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Restriction map in a regular reduction of $\mathbf{SU}(n)^{2g}$

Sébastien Racanière

Abstract. The quasi-Hamiltonian reduction of $\mathbf{SU}(n)^{2g}$ at a regular value, in the centre of $\mathbf{SU}(n)$, of the moment map is isomorphic to a moduli-space of semi-stable vector bundles over a Riemann surface. We describe the restriction map from the equivariant cohomology of $\mathbf{SU}(n)^{2g}$ to the cohomology of the moduli space in terms of natural multiplicative generators of these cohomologies.

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Keywords. Moduli spaces, symplectic reduction, quasi-Hamiltonian spaces, $\mathbf{SU}(n)$.

Motivations

All cohomologies will be taken with coefficients in the field \mathbf{Q} of rational numbers. For a compact connected Lie group G , we denote $EG \rightarrow BG$ the universal principal G -bundle. If G acts on a manifold M , we denote $(M)_G$ the space $M \times_G EG$. The equivariant cohomology $H_G^*(M)$ of M with respect to the action of G is by definition the Čech cohomology of $(M)_G$. For an account of equivariant cohomology see [6] and [14].

Let g be an integer bigger than 1. Let π be the group

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; \prod_{k=1}^g [a_k, b_k] = c, c^n = 1 \rangle.$$

Let n and d be integers with n bigger than 1 and let ζ be the n -th root of unity $\zeta = e^{-2\pi i \frac{d}{n}}$. Put $\beta = \zeta \mathbf{I}$, where \mathbf{I} is the identity matrix in the special unitary group $\mathbf{SU}(n)$. We define

$$S_\beta = \{\rho \in \text{Hom}(\pi, \mathbf{SU}(n)) \mid \rho(c) = \beta\}$$

the space of $\mathbf{SU}(n)$ -representations of π such that β is the image of c . Because β is in the centre of $\mathbf{SU}(n)$, the group $\mathbf{SU}(n)$ acts on S_β and its quotient

$$\mathfrak{m}_\beta = S_\beta / \mathbf{SU}(n)$$

is the moduli space of $\mathbf{SU}(n)$ -representations of π that send c to β .

Narasimhan and Seshadri [17] have shown that \mathfrak{m}_β is isomorphic to the moduli space of holomorphic semi-stable vector bundles of rank n , degree d , and fixed determinant over a compact Riemann surface X of genus g . For d and n co prime, \mathfrak{m}_β is compact and smooth. In this case, Atiyah and Bott [1] showed that this space is symplectic, proposed a family of multiplicative generators of its cohomology and gave an inductive formula (on the rank n) for the Betti numbers of \mathfrak{m}_β . Their method consists in studying an infinite dimensional Hamiltonian space. In 1993, Huebschmann [8] and Jeffrey [10] independently gave a group cohomology construction of the symplectic form on \mathfrak{m}_β (their results are summarised in a joint paper [9]). In 1998, Alekseev, Malkin and Meinrenken [2] showed that Huebschmann and Jeffrey's construction fits in a more general setting: one can get the moduli space \mathfrak{m}_β (and many others, moduli spaces of flat connections on a principal bundle) as the Marsden–Weinstein reduction of a quasi-Hamiltonian space. This space is $\mathbf{SU}(n)^{2g}$, it is relatively simple in contrast with the usual descriptions of \mathfrak{m}_β as a Hamiltonian reduction. A quasi-Hamiltonian (or q-Hamiltonian for short) space is a Hamiltonian space with a group valued moment map. Its 2-form is not symplectic in general but the Marsden–Weinstein reduction is well defined and the reduced space is symplectic.

An important result about Hamiltonian spaces is the

Theorem 0.1 (Kirwan). *Let M be a symplectic manifold. Assume G is a compact Lie group acting symplectically on M and assume there exists a moment map ϕ for this action. Let 0 be the null vector of the dual of the Lie algebra of G . The restriction map*

$$H_G^*(M) \longrightarrow H_G^*(\phi^{-1}(0))$$

is surjective.

It is a natural question to ask if this theorem is still true for q-Hamiltonian spaces. It is quite easy to see that the answer is no. For example, to get \mathfrak{m}_β , one considers $\mathbf{SU}(n)^{2g}$ with moment map μ

$$\begin{aligned} \mathbf{SU}(n)^{2g} &\longrightarrow \mathbf{SU}(n) \\ (A_1, B_1, \dots, A_g, B_g) &\longmapsto \prod_{k=1}^g [A_k, B_k] \end{aligned}$$

the product of the commutators and a certain 2-form (see [2] for more details). Then the reduced space at β being symplectic and compact, its degree two cohomology (which is isomorphic to $H_{\mathbf{SU}(n)}^2(\mu^{-1}(\beta))$) contains a non trivial class whereas $H_{\mathbf{SU}(n)}^2(\mathbf{SU}(n)^{2g}) = \{0\}$. Thus the map

$$r : H_{\mathbf{SU}(n)}^*(\mathbf{SU}(n)^{2g}) \longrightarrow H_{\mathbf{SU}(n)}^*(\mu^{-1}(\beta))$$

is not surjective.

Our aim is to give a description of this last map r (Theorem 5.1) when d and n are co prime. Note that in [3], a theorem of localisation in the context of quasi-

Hamiltonian spaces is given. It may be interesting to see how our theorem could be used to apply this localisation theorem to the reduction of $\mathbf{SU}(n)^{2g}$ at β .

This paper is organised in the following way. Section 2 gives a (very short) review of the prerequisites on q-Hamiltonian spaces and semi-stable bundles. In particular, Narasimhan and Seshadri's theorem (see Theorems 2.9 and 2.13) is used throughout this article to identify \mathfrak{m}_β with $\mu^{-1}(\beta)/\mathbf{SU}(n)$ and $H^*(\mathfrak{m}_\beta)$ with $H_{\mathbf{SU}(n)}^*(\mu^{-1}(\beta))$. In Section 3, we give a construction of a universal bundle on $\mathfrak{m}_\beta \times X$, we then recall how Biswas and Raghavendra [4] use this bundle to define a set $\{a_k, b_{k,j}, d_k, 2 \leq k \leq n, 1 \leq j \leq 2g\}$ of canonical multiplicative generators of the cohomology of \mathfrak{m}_β (Theorem 3.4). In the next section we define a bundle on $\mathbf{SU}(n)^{2g} \times X - \{\text{point}\}$ and use it to get a set $\{c_k, \sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq g\}$ of multiplicative generators for the equivariant cohomology of $\mathbf{SU}(n)^{2g}$ (Theorems 4.4 and 4.6). Finally in Section 5 we prove the

Theorem 5.1. *The restriction map*

$$r : H_{\mathbf{SU}(n)}^*(\mathbf{SU}(n)^{2g}) \longrightarrow H_{\mathbf{SU}(n)}^*(\mu^{-1}(\zeta\mathbf{I}))$$

is given by

$$\begin{aligned} r(c_k) &= a_k \text{ for } k = 2, \dots, n \\ r(\sigma_{k,j}) &= b_{k,j} \text{ for } k = 2, \dots, n, j = 1, \dots, 2g. \end{aligned}$$

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2. Prerequisites

In paragraph 2.1, we recall the definition of semi-stability for holomorphic vector bundles and state Narasimhan and Seshadri's theorem (Theorem 2.9). Then in §2.2, we give the definition of a q-Hamiltonian space and restate Narasimhan and Seshadri's result in the language of q-Hamiltonian spaces (Theorem 2.13).

2.1. Semi-stable bundle

The following constructions are due to Narasimhan and Seshadri [17]. Apart from the proof of Proposition 2.1, everything in this paragraph is from their article.

Let X be a Riemann surface of genus g , $g \geq 2$. Fix a point x_0 of X . We will first give a construction of a ramified covering $Y \rightarrow X$ used in [17].

