and co-augmented (by $\pi_2(X) \rightarrow C_2(\tilde{X})$) whereas $C_*(X)$ is merely augmented. In comparing the two, we should, strictly speaking, drop the co-augmentation (which will then be remembered simply as $\text{Ker}(\partial_2 : C_2(\tilde{X})/\text{Im}(\partial_3) \rightarrow C_1(\tilde{X}))$). There is a transformation of (augmented, coaugmented) chain complexes $\nu : C_*(K) \rightarrow \langle X \rangle$, natural in the cell structure of X , described by the following diagram:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \pi_2(K) & \rightarrow & C_2(\tilde{K}) & \rightarrow & C_1(\tilde{K}) & \rightarrow & C_0(\tilde{K}) & \rightarrow & \mathbf{Z} & \rightarrow & 0 \\ & & \downarrow \nu & & \downarrow \nu_2 & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \\ 0 & \rightarrow & \pi_2(K)/\text{Im}(\partial_3) & \rightarrow & C_2(\tilde{X})/\text{Im}(\partial_3) & \rightarrow & C_1(\tilde{X}) & \rightarrow & C_0(\tilde{X}) & \rightarrow & \mathbf{Z} & \rightarrow & 0. \end{array}$$

Theorem 6.7. *Let G be a finite group, and let X be a finite cohomologically 2-dimensional 3-complex with $\pi_1(X) \cong G$; then X is homotopy equivalent to a finite 2-complex if and only if the algebraic 2-complex $\langle X \rangle$ is geometrically realized.*

Proof. Suppose that L is a finite 2-complex, and that $\psi : L \rightarrow X$ is a homotopy equivalence over the identity. Then $\psi : \pi_2(L) \rightarrow \pi_2(X)$ is an isomorphism. Hence $\varphi \circ \psi_* : \pi_2(L) \xrightarrow{\cong} \pi_2(\langle X \rangle)$ is an isomorphism; thus $\varphi \circ \psi_* : C_*(L) \rightarrow \langle X \rangle$ is a weak homotopy equivalence, and so $\langle X \rangle$ is geometrically realized.

Conversely, suppose that $\langle X \rangle$ is geometrically realized. This means there is a finite 2-complex L with $\pi_1(L) = G$ and a weak homotopy equivalence $g : C_*(L) \rightarrow \langle X \rangle$. By symmetry of weak homotopy equivalence on algebraic 2-complexes, we may suppose there is also a weak homotopy equivalence $f : \langle X \rangle \rightarrow C_*(L)$. As usual, let K denote the 2-skeleton of X ; without loss of generality, we may suppose that:

$$(i) L^{(1)} = K^{(1)},$$

and that

$$(ii) f_r = \text{Id} : C_r(\tilde{K}) \equiv C_r(\tilde{L}) \text{ for } r \leq 1.$$

Furthermore, assuming X has N cells of dimension three, we may write:

$$X = K \cup_{\alpha_1} E_1^{(3)} \cup_{\alpha_2} E_2^{(3)} \cdots \cup_{\alpha_N} E_N^{(3)}.$$

Observe that f induces an isomorphism $f_* : \pi_2(\langle X \rangle) = \pi_2(X) \xrightarrow{\cong} \pi_2(L)$. The chain map $f \circ \nu : C_*(K) \rightarrow C_*(L)$ has the property that $(f \circ \nu)_r = \text{Id} : C_r(\tilde{K}) \equiv C_r(\tilde{L})$ for $r \leq 1$. By (2.3) there exists a cellular map $g : K \rightarrow L$ such that

$$(i) f_* = \text{Id} : \pi_1(K) \rightarrow \pi_1(L), \text{ and}$$

$$(ii) g_* = f_* \circ \nu_* : \pi_2(K) \rightarrow \pi_2(L).$$

In particular, since f_* is an isomorphism, we obtain.

$$\text{Ker}(g_* : \pi_2(K) \rightarrow \pi_2(L)) = \text{Ker}(\nu_*) = \text{Im}(\partial_3 : C_3(\tilde{X}) \rightarrow \pi_2(K)) \cong \mathbf{Z}[G]^N.$$

Each homotopy class $[\alpha_j]$ belongs to $\text{Im}(\partial_3) = \text{Ker}(g_* : \pi_2(K) \rightarrow \pi_2(L))$, so that $g : K \rightarrow L$ extends over the 3-cells of X to a map $h : X \rightarrow L$ such that

$$(iii) h_* = \text{Id} : \pi_1(X) \rightarrow \pi_1(L) \text{ and}$$

$$(iv) h_* = f_* : \pi_2(X) \rightarrow \pi_2(L).$$

Since $H_r(\tilde{X}; \mathbf{Z}) = H_r(\tilde{L}; \mathbf{Z}) = 0$ for $2 < r$, we see by Whitehead's Theorem that h is a homotopy equivalence as required. \square