Proposition 2.1. *There exists a simply connected covering*

$$p : Y \rightarrow X$$

with only one point of ramification x_0 of order n . Outside of this point, the map

$$Y - \{p^{-1}(x_0)\} \rightarrow X - \{x_0\}$$

is a covering with group

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; c = \prod_{k=1}^g [a_k, b_k], c^n = 1 \rangle.$$

Proof. We start by constructing Y , then we show that it is simply connected. Let D be an open neighbourhood of x_0 biholomorphic to an open disc in \mathbf{C} centred at zero. Let $D' = D - \{x_0\}$. The fundamental group of $X' = X - \{x_0\}$ has a presentation

$$\pi_1(X') = \langle a_1, b_1, \dots, a_g, b_g \rangle,$$

such that the element $\prod_{k=1}^g [a_k, b_k]$ is the class of a small circle γ included in D' and going counter clockwise around x_0 . Let π be the group

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; c = \prod_{k=1}^g [a_k, b_k], c^n = 1 \rangle.$$

The natural surjective map

$$\pi_1(X') \rightarrow \pi$$

defines a galoisian covering

$$p : Y' \rightarrow X'$$

with group π . Fix a point x_1 of D' . We take it as the base point for the fundamental groups of X, X' and D' . Let us decompose $p^{-1}(D')$ in its different connected components

$$p^{-1}(D') = \bigcup_{\alpha \in \Lambda} V_\alpha.$$

Each

$$V_\alpha \xrightarrow{p} D'$$

is a connected covering. As D' is a disc minus its centre, this last covering group is generated by the element in π corresponding to the loop γ . The connected covering of a disc with its centre removed is either the upper half complex plane with the exponential as a projective map or a disc with its centre removed and

a projection map of the type $z \mapsto z^m$, where m is a positive integer. Here, the class of γ acts as c , thus V_α is a disc with its centre removed and, for each α ,

$$V_\alpha \xrightarrow{p} D'$$

is the map $z \mapsto z^n$. Let

$$Y = (Y' \bigcup_{\alpha \in \Lambda} (D, \alpha)) / \sim$$

where for $y \in Y'$ and $(x, \alpha) \in (D, \alpha)$

$$y \sim (x, \alpha) \text{ if and only if } p(x) = y.$$

The natural projection

$$Y \longrightarrow X$$

is a covering with a unique ramification point at x_0 with order n . We now have to check that Y is simply connected. We get Y' from Y by removing a discrete set of points. Hence the map

$$\pi_1(Y') \longrightarrow \pi_1(Y)$$

is surjective. The sequence

$$\{1\} \longrightarrow \pi_1(Y') \longrightarrow \pi_1(X') \longrightarrow \pi \longrightarrow \{1\}$$

is exact. The kernel of $\pi_1(X') \longrightarrow \pi$ is the normal subgroup generated by c^n . Let a in $\pi_1(X')$ be the class of a loop $\eta : [0, 1] \longrightarrow X'$. The class of γ^n is c^n . Let us lift $\eta \cdot \gamma^n \cdot \eta^{-1}$ in Y' . To do so we have to take a lift $\tilde{\eta}$ of η in Y' and then take a lift $\tilde{\gamma}^n$ of γ^n satisfying $\tilde{\gamma}^n(0) = \tilde{\eta}(1)$. The loop we wanted is $\tilde{\eta} \cdot \tilde{\gamma}^n \cdot \tilde{\eta}^{-1}$. There exists α such that

$$\tilde{\gamma}^n \subset V_\alpha,$$

thus $\tilde{\gamma}^n$ is homotopic to the constant loop in Y . The image of $a \cdot c^n \cdot a^{-1}$ by $\pi_1(Y') \longrightarrow \pi_1(Y)$ is 1, hence the image of $\pi_1(Y') \longrightarrow \pi_1(Y)$ is $\{1\}$ and

$$\pi_1(Y) = \{1\}.$$

□

Choose a y_0 in $p^{-1}(x_0)$. In the presentation

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; \prod_{k=1}^g [a_k, b_k] = c, c^n = 1 \rangle$$

of π , we can assume that c is a generator of the isotropy group π_{y_0} of y_0 .

For a representation $\rho : \pi \rightarrow \mathbf{GL}(n)$ of π , we denote $E_\pi(\rho)$ the vector bundle

$$Y \times \mathbf{C}^n \longrightarrow Y$$

with the action:

$$\begin{aligned} \pi \times (Y \times \mathbf{C}^n) &\longrightarrow Y \\ (\gamma, (y, v)) &\longmapsto (\gamma \cdot y, \rho(\gamma)v). \end{aligned}$$

Let \mathbf{E} be the sheaf of germs of holomorphic sections of $E_\pi(\rho)$. The group π acts on the image sheaf $p_*(\mathbf{E})$. Let $p_*^\pi(\mathbf{E})$ be the subsheaf of π -invariant elements of $p_*(\mathbf{E})$. It is a rank n locally free sheaf of \mathbf{O}_X -modules. It defines a holomorphic vector bundle, $p_*^\pi(E_\rho)$, of rank n on X . A set of transition functions is obtained in the following way. Let $\{U_i\}_{i=0}^m$ be a finite open covering of X satisfying:

- (1) all non-empty intersections of sets of the type U_i is contractible,
- (2) $x_0 \in U_0$ and $\bigcup_{i=1}^m U_i = X - \{x_0\}$,
- (3) there exist discs $\{D_i\}_{i=0}^m$ in Y such that $y_0 \in D_0$ and U_0 is the quotient of D_0 by π_{y_0} , the restriction $p|_{D_i}$ is an homeomorphism of D_i with U_i , for all non zero i .

For each triplet i, j, k , choose a connected component $W_{ij,k}$ of $p^{-1}(U_i \cap U_j) \cap D_k$. If $U_i \cap U_j$ is not empty, we denote $\gamma_{i,j}$ the element of π satisfying $\gamma_{i,j}W_{ij,j} = W_{ji,i}$. According to [17, p. 550] :

Proposition 2.2. *On each $\{U_i\}_{i=0}^m$, the bundle $p_*^\pi(E_\rho)$ is trivial and a set of transition functions is given by:*

$$\begin{cases} g_{i,j} = \rho(\gamma_{i,j}) & \text{in } U_i \cap U_j, \text{ for } i, j \neq 0 \\ g_{0,i} = f_{0,i}\rho(\gamma_{0,i}) & \text{in } U_0 \cap U_i, \text{ for } i \neq 0 \end{cases}$$

where $f_{0,i} : U_0 \cap U_i \rightarrow \mathbf{C}^*$ depends only on τ .

Definition 2.3. Let W be a degree $d(W)$ and rank $r(W)$ holomorphic vector bundle on X . It is said to be stable, resp. semi-stable, if for each proper subbundle V , we have

$$\frac{d(V)}{r(V)} < \frac{d(W)}{r(W)}, \text{ resp. } \frac{d(V)}{r(V)} \leq \frac{d(W)}{r(W)}.$$

Remark 2.4. If $d(W)$ and $r(W)$ are co prime then the notions of stability and semi-stability are equivalent.

Recall that d is an integer, $0 \leq d \leq n-1$, and $\zeta = e^{-2\pi i \frac{d}{n}}$ is an n -th root of unity. Let z be a coordinate in a neighbourhood of y_0 such that π_{y_0} is the group of multiplications by ζ^k . Up to a change of generator c of π_{y_0} , we can assume that c acts by multiplication by $e^{\frac{2i\pi}{n}}$. Let τ be the character of π_{y_0} defined by $\tau(c) = \zeta$. A representation $\rho : \pi \rightarrow \mathbf{U}(n)$ is said to be of type τ if for all $\gamma \in \pi_{y_0}$, we have $\rho(\gamma) = \tau(\gamma)\mathbf{I}$. For any representation ρ of type τ we have:

$$d(p_*^\pi(E_\rho)) = d - n \quad (\text{see [5, p. 13]}).$$

Again, according to [17]:

Theorem 2.5. *A holomorphic vector bundle of rank n and degree $d - n$ on X is semi-stable if and only if it is isomorphic to a $p_*^\pi(E_\rho)$, where $\rho : \pi \rightarrow \mathbf{U}(n)$ is a unitary representation of type τ . This bundle is stable if and only if the*

representation ρ is irreducible. Moreover, two such bundles are isomorphic if and only if their corresponding unitary representations are isomorphic.

Remark 2.6. For d and n co prime, any representation $\rho : \pi \rightarrow \mathbf{U}(n)$ of type τ is irreducible [17, Prop. 9.3].

Let \mathbf{n} be the moduli space of rank n , degree d , stable holomorphic vector bundles over X .

Remark 2.7. Let M be a holomorphic line bundle of degree 1 over X (it always exists). The moduli space of rank n stable holomorphic vector bundles with fixed determinant of degree $d - n$ over X is isomorphic to the moduli space of rank n stable holomorphic vector bundles with fixed determinant of degree d over X . The isomorphism is induced by the map which to a bundle $E \rightarrow X$ associates $E \otimes M$.

We fix such a bundle M and use it to identify the two moduli spaces of Remark 2.7. Thus we have

Theorem 2.8. *The moduli space \mathbf{n} is isomorphic to the quotient of the space of unitary representations of type τ of π by the action of $\mathbf{U}(n)$.*

The map which to a class of bundles in \mathbf{n} associates its determinant is a fibration over the moduli space of line bundles of degree d . Its fibre is called the moduli space of rank n stable holomorphic line bundles over X with fixed determinant (of degree d). We get all such bundles by taking only representations $\rho : \pi \rightarrow \mathbf{SU}(n)$ of type τ . Let S be the set of such representations. We identify it with $\{(A_1, B_1, \dots, A_g, B_g) \in \mathbf{SU}(n)^{2g} \mid \prod_{k=1}^g [A_k, B_k] = \zeta \mathbf{I}\}$ by:

$$\begin{aligned} S &\longrightarrow \{(A_1, B_1, \dots, A_g, B_g) \in \mathbf{SU}(n)^{2g} \mid \prod_{k=1}^g [A_k, B_k] = \zeta \mathbf{I}\} \\ \rho &\longmapsto (\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)). \end{aligned}$$

The action of $\mathbf{SU}(n)$ on the representations becomes, under this identification, the diagonal action by conjugation of $\mathbf{SU}(n)$ on $\mathbf{SU}(n)^{2g}$. In this article we work with \mathbf{m} rather than \mathbf{n} . We have:

Theorem 2.9. *Let d be an integer, $1 \leq d \leq n - 1$, co prime with n . Let \mathbf{m} be the moduli space of rank n holomorphic stable vector bundles over X with fixed determinant (of degree d). The map*

$$\begin{aligned} S &\longrightarrow \mathbf{m} \\ \rho &\longmapsto p_*^\pi(E_\rho) \end{aligned}$$

is a $\mathbf{PSU}(n)$ -principal bundle.

Proof. The only thing that is left to check is that for a representation $\rho = (A_1, B_1, \dots, A_g, B_g) \in S$ (recall we have identified S with a set of matrices), its stabiliser $\text{Stab}(\rho)$ is the centre of $\mathbf{SU}(n)$. Let C be in the stabiliser of ρ . Let λ be an eigenvalue of C and let E_λ be its eigenspace. As C commutes with each of the A_i, B_i , the subspace E_λ is stable by the unitary representation ρ . As ρ is irreducible, $E_\lambda = \mathbf{C}^n$ and C is in the centre of $\mathbf{SU}(n)$. On the other hand, any matrix in the centre of $\mathbf{SU}(n)$ does leave ρ invariant. We have indeed a free action of $\mathbf{PSU}(n)$ on S . \square

2.2. Quasi-Hamiltonian spaces

The definition of a q-Hamiltonian space is due to Alekseev, Malkin and Meinrenken. Roughly speaking this is a Hamiltonian space with a group valued moment map. When the group is a torus, the definition reduces to the usual one of a Hamiltonian torus action whose moment map takes its values in the torus itself (see McDuff [15] and Weitsmann [19]).

Let G be a compact Lie group. Let θ and $\bar{\theta}$ be respectively the left invariant and right invariant Maurer-Cartan forms on G . Choose a G -invariant scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{g} of G . Define a 3-form χ on G by

$$\chi = \frac{1}{12} \langle [\theta, \theta], \theta \rangle.$$

Definition 2.10 ([2]). Let (M, G, ω, μ) be a 4-tuple where M is a manifold acted on by a compact Lie group G , ω is a G -invariant 2-form on M and μ is an equivariant map from M to G (for the action by conjugation of G on itself). This 4-tuple (or simply M if there is no risk of confusion) is a q-Hamiltonian space if

- (B1) $d\omega = -\mu^*\chi$
- (B2) $\iota(v_\xi)\omega = \frac{1}{2}\mu^*\langle \theta + \bar{\theta}, \xi \rangle$
- (B3) $\ker \omega_x = \{v_\xi(x) \mid \xi \in \ker \text{Ad}_{\mu(x)} + 1\}$.

The map μ is called the moment map.

This definition is a generalisation of the definition of a Hamiltonian space in the sense that any compact Hamiltonian space can be endowed with a q-Hamiltonian structure (this is an easy corollary of [2, Prop. 3.4.]).

A first example of a q-Hamiltonian space is a conjugacy class in a Lie group with moment map the inclusion of the conjugacy class in the group (see [2, §3]). The example that will be of interest to us is

Theorem 2.11 ([2]). *Let G be a compact Lie group and $g \geq 1$ an integer. There exists a 2-form ω on G^{2g} such that the map*

$$\mu : \begin{array}{ccc} G^{2g} & \longrightarrow & G \\ (a_1, b_1, \dots, a_g, b_g) & \longmapsto & \prod_{k=1}^g [a_k, b_k] \end{array}$$

and the diagonal action of G on G^{2g} by conjugation makes (G^{2g}, G, ω, μ) into a q -Hamiltonian space.

In particular we will apply this theorem with $G = \mathbf{SU}(n)$. An important fact about q -Hamiltonian spaces is that one can take their Marsden–Weinstein reduction. More precisely:

Theorem 2.12 ([2]). *Let (M, G, ω, μ) be a q -Hamiltonian space. Let h be in the centre of G . The moment map μ is a submersion at $x \in M$ if and only if the stabiliser of x in G is finite. If this is the case for any point of $\mu^{-1}(h)$, the reduced space $\mu^{-1}(h)/G$ is an orbifold (a manifold if the action of G on $\mu^{-1}(h)$ is principal) on which the restriction of ω to $\mu^{-1}(h)$ descends to define a symplectic form. We call this space the reduction of M at h .*

As a corollary of Theorems 2.2, 2.12 and 2.9 we have:

Theorem 2.13. *Let n, d be co prime integers, $n \geq 2$ and $0 \leq d \leq n - 1$. Let $\zeta = e^{-2\pi i \frac{d}{n}}$ be an n -th root of unity and $\beta = \zeta \mathbf{I}$ in the centre of $\mathbf{SU}(n)$. The moduli space \mathfrak{m}_β of rank n stable holomorphic vector bundles with fixed determinant (and degree d) over a Riemann surface X of genus g is isomorphic to the reduction of the q -Hamiltonian space $\mathbf{SU}(n)^{2g}$ at β . It is a compact smooth symplectic manifold.*

2.3. Characteristic classes of principal bundles

Following Biswas and Raghavendra [4], we define in this section some characteristic classes of a projective bundle. We will see that when the projective bundle comes from a vector bundle of degree 0, these characteristic classes are the same as the Chern classes of the vector bundle.

Let $\mathbf{Q}[X_1, \dots, X_n]$ be a polynomial ring in n variables. The cohomology of $BU(n)$ is isomorphic to the subalgebra of invariant polynomials in the algebra $\mathbf{Q}[X_1, \dots, X_n]$, under the action of the symmetric group S_n on the variables. For k an integer in $[1, n]$, the Chern class c_k in $H^*(BU(n))$ corresponds to the Schur polynomial

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} X_{i_1} \dots X_{i_k}.$$

The projection from $\mathbf{U}(n)$ to $\mathbf{PU}(n)$ defines a fibration $BU(n) \longrightarrow BPU(n)$ with fiber $BU(1)$. This fibration is cohomologically trivial and $H^*(BPU(n))$ injects into $H^*(BU(n))$. Let us define

$$Y_k = X_k - \frac{1}{n} \sum_{k=1}^n X_k.$$

The image of $H^*(BPU(n))$ in $H^*(BU(n))$ is the ideal generated by the polyno-

mials

$$p_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} Y_{i_1} \dots Y_{i_k}, \text{ for } 2 \leq k \leq n.$$

The k -th characteristic class of a projective bundle over a manifold M is the pull-back of p_k under the classifying map $M \rightarrow B\mathbf{PU}(n)$.

For a vector bundle F of degree 0, that is when the first Chern class vanishes, we have $p_k(F) = c_k(F)$ for k in $[2, n]$. It will be the case in particular if the structure group of the vector bundle is $\mathbf{SU}(n)$. This corresponds to the fact that the projection $B\mathbf{SU}(n) \rightarrow B\mathbf{PU}(n)$ defines an isomorphism in cohomology.