A 2-tame group G is precisely one over which every algebraic 2-complex is geometrically realized; thus we obtain:

Corollary 6.8. *If G is 2-tame then the $D(2)$ -property holds for G .*

In [16], we give examples of finite groups which are 2-tame; in the absence of positive evidence to the contrary, one can presently hope that every finite group

is 2-tame, and so possesses the D(2)-property. Nevertheless, the possibility of a constructing a counterexample is an intriguing one. So far, we have had no recourse to Browning's results. However, using the first of Browning's results, the Stability Theorem, we can establish the converse to (6.7). First we need a criterion for recognising cohomologically 2-dimensional 3-complexes. We adopt the following notation:

- K : a finite 2-complex with $\pi_1(K) \cong G$;
- m : an integer ≥ 1 ;
- α : a $\mathbf{Z}[G]$ -homomorphism $\mathbf{Z}[G]^m \rightarrow \pi_2(K)$ will be;
- $\{\epsilon_j\}_{1 \leq j \leq m}$: the canonical $\mathbf{Z}[G]$ -basis for $\mathbf{Z}[G]^m$;
- α_j : a map $S^2 \rightarrow K$ in the homotopy class $\alpha(\epsilon_j)$.

Furthermore, $X(\alpha)$ will denote the 3-complex obtained by attaching 3-cells to K by means of $\alpha_1, \alpha_2, \dots, \alpha_m$; that is,

$$X(\alpha) = K \cup_{\alpha_1} E_1^{(3)} \cup_{\alpha_2} E_2^{(3)} \dots \cup_{\alpha_m} E_m^{(3)}.$$

Lemma 6.9. *Let K be a finite 2-complex with $\pi_1(K) \cong G$, and let $\alpha : \mathbf{Z}[G]^m \rightarrow \pi_2(K)$ be an injective $\mathbf{Z}[G]$ -homomorphism such that $\pi_2(K)/\text{Im}(\alpha)$ is torsion free over \mathbf{Z} . Then $H^3(X(\alpha); \mathcal{B}) = 0$ for all local coefficient systems \mathcal{B} on $X(\alpha)$.*

Proof. We can make the following identifications:

$$\begin{aligned} C_3(\widetilde{X(\alpha)}) &\longleftrightarrow \mathbf{Z}[G]^m \\ C_2(\widetilde{X(\alpha)}) &\longleftrightarrow C_2(\tilde{K}) \\ \pi_2(K) &\longleftrightarrow Z_2(\tilde{K}) \\ \partial_3 : C_3(\widetilde{X(\alpha)}) \rightarrow C_2(\widetilde{X(\alpha)}) &\longleftrightarrow i \circ \alpha : \mathbf{Z}[G]^m \rightarrow C_2(\tilde{K}) \\ H^3(X(\alpha); \mathcal{B}) &\longleftrightarrow (\mathbf{Z}[G]^M)^* \otimes \mathcal{B} / \text{Im}((i \circ \alpha)^* \otimes 1_{\mathcal{B}}). \end{aligned}$$

where $i : \pi_2(K) = Z_2(\tilde{K}) \subset C_2(\tilde{K})$ is the inclusion. Since α is injective and $\pi_2(K)/\text{Im}(\alpha)$ is \mathbf{Z} -free, the exact sequence

$$0 \rightarrow \mathbf{Z}[G]^m \xrightarrow{\alpha} \pi_2(K) \rightarrow \pi_2(K)/\text{Im}(\alpha) \rightarrow 0$$

lies within $\mathcal{F}(\mathbf{Z})[G]$; hence its dual sequence is exact, and $\alpha^* : \pi_2(K)^* \rightarrow (\mathbf{Z}[G]^m)^*$ is surjective. Likewise, the exact sequence

$$0 \rightarrow \pi_2(K) \rightarrow C_2(\tilde{K}) \rightarrow C_1(\tilde{K}) \rightarrow C_0(\tilde{K}) \rightarrow \mathbf{Z} \rightarrow 0$$

lies within $\mathcal{F}(\mathbf{Z})[G]$, so its dual sequence too is exact, and $i^* : C_2(\tilde{K})^* \rightarrow \pi_2(K)^*$ is surjective. Hence $(\alpha \circ i)^* : C_2(\tilde{K})^* \rightarrow (\mathbf{Z}[G]^m)^*$ is surjective. Thus $(i \circ \alpha)^* \otimes 1_{\mathcal{B}} :$