3. Construction of a universal bundle

In this section, we fix d , n , ζ and β as in Theorem 2.13. We use the notations of that theorem and of Theorem 2.2 with $G = \mathbf{SU}(n)$. We construct a universal bundle on \mathfrak{m}_β , that is a vector bundle U over $\mathfrak{m}_\beta \times X$, holomorphic in the X direction, such that for any class $[E]$ in \mathfrak{m}_β , the restriction of U to $\{[E]\} \times X$ is in the class $[E]$. We then use this bundle to define natural multiplicative generators of the cohomology of \mathfrak{m}_β .

Recall that we defined page 399 an open covering of X by subsets $\{U_i\}_{i=0}^m$. Define a complex vector bundle T over $S \times X$ (where we have identified S to $\mu^{-1}(\zeta\mathbf{I})$) as being trivial over the $S \times U_i$ and with transition functions:

$$\begin{aligned} (S \times U_i) \cap (S \times U_j) &\longrightarrow \mathbf{U}(n) \\ (\rho, x) &\longmapsto \begin{cases} \rho(\gamma_{i,j}) & x \in U_i \cap U_j, \ i, j \neq 0 \\ f_{0,i}(x)\rho(\gamma_{0,i}) & x \in U_0 \cap U_i, \ i \neq 0 \end{cases}. \end{aligned}$$

According to Proposition 2.2:

Proposition 3.1. *The bundle T satisfies: for all ρ in S*

$$T|_{\{\rho\} \times X} \cong p_*^\pi(E_\rho).$$

Define an action of $\mathbf{SU}(n)$ on T by defining it on each $T|_{S \times U_i}$ by

$$\begin{aligned} \mathbf{SU}(n) \times (S \times U_i \times \mathbf{C}^n) &\longrightarrow S \times U_i \times \mathbf{C}^n \\ (g, (\rho, x, u)) &\longmapsto (g \cdot \rho, x, g(u)). \end{aligned}$$

This action is well defined because if $x \in U_j \cap U_i$ and $t = (\rho, x, u)$ is in $S \times U_i \times \mathbf{C}^n$, then in the trivialisation $S \times U_j \times \mathbf{C}^n$, t is written $t = (\rho, x, v(x)\rho(\gamma_{i,j})(u))$ where $v(x)$ is a scalar and

$$\begin{aligned} g \cdot (\rho, x, v(x)\rho(\gamma_{i,j})(u)) &= (g \cdot \rho, x, g(v(x)\rho(\gamma_{i,j})(u))) \\ &= (g \cdot \rho, x, v(x)g\rho(\gamma_{i,j})g^{-1}g(u)). \end{aligned}$$

This last term is $(g \cdot \rho, x, g(u))$ written in $S \times U_j \times \mathbf{C}^n$. This action is a lift for the action of $\mathbf{SU}(n)$ on $S \times X$. Unfortunately it does not come from an action of

$\mathbf{PSU}(n)$ and the bundle T does not descend to a bundle on $\mathfrak{m} \times X$. Indeed the centre $\mathbf{Z}/n\mathbf{Z}$ of $\mathbf{SU}(n)$ acts trivially on S but the generator $\zeta\mathbf{I}$ of $\mathbf{Z}/n\mathbf{Z}$ acts by multiplication by ζ in the fibres. To overcome this problem, we can construct a line bundle L on S with an action of $\mathbf{SU}(n)$ lifting the action on S and such that $\zeta\mathbf{I}$ also acts by multiplication by ζ in the fibres. We will also denote L the induced bundle on $S \times X$. The bundle $T \otimes L^*$ has the property of Proposition 3.1 but the action of $\mathbf{SU}(n)$ reduces to an action of $\mathbf{PSU}(n)$. By taking the quotient we get

Proposition 3.2. *Let M be the line bundle of Remark 2.7. The bundle*

$$U = M \otimes (T \otimes L^*) / \mathbf{PSU}(n) \longrightarrow \mathfrak{m} \times X$$

is a universal bundle for \mathfrak{m}_β . That is, if $[E] \in \mathfrak{m}_\beta$ is the class of a bundle $E \rightarrow X$ then $U|_{[E] \times X}$ is isomorphic to E .

We still have to prove the existence of the bundle L .

Lemma 3.3. *There exists a line bundle L over S with an action of $\mathbf{SU}(n)$ lifting the one of $\mathbf{SU}(n)$ on S . This action satisfies: $\zeta\mathbf{I}$ acts by multiplication by ζ in the fibres.*

Proof. The proof is inspired from [16].

The bundle $M \otimes T$ is a family (parameterised by S) of rank n , degree d stable holomorphic vector bundles. Let E be in this family and let k be an integer. By Serre duality,

$$H^1(E \otimes (\Omega_X^1)^k) = H^0(E^\vee \otimes (\Omega_X^1)^{1-k})^*$$

and this is the null vector space. Otherwise there would exist a non zero homomorphism $(\Omega_X^1)^{k-1} \rightarrow E^\vee$ and thus a subbundle of E^\vee of degree bigger than or equal to $2(g-1)(k-1) \geq 0$. This is impossible because E is stable.

The $H^0(E \otimes (\Omega_X^1)^k)$ form a holomorphic bundle (see [13]) A_k over S of rank u_k the dimension of $H^0(E \otimes (\Omega_X^1)^k)$. By Riemann-Roch, we have

$$\begin{aligned} u_k &= d(E \otimes (\Omega_X^1)^k) + n(1-g) \\ &= d(E) + 2nk(g-1) + n(1-g) \\ &= d + n(g-1)(2k-1) \\ &= 2hk + d - h \quad (\text{where } h = n(g-1)). \end{aligned}$$

We have

$$\begin{aligned} (u_2, u_1) = 1 &\Leftrightarrow (d + 3h, d + h) = 1 \Leftrightarrow (2h, d + h) = 1 \\ &\Leftrightarrow d + h \text{ is odd and } (d, h) = 1. \end{aligned}$$

As d and n are co prime, d and h are co prime if and only if d and $g-1$ are co prime. If in addition we assume $g-1$ is odd then $d+n(g-1)$ is odd (d and n have different parities). In this case, there exist integers a and b such that $au_1 + bu_2 = 1$ and we can take

$$L = (\wedge^{u_1} A_1)^a \otimes (\wedge^{u_2} A_2)^b.$$

Otherwise, there exists $g' \geq g$ such that $g' - 1$ is odd and $(d, g' - 1) = 1$. The injection

$$\begin{aligned} \mathbf{SU}(n)^{2g} &\longrightarrow \mathbf{SU}(n)^{2g'} \\ (A_1, B_1, \dots, A_g, B_g) &\longmapsto (A_1, B_1, \dots, A_g, B_g, 1, 1, \dots, 1, 1) \end{aligned}$$

restricts to an equivariant injection

$$S \rightarrow S'$$

where S' is the set of $2g'$ -tuple of matrices

$$S' = \{(A_1, B_1, \dots, A_{g'}, B_{g'}), \prod_{k=1}^{g'} [A_k, B_k] = \zeta \mathbf{I}\}.$$

We have seen we can construct on S' a line bundle with the required properties. We take L to be the restriction of this bundle to S . \square

Let us use the universal bundle to define classes in $H^*(\mathfrak{m}_\beta)$.

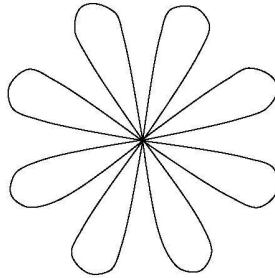


FIGURE 1. Bouquet of $2g$ circles (with $g = 4$)

Let B be a bouquet of $2g$ circles (Figure 1) embedded in X' in such a way that X' retracts on B . Each of the $2g$ circles defines a class in $H_1(X)$. Let $\alpha_1, \dots, \alpha_{2g}$ be their Poincaré duals. They form a basis of $H^1(X)$. Let κ be the class of the volume form on X of volume 1. Let us decompose the characteristic classes of the projective bundle $P(U)$. For k in $[2, n]$:

$$p_k(P(U)) = a_k \otimes \mathbf{1} + \sum_{j=1}^{2g} b_{k,j} \otimes \alpha_j + d_k \otimes \kappa.$$

Then, according to Biswas and Raghavendra [4], we have

Theorem 3.4. *The family*

$$\{a_k, b_{k,j}, d_k, 2 \leq k \leq n, 1 \leq j \leq 2g\}$$

is a multiplicative system of generators of $H^(\mathfrak{m}_\beta) \simeq H_{\mathbf{SU}(n)}^*(\mu^{-1}(\beta))$.*

4. A bundle over $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ and its Chern classes

Let B be a bouquet of $2g$ circles (Figure 1) embedded in X' in such a way that X' retracts on B . The theory of vector bundles with their Chern classes is the same on B and X' . We want to construct a complex vector bundle on $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$. Denote B' the star with $2g$ branches (see Figure 2), that is $B' = (\cup_{i=1}^{2g} [0, 1]_i) / \sim$,

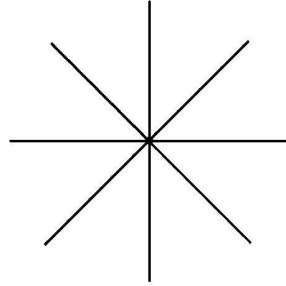


FIGURE 2. A star with $2g$ branches (with again $g = 4$)

where \sim is the equivalence relation that identifies all the 0 to a point. There is a natural map

$$\eta : B' \longrightarrow B.$$

It is defined by means of the exponential $\exp : [0, 1] \rightarrow S^1$. Denote

$$D_n = (\mathbf{SU}(n)^{2g} \times E\mathbf{U}(n) \times B' \times \mathbf{C}^n) / \sim$$

where \sim is the relation:

$$\begin{aligned} ((\rho_1, \dots, \rho_{2g}), e, 0, v) &\sim ((\text{Ad}_A \rho_1, \dots, \text{Ad}_A \rho_{2g}), A \cdot e, 1_i, A \circ \rho_i(v)), \\ &\forall i \in [1, 2g], \forall A \in \mathbf{SU}(n). \end{aligned}$$

The projection

$$D_n \longrightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$$

makes D_n into a rank n complex vector bundle over $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$. We wish to compute the characteristic classes of the projectivised bundle $P(D_n)$ of D_n . Notice that as the structure group of D_n reduces to $\mathbf{SU}(n)$, the classes $p_k(P(D))$ are equal to the Chern classes $c_k(D)$.

Let us describe the cohomology of $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$. By the Künneth formula, we have

$$H^*((\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B) = H^*_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) \otimes H^*(B).$$

Proposition 4.1. *Let G be a compact Lie group. Let k be an integer bigger than 0. Let G act on G^k diagonally by conjugation. The equivariant cohomology of G^k is isomorphic, as a graded algebra, to $H^*(G^k) \otimes H^*(BG)$.*

Proof. The fibration $(G^k)_G \longrightarrow BG$ is cohomologically trivial (see [3]) so that we have an isomorphism of graded vector spaces between $H^*_G(G^k)$ and $H^*(G^k) \otimes H^*(BG)$. The proposition then follows from the fact that for any compact Lie group, its cohomology is an exterior algebra on a finite number of elements and from the

Lemma 4.2. *Let $q : N \longrightarrow M$ be a cohomologically trivial fibration with fiber F . Assume that the cohomology of F is an exterior algebra on a family $\{\xi_1, \dots, \xi_r\}$. Then the cohomology of N is isomorphic, as a graded algebra, to the tensor product of $H^*(F)$ and $H^*(M)$.*

Proof. Let \mathfrak{J} be the set of strictly increasing sequences of integers $I = (i_1, \dots, i_p)$ such that $i_1 \geq 1$ and $i_p \leq r$. For $I \in \mathfrak{J}$ with $I = (i_1, \dots, i_p)$, let

$$\xi_I = \xi_{i_1} \wedge \dots \wedge \xi_{i_p}.$$

The family $\{\xi_I\}_{I \in \mathfrak{J}}$ forms a basis of $H^*(F)$.

To say that the fibration $N \longrightarrow M$ is cohomologically trivial is equivalent (by the Leray-Hirsch Theorem) to saying that the inclusion of a fiber F into N induces a surjection $H^*(N) \longrightarrow H^*(F)$. For $i \in [1, r]$, let ζ_i , in $H^*(N)$, be a pre-image of ξ_i . For $I \in \mathfrak{J}$ with $I = (i_1, \dots, i_p)$, let

$$\zeta_I = \zeta_{i_1} \wedge \dots \wedge \zeta_{i_p}.$$

The map

$$\begin{aligned} H^*(F) &\longrightarrow H^*(N) \\ \sum \lambda_I \xi_I &\longmapsto \sum \lambda_I \zeta_I \end{aligned}$$

is a morphism of algebra and the map

$$\begin{aligned} H^*(F) \otimes H^*(M) &\longrightarrow H^*(N) \\ (\sum \lambda_I \xi_I) \otimes \chi &\longmapsto (\sum \lambda_I \zeta_I) \otimes q^*(\chi) \end{aligned}$$

is an isomorphism of graded algebra. □

□

According to the previous proposition, we have isomorphisms

$$\begin{aligned} H_{\mathbf{SU}(n)}^*(\mathbf{SU}(n)^{2g}) &\simeq H^*(\mathbf{SU}(n)^{2g}) \otimes H^*(BSU(n)) \\ &\simeq \otimes_{j=1}^{2g} H^*(\mathbf{SU}(n)) \otimes H^*(BSU(n)). \end{aligned} \quad (3.1)$$

For all $k \geq 2$, the fibration $\mathbf{SU}(k) \longrightarrow S^{2k-1}$ with fiber $\mathbf{SU}(k-1)$ is cohomologically trivial (see Hatcher [7]). Let γ_k be the volume form of volume 1 on S^{2k-1} . The cohomology of $\mathbf{SU}(n)$ is the exterior algebra freely generated by the family $\{\sigma_k, 2 \leq k \leq n\}$, where σ_k is a class of degree $2k-1$ which pulls-back under the restriction $\mathbf{SU}(k) \longrightarrow \mathbf{SU}(n)$ to the image of γ_k by $H^{2k-1}(S^{2k-1}) \longrightarrow H^{2k-1}(\mathbf{SU}(k))$. Denote $\sigma_{k,j}$ the image of $\sigma_k \in H^{2k-1}(\mathbf{SU}(n))$ by the homomorphism $H^*(\mathbf{SU}(n)) \rightarrow H^*(\mathbf{SU}(n)^{2g})$ induced by the projection on the j -th factor $\mathbf{SU}(n)^{2g} \rightarrow \mathbf{SU}(n)$. We have

Lemma 4.3. *The algebra $H^*(\mathbf{SU}(n)^{2g})$ is the exterior algebra freely generated by the family $\{\sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq 2g, \deg \sigma_{k,j} = 2k-1\}$.*

In addition, we know that $H^*(BSU(n)) = \mathbf{Q}[c_2, \dots, c_n]$. From the preceding lemma and Proposition 4.1, we deduce

Theorem 4.4. *Let Λ be the exterior algebra freely generated by the family $\{\sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq 2g, \deg \sigma_{k,j} = 2k-1\}$. The $\mathbf{SU}(n)$ -equivariant cohomology of $\mathbf{SU}(n)^{2g}$ is isomorphic, as a graded algebra, to $\Lambda \otimes \mathbf{Q}[c_2, \dots, c_n]$.*

When there is no risk of confusion, we will write c_k and $\sigma_{k,j}$ instead of respectively $1 \otimes c_k$ and $\sigma_{k,j} \otimes 1$.