$C_2(\tilde{K})^* \otimes \mathcal{B} \longrightarrow \mathbf{Z}[G]^m \otimes \mathcal{B}$ is surjective, by right exactness of $-\otimes_{\mathbf{Z}[G]} \mathcal{B}$. From the correspondence

$$H^3(X(\alpha); \mathcal{B}) \longleftarrow (\mathbf{Z}[G]^M)^* \otimes \mathcal{B} / \text{Im}((i \circ \alpha)^* \otimes 1_{\mathcal{B}})$$

we see that $H^3(X(\alpha); \mathcal{B}) = 0$ for all local coefficient systems \mathcal{B} . \square

Observe that if \mathbf{E} is any algebraic 2-complex, there is a natural chain map $\psi : \mathcal{S}(\mathbf{E}) \rightarrow \mathbf{E}$ which is the projection on the top two terms, but is otherwise the identity. The following is an obvious consequence of Browning's Stability Theorem [2], [11].

Proposition 6.10. *If $\mathbf{E}_1, \mathbf{E}_2$ are algebraic 2-complexes over $\mathbf{Z}[G]$ then*

$$\mathcal{S}^n(\mathbf{E}_1) \simeq \mathcal{S}^m(\mathbf{E}_2)$$

for some $n, m \geq 1$.

Corresponding to any algebraic 2-complex over $\mathbf{Z}[G]$, we can now construct a three dimensional cell complex which is cohomologically 2-dimensional.

Theorem 6.11. *Let \mathbf{E} be an algebraic 2-complex over the finite group G ; then there exists a finite 3-dimensional cell complex X such that*

- (i) $\pi_1(X) \cong G$;
- (ii) $H^3(X; \mathcal{B}) = 0$ for all local coefficient systems \mathcal{B} ; and
- (iii) $\langle X \rangle$ is homotopy equivalent to \mathbf{E} .

Proof. Write \mathbf{E} thus

$$\mathbf{E} = (0 \rightarrow J \rightarrow \mathbf{Z}[G]^\gamma \xrightarrow{\delta_2} \mathbf{Z}[G]^\beta \xrightarrow{\delta_1} \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \rightarrow 0).$$

Let L be a finite 2-complex with $\pi_1(L) \cong G$; then by (6.10),

$$\mathcal{S}^n(\mathbf{E}) \simeq \mathcal{S}^m(\mathcal{C}(L))$$

for some $n, m \geq 1$. However, $\mathcal{S}^m(\mathcal{C}(L)) \cong \mathcal{C}(L \vee mS^2)$. Put $K = L \vee mS^2$, and let $\varphi : \mathcal{C}(K) \rightarrow \mathcal{S}^n(\mathbf{E})$ be a weak homotopy equivalence. In particular, φ induces an isomorphism

$$\varphi : \pi_2(K) \xrightarrow{\cong} J \oplus \mathbf{Z}[G]^n.$$

If $\psi : \mathcal{S}^n(\mathbf{E}) \rightarrow \mathbf{E}$ is the natural chain projection, then $\psi \circ \varphi : \mathcal{C}(K) \rightarrow \mathbf{E}$ has the property that $\text{Ker}(\psi \circ \varphi : \pi_2(K) \rightarrow J)$ is isomorphic to $\mathbf{Z}[G]^n$. Choose a $\mathbf{Z}[G]$ basis $\alpha_1, \alpha_2, \dots, \alpha_m$ for $\text{Ker}(f)$, and form $X(\alpha)$ by attaching 3-cells $E_1^{(3)}, E_2^{(3)}, \dots, E_m^{(3)}$ to K by means of $\alpha_1, \alpha_2, \dots, \alpha_m$; that is,

$$X(\alpha) = K \cup_{\alpha_1} E_1^{(3)} \cup_{\alpha_2} E_2^{(3)} \dots \cup_{\alpha_m} E_m^{(3)}.$$

It follows from (6.9) that $H^r(X(\alpha); \mathcal{B}) = 0$ for all $r \geq 3$, and it is straightforward to see that $\langle X \rangle$ is homotopy equivalent to \mathbf{E} . \square

From (6.7) and (6.11) we obtain:

Realization Criterion. The D(2) property holds for the finite group G if and only if every algebraic 2-complex over $\mathbf{Z}[G]$ is geometrically realized.

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