Remark 4.5. The injection ι of $\mathbf{SU}(n)$ into $\mathbf{SU}(n+1)$ and the map $BSU(n) \rightarrow BSU(n+1)$ induce isomorphisms

$$H^k(\mathbf{SU}(n+1)) \xrightarrow{\sim} H^k(\mathbf{SU}(n)) \text{ for } k \leq 2n \text{ and } k = 2n+2 \quad (4.1)$$

and

$$H^k(BSU(n+1)) \xrightarrow{\sim} H^k(BSU(n)) \text{ for } k \leq 2n. \quad (4.2)$$

With the notations of Theorem 4.4, we have

Proposition 4.6. *The Chern classes of D_n are:*

$$\begin{aligned} c_0(D_n) &= 1, \\ c_1(D_n) &= 0, \\ c_k(D_n) &= (1 \otimes c_k) \otimes 1 + \sum_{j=1}^{2g} (\sigma_{k,j} \otimes 1) \otimes \alpha_j \text{ for } k \geq 2. \end{aligned}$$

Proof. The classes $c_0(D_n)$ and $c_1(D_n)$ are trivially 1 and 0 (the structure group is $\mathbf{SU}(n)$). Assume from now on that $k \geq 2$. Let us write the Chern classes of D_n

in $H^*((\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B)$ as

$$c_k(D_n) = \gamma_k^{(n)} \otimes 1 + \sum_{j=1}^{2g} \beta_{k,j}^{(n)} \otimes \alpha_j.$$

We will prove the proposition by induction on n . For $n = 1$, $\mathbf{SU}(1)$ is just a point, the bundle D_1 is trivial and we are already done. Suppose the proposition to be true for a given n , $n \geq 1$ and let us prove it for $n + 1$. We need to prove that

$$\gamma_k^{(n+1)} = 1 \otimes c_k \text{ and } \beta_{k,j}^{(n+1)} = \sigma_{k,j} \otimes 1.$$

Let

$$m : (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)},$$

the map induced by the inclusion $\mathbf{SU}(n) \longrightarrow \mathbf{SU}(n+1)$ and

$$\ell = m \times \text{id}_B : (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)} \times B.$$

The bundle $\ell^* D_{n+1}$ is isomorphic to $D_n \oplus \mathbf{C}$. Hence, for all k , we have $c_k(\ell^* D_{n+1}) = c_k(D_n)$. Thus

$$(m^* \gamma_k^{(n+1)}) \otimes 1 + \sum_{j=1}^{2g} (m^* \beta_{k,j}^{(n+1)}) \otimes \alpha_j = \gamma_k^{(n)} \otimes 1 + \sum_{j=1}^{2g} \beta_{k,j}^{(n)} \otimes \alpha_j.$$

From this we deduce

$$m^* \gamma_k^{(n+1)} = \gamma_k^{(n)} \text{ and } m^* \beta_{k,j}^{(n+1)} = \beta_{k,j}^{(n)}.$$

Because of the isomorphisms (4.1), (4.2) and the induction hypothesis, we have:

$$\begin{aligned} \gamma_k^{(n+1)} &= 1 \otimes c_k \text{ for } k \leq n, \\ \beta_{k,j}^{(n+1)} &= \sigma_{k,j} \otimes 1 \text{ for } k \leq n. \end{aligned}$$

There only remains to compute $\gamma_{n+1}^{(n+1)}$ and the $\beta_{n+1,j}^{(n+1)}$. The class $\gamma_{n+1}^{(n+1)}$ belongs to

$$H_{\mathbf{SU}(n+1)}^{2n+2}(\mathbf{SU}(n+1)^{2g}) = \bigoplus_{p+q=2n+2} H^p(\mathbf{SU}(n+1)^{2g}) \otimes H^q(B\mathbf{SU}(n+1)).$$

Let us decompose it

$$\gamma_{n+1}^{(n+1)} = \sum_{k=0}^{n+1} \varepsilon_k^{(n+1)} \otimes c_k$$

where $\varepsilon_k^{(n+1)}$ is in $H^{2n+2-2k}(\mathbf{SU}(n+1)^{2g})$ and where we have put $c_0 = 1$ in $H^0(B\mathbf{SU}(n))$, $c_1 = 0$. The classes $\beta_{n+1,j}^{(n+1)}$ are in

$$H_{\mathbf{SU}(n+1)}^{2n+1}(\mathbf{SU}(n+1)^{2g}) = \bigoplus_{p+q=2n+1} H^p(\mathbf{SU}(n+1)^{2g}) \otimes H^q(B\mathbf{SU}(n+1)).$$

We decompose them in

$$\beta_{n+1,j}^{(n+1)} = \sum_{k=0}^n \delta_{k,j}^{(n+1)} \otimes c_k$$

where $\delta_{k,j}^{(n+1)}$ belongs to $H^{2n+1-2k}(\mathbf{SU}(n+1)^{2g})$. The bundle $\ell^* D_{n+1} = D_n \oplus \mathbf{C}$ has a nowhere vanishing section, hence its Euler class $\ell^* c_{n+1}(D_{n+1})$ vanishes. Because of the isomorphisms (4.1) and (4.2), we deduce that the $\{\varepsilon_k^{(n+1)}, 1 \leq k \leq n\}$ and the $\{\delta_{k,j}^{(n+1)}, 1 \leq k \leq n, 1 \leq j \leq 2g\}$ vanish. Remark that the $\{\delta_{n+1,j}^{(n+1)}, 1 \leq j \leq 2g\}$ are linear combinations of the $\sigma_{2n+1,j}, 1 \leq j \leq 2g$. Let us define a section

$$\begin{aligned} s : B\mathbf{SU}(n+1) &\longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)} \times B \\ [e] &\longmapsto ([(\mathbf{I}, \dots, \mathbf{I}), e], 1) \end{aligned}$$

where, for e in $E\mathbf{U}(n+1)$, we denote by $[e]$ its class in $B\mathbf{SU}(n+1)$. The Euler class of the bundle $s^* D_{n+1}$ is εX_{n+1} . Since $s^* D_{n+1}$ is equal to $E\mathbf{U}(n+1) \times_{\mathbf{SU}(n+1)} \mathbf{C}^{n+1}$ we have $\varepsilon = 1$. As a conclusion we have

$$\gamma_{n+1}^{(n+1)} = 1 \otimes X_{n+1}.$$

Let

$$h : \mathbf{SU}(n+1)^{2g} \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)}$$

be the inclusion of a fiber (we will always write h this application, omitting the subscript n). The bundle

$$F_{n+1}^{2g} := (h \times \text{id}_B)^* D_{n+1}$$

is isomorphic to

$$F_{n+1}^{2g} \cong (\mathbf{SU}(n+1)^{2g} \times B' \times \mathbf{C}^{n+1}) / \sim,$$

where \sim is the relation:

$$((\rho_1, \dots, \rho_{2g}), 1_j, v) \sim ((\rho_1, \dots, \rho_{2g}), 0, \rho_j^{-1}(v)), \text{ for all } j \text{ in } [1, 2g].$$

The Euler class of F_{n+1}^{2g} is

$$c_{n+1}(F_{n+1}^{2g}) = \sum_{j=1}^{2g} \beta_{n+1,j}^{(n+1)} \otimes \alpha_j.$$

Let $f_j : S^1 \rightarrow B$ (resp. $g_j : \mathbf{SU}(n+1) \rightarrow \mathbf{SU}(n+1)^{2g}$) be the inclusion of the j -th circle (resp. $\mathbf{SU}(n+1)$) in B (resp. $\mathbf{SU}(n+1)^{2g}$). The $\beta_{n+1,j}^{(n+1)}$ are characterised by:

$$c_{n+1}((\text{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}) = \beta_{n+1,j}^{(n+1)} \otimes f_j^* \alpha_j,$$

or

$$c_{n+1}((\text{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}) = \beta_{n+1,j}^{(n+1)} \otimes \frac{d\theta}{2\pi}. \quad (4.3)$$

Let us define a vector bundle E over $\mathbf{SU}(n+1) \times S^1$ by

$$E = (\mathbf{SU}(n+1) \times [0, 1] \times \mathbf{C}) / \sim,$$

where \sim is the relation

$$(\rho, 1, v) \sim (\rho, 0, \rho^{-1}(v)).$$

The bundle $(\mathrm{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}$ is isomorphic to $(g_j \times \mathrm{id}_{S^1})^* E$. Hence there exists a real λ such that

$$c_{n+1}(E) = \lambda \sigma_{2n+1} \otimes \frac{d\theta}{2\pi}.$$

If (ρ, t, v) belongs to $\mathbf{SU}(n+1) \times [0, 1] \times \mathbf{C}^{n+1}$, let us write $[\rho, t, v]$ for its class in E . Let (e_1, \dots, e_{n+1}) be the canonical basis, over the field \mathbf{C} , of \mathbf{C}^{n+1} . The family $(e_1, ie_1, \dots, e_{n+1}, ie_{n+1})$ is then a basis of \mathbf{C}^{n+1} over \mathbf{R} . A section of E is given by:

$$\begin{aligned} s : \mathbf{SU}(n+1) \times S^1 &\longrightarrow E \\ (A, e^{2i\pi\theta}) &\longmapsto [A, \theta, (\theta A + (1-\theta)\mathrm{id})e_1]. \end{aligned}$$

Let us determine its zeros. The vector $(\theta A + (1-\theta)\mathrm{id})e_1$ vanishes if $\theta = \frac{1}{2}$ and $A = \begin{bmatrix} -1 & 0 \\ 0 & \tilde{A} \end{bmatrix}$, $\tilde{A} \in \mathbf{U}(n)$, $\det \tilde{A} = -1$. Fix ξ an n -th root of -1 . The zero set Z of s is

$$Z = \left\{ \left(\begin{bmatrix} -1 & 0 \\ 0 & \xi \tilde{A} \end{bmatrix}, \frac{1}{2} \right), \tilde{A} \in \mathbf{SU}(n) \right\}.$$

Lemma 4.7. *The section s intersects the zero section s_0 transversally.*

Proof. We want to prove that for all x of Z

$$T_{s(x)} \mathrm{Im} s + T_{s(x)} \mathrm{Im} s_0 = T_{(x,0)} E.$$

We have

$$T_{(x,0)} E \simeq T_x(\mathbf{SU}(n+1) \times S^1) \oplus \mathbf{C}^{n+1} \simeq \mathfrak{su}(n+1) \oplus \mathbf{R} \oplus \mathbf{C}^{n+1}$$

and

$$T_{s(x)} \mathrm{Im} s_0 = \mathfrak{su}(n+1) \oplus \mathbf{R} \oplus \{0\},$$

$$T_{s(x)} \mathrm{Im} s = T_x s(T_x(\mathbf{SU}(n+1) \times S^1)).$$

Let x be the point $(A = \begin{bmatrix} -1 & 0 \\ 0 & \xi \tilde{A} \end{bmatrix}, \frac{1}{2})$,

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} s(A, \tfrac{1}{2} + \varepsilon) &= [A, \tfrac{1}{2} + \varepsilon, ((\tfrac{1}{2} + \varepsilon)A + (\tfrac{1}{2} - \varepsilon)\mathrm{id})e_1] \\ &= [A, \tfrac{1}{2} + \varepsilon, -2\varepsilon e_1] \\ &= (0, 1, -2e_1). \end{aligned}$$

Let J be in $\mathfrak{su}(n+1)$,

$$\begin{aligned} \frac{d}{d\varepsilon}|_{\varepsilon=0} s(\exp(\varepsilon J)A, \tfrac{1}{2}) &= \frac{d}{d\varepsilon}|_{\varepsilon=0} [\exp(\varepsilon J)A, \tfrac{1}{2}, \tfrac{1}{2}(\exp(\varepsilon J)A + \text{id})e_1] \\ &= \frac{d}{d\varepsilon}|_{\varepsilon=0} [\exp(\varepsilon J)A, \tfrac{1}{2}, \tfrac{1}{2}(\exp(\varepsilon J)(-e_1) + e_1)] \\ &= (J \cdot A, 0, \tfrac{1}{2}(-Je_1 + e_1)). \end{aligned}$$

We conclude the proof of Lemma 4.7 by noticing that, for any k , it is possible to find J in $\mathfrak{su}(n+1)$ such that Je_1 is equal to e_k or ie_k . \square

Lemma 4.8. *The Euler class of the bundle E is*

$$c_{n+1}(E) = \sigma_{2n+1} \otimes \frac{d\theta}{2\pi}.$$

Proof. According to the preceding lemma, the Euler class of E is Poincaré dual of Z , that is it is characterised by

$$\forall \nu \in H^{n^2-1}(\mathbf{SU}(n+1) \times S^1), \int_Z \nu = \int_{\mathbf{SU}(n+1) \times S^1} \nu \wedge c_{n+1}(E)$$

where $n^2 - 1 = \dim(\mathbf{SU}(n+1) \times S^1) - 2(n+1)$. This Euler class is of the type

$$c_{n+1}(E) = \lambda \sigma_{2n+1} \otimes \frac{d\theta}{2\pi}$$

where λ is a real we are going to compute. The injection

$$\begin{aligned} \mathbf{SU}(n) &\longrightarrow \mathbf{SU}(n+1) \\ A &\longmapsto \begin{bmatrix} -1 & 0 \\ 0 & \xi A \end{bmatrix} \end{aligned}$$

identifies $\mathbf{SU}(n)$ to the fibre above $(-1, 0, \dots, 0)$ of the projection $\mathbf{SU}(n+1) \rightarrow S^{2n+1}$, that is Z . Let γ be the cohomology class of a volume form of volume 1 over $\mathbf{SU}(n)$. The decomposition $H^*(\mathbf{SU}(n+1)) = H^*(\mathbf{SU}(n)) \otimes H^*(S^{2n+1})$ defines a class

$$\nu = \gamma \otimes 1.$$

As the integral of ν on Z is 1, we have

$$\int_{\mathbf{SU}(n+1) \times S^1} \nu \wedge c_{n+1}(E) = 1,$$

that is

$$\lambda \int_{\mathbf{SU}(n+1) \times S^1} (\gamma \otimes 1) \wedge (\sigma_{2n+1} \otimes \frac{d\theta}{2\pi}) = 1.$$

The conclusion follows since the integral in the left-hand side of the equality is equal to 1. \square

Proposition 4.6 follows from this lemma. \square

5. Description of the restriction map

Using results of the previous sections, we wish to prove:

Theorem 5.1. *The restriction map r is described by*

$$\begin{aligned} r(c_k) &= a_k \text{ for } k = 2, \dots, n \\ r(\sigma_{k,j}) &= b_{k,j} \text{ for } k = 2, \dots, n, j = 1, \dots, 2g. \end{aligned}$$

In particular, $\text{Im}(r)$ is multiplicatively generated by

$$\text{Im}(r) = \langle a_k, b_{k,j}, k = 2, \dots, n, j = 1, \dots, 2g \rangle.$$

Notice that for n equals 2, we get that r is surjective modulo the symplectic form on \mathfrak{m}_β (this result has been in [18]).

It is also very interesting to compare this theorem with [11, Theo. 7.1] where a group cohomological construction of multiplicative generators of $H^*(\mathfrak{m}_\beta)$ is given.

Proof. The key point of the proof is to compare the bundles U of Section 3 and D_n of Section 4.

From now on, if $g \in \mathbf{SU}(n)$, we denote \bar{g} its class in $\mathbf{PSU}(n)$. Over each $S \times U_i$, $i = 0, \dots, m$, the bundle $M \otimes T \otimes L^*$ is trivial. In each of these sets, the action of $\mathbf{PSU}(n)$ on $M \otimes T \otimes L^*$ is

$$\begin{aligned} \mathbf{PSU}(n) \times M \otimes (S \times U_i \times \mathbf{C}^n) \otimes L^* &\longrightarrow M \otimes (S \times U_i \times \mathbf{C}^n) \otimes L^* \\ m \otimes (\bar{g}, (\rho, x, u) \otimes l) &\longmapsto m \otimes (g \cdot \rho, x, g(u)) \otimes (g \cdot l). \end{aligned}$$

Lemma 5.2. *We have*

$$\mathbf{P}(U) = \mathbf{P}(M \otimes (T \otimes L^*)/\mathbf{PSU}(n)) \cong \mathbf{P}(T)/\mathbf{PSU}(n).$$

Proof. This time, $\mathbf{PSU}(n)$ acts on $\mathbf{P}(T)$ by

$$\begin{aligned} \mathbf{PSU}(n) \times (S \times U_i \times \mathbf{CP}^n) &\longrightarrow (S \times U_i \times \mathbf{CP}^n) \\ (\bar{g}, (\rho, x, \bar{u})) &\longmapsto (g \cdot \rho, x, \overline{g(u)}) \end{aligned}$$

and the announced isomorphism is

$$\begin{aligned} \mathbf{P}(U) &\xrightarrow{\cong} \mathbf{P}(T)/\mathbf{PSU}(n) \\ \text{class of } m \otimes (\rho, x, u) \otimes l &\longmapsto \text{class of } (\rho, x, u). \end{aligned}$$

□

Lemma 5.3. *There exists an action of $\pi \times \mathbf{PSU}(n)$ on $S \times Y' \times \mathbf{CP}^{n-1}$ such that the quotient*

$$(S \times Y' \times \mathbf{CP}^{n-1})/(\pi \times \mathbf{PSU}(n))$$

is isomorphic to

$$\mathbf{P}(U)|_{\mathfrak{m}_\beta \times X'}.$$

Proof. The bundle T restricted to $S \times X'$ is trivial on each $S \times U_i$, $i \neq 0$ and transition functions are given by

$$\begin{aligned} (S \times U_i) \cap (S \times U_j) &\longrightarrow \mathbf{SU}(n) \\ (\rho, x) &\longmapsto \rho(\gamma_{i,j}). \end{aligned}$$

The group π acts freely on Y' and $T|_{S \times X'}$ is $(S \times Y' \times \mathbf{C}^n)/\pi$, where the action of π is

$$\begin{aligned} \pi \times (S \times Y' \times \mathbf{C}^n) &\longrightarrow S \times Y' \times \mathbf{C}^n \\ (\gamma, (\rho, y, u)) &\longmapsto (\rho, \gamma \cdot y, \rho(\gamma)u). \end{aligned}$$

Let us consider the projective bundle $P(T)|_{S \times X'}$. It is isomorphic to $(S \times Y' \times \mathbf{CP}^{n-1})/\pi$. The subspace $P(T)|_{S \times X'}$ is stable by $\mathbf{PSU}(n)$ and the action comes from an action of $\mathbf{PSU}(n)$ on $S \times Y' \times \mathbf{CP}^{n-1}$. That is

$$\begin{aligned} \mathbf{PSU}(n) \times (S \times Y' \times \mathbf{CP}^{n-1}) &\longrightarrow S \times Y' \times \mathbf{CP}^{n-1} \\ (\bar{g}, (\rho, y, \bar{u})) &\longmapsto (g \cdot \rho, y, \overline{g(u)}). \end{aligned}$$

This action commutes indeed with the one of π , the result follows. \square

The pull-back of the bundle $U \rightarrow (S/\mathbf{PSU}(n)) \times X'$ to $(S)_{\mathbf{SU}(n)} \times X'$ by the natural map

$$f : (S)_{\mathbf{SU}(n)} \times X' \longrightarrow (S/\mathbf{PSU}(n)) \times X'$$

is a vector bundle, we will denote it F . Its projectivised bundle is

$$P(F) = (P(T))_{\mathbf{SU}(n)} \longrightarrow (S)_{\mathbf{SU}(n)} \times X'.$$

We will now state a proposition which will be our main tool in the study of the map r :

Proposition 5.4. *There is a projective bundle $P(D)$ over $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ whose restriction to $(S)_{\mathbf{SU}(n)} \times X'$ is isomorphic to $P(F)$.*

First proof. The projection $p : Y' \rightarrow X'$ is a covering. Its group is π . Let $q : \tilde{Y}' \rightarrow Y'$ be the universal covering of Y' . The composed map $\tilde{p} = p \circ q : \tilde{Y}' \rightarrow X'$ is the universal covering of X' . Its group is

$$\pi_1(X') = \langle a_1, b_1, \dots, a_g, b_g \rangle$$

and we have a projection $\pi_1(X') \xrightarrow{\theta} \pi$ whose kernel is the group of the covering $\tilde{Y}' \rightarrow Y'$.

The open covering of X' by the $\{U_i\}_{i=1}^m$ is such that any intersection of open sets of the type U_i is contractible. In particular, for all i , there exists a disc \tilde{D}_i in \tilde{Y}' such that $\tilde{p} : \tilde{D}_i \rightarrow U_i$ is a diffeomorphism. Choose, for all i, j, k , a connected component $W_{ij,k}$ of $\tilde{p}^{-1}(U_i \cap U_j) \cap \tilde{D}_k$. If $U_i \cap U_j \neq \emptyset$, let $\tilde{\gamma}_{i,j}$ be the element of $\pi_1(X')$ such that $\tilde{\gamma}_{i,j} \tilde{W}_{ij,j} = \tilde{W}_{ji,i}$. In Proposition 2.2, we can take the $W_{ij,k}$ and $\gamma_{i,j}$ such that

$$W_{ij,k} = \tilde{p}(\tilde{W}_{ij,k}), \gamma_{i,j} = \theta(\tilde{\gamma}_{i,j}).$$

Let us identify the set of representations $\rho : \pi_1(X') \rightarrow \mathbf{SU}(n)$ to $\mathbf{SU}(n)^{2g}$ by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)).$$

Let

$$T' \rightarrow \mathbf{SU}(n)^{2g} \times X'$$

be the rank n complex vector bundle defined by the following properties:

- (1) $T'|_{\mathbf{SU}(n)^{2g} \times U_i}$ is trivial,
- (2) the transition functions are

$$g_{i,j} = \rho(\tilde{\gamma}_{i,j}) \text{ on } \mathbf{SU}(n)^{2g} \times (U_i \cap U_j).$$

The restriction of this bundle to $S \times X'$ is $T|_{S \times X'}$. The action of $\mathbf{SU}(n)$ on $T|_{S \times X'}$ is then the restriction of the $\mathbf{SU}(n)$ action on T' defined on each $T'|_{\mathbf{SU}(n)^{2g} \times U_i}$ by

$$\begin{aligned} \mathbf{SU}(n) \times (\mathbf{SU}(n)^{2g} \times U_i \times \mathbf{C}^n) &\longrightarrow \mathbf{SU}(n)^{2g} \times U_i \times \mathbf{C}^n \\ (g, (\rho, x, u)) &\longmapsto (g \cdot \rho, x, g(u)). \end{aligned}$$

Notice that this action is a lift of the action of $\mathbf{SU}(n)$ on $\mathbf{SU}(n)^{2g} \times X'$. Thus the bundle

$$P(F) = (P(T))_{\mathbf{SU}(n)} \rightarrow (S)_{\mathbf{SU}(n)} \times X'$$

is the restriction of the bundle

$$(P(T'))_{\mathbf{SU}(n)} \rightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'.$$

Second proof. We have seen that

$$P(U)|_{\mathfrak{m}_\beta \times X'} \cong (S \times Y' \times \mathbf{CP}^{n-1})/(\pi \times \mathbf{PSU}(n)),$$

hence

$$P(F) \cong (S \times E\mathbf{U}(n) \times Y' \times \mathbf{CP}^{n-1})/(\pi \times \mathbf{SU}(n)).$$

Let us define, in a similar way as before, an action of $\pi_1(X')$ on $\mathbf{SU}(n)^{2g} \times E\mathbf{U}(n) \times \widetilde{Y}' \times \mathbf{C}^n$ and denote D the bundle we obtain when quotienting by $\pi_1(X') \times \mathbf{SU}(n)$. The projection $S \times E\mathbf{U}(n) \times \widetilde{Y}' \times \mathbf{C}^n \rightarrow S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}^n$ is equivariant for the respective actions of $\pi_1(X')$ and π . It induces an action on the quotient and defines an isomorphism between

$$(S \times E\mathbf{U}(n) \times \widetilde{Y}' \times \mathbf{C}^n)/(\pi_1(X') \times \mathbf{SU}(n))$$

and

$$(S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}^n)/(\pi \times \mathbf{SU}(n)).$$

We deduce that $P(F)$ is isomorphic to $P(D)|_{(S)_{\mathbf{SU}(n)} \times X'}$. □

Remark 5.5. The bundle $D \rightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ is isomorphic to $(T' \times E\mathbf{U}(n))/\mathbf{SU}(n)$.

When restricted to $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$, the bundle D is isomorphic to D_n (restricted to $(\mu^{-1}(\zeta\mathbf{I}))_{\mathbf{SU}(n)} \times B$). Denote w the injection of $(S)_{\mathbf{SU}(n)} \times X'$ in $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$. The induced map w^* in cohomology is $r \times \text{id}_{H^*(X')}$. The restriction w^*D_n of D_n to $(S)_{\mathbf{SU}(n)} \times X'$ has the same projectivisation as F . Thus, because of Proposition 4.6, we have for every k

$$p_k(P(F)) = a_k \otimes 1 + \sum_{j=1}^{2g} b_k^j \otimes \alpha_j \quad (5.1)$$

$$\begin{aligned} &= p_k(P(w^*D_n)) \\ &= w^*p_k(P(D_n)) \\ &= r(1 \otimes p_k) \otimes 1 + \sum_{j=1}^{2g} r(\sigma_{k,j} \otimes 1) \otimes \alpha_j. \end{aligned} \quad (5.2)$$

Theorem 5.1 follows from the comparison of Line (5.1) and Line (5.2). \square

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