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A perturbative $SU(3)$ Casson invariant

S. E. Cappell, R. Lee and E. Y. Miller

Abstract. A perturbative $SU(3)$ Casson invariant $\Lambda_{SU(3)}(X)$ for integral homology 3-sphere is defined. Besides being fully perturbative, it has the nice properties: (1) $4 \cdot \Lambda_{SU(3)}$ is an integer. (2) It is preserved under orientation change. (3) A connect sum formula. Explicit calculations of the invariant for $1/k$ surgery of $(2,q)$ torus knot are presented and compared with Boden–Herald's different $SU(3)$ generalization of Casson's invariant. For those cases computed, the invariant defined here is a quadratic polynomial in k for $k > 0$ and a different quadratic polynomial for $k < 0$.

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1. Introduction

From a gauge theoretic viewpoint, the well-known $SU(2)$-Casson invariant $\lambda_{SU(2)}(X)$ of an integral homology 3-sphere X can be regarded as the number, counted with sign, of flat $SU(2)$-connections on X after making a suitable perturbation of the curvature equation [T]. In Casson's original treatment, $\lambda_{SU(2)}$ was obtained from a finite dimensional, symplectic setting, as the intersection number in a representation variety of two perturbed Lagrangian subvarieties associated to a Heegaard decomposition of X (see [AM]). In both these gauge-theoretic and symplectic settings, the fact that perturbations were used in the definition and that large scale perturbations are permissible underlay remarkable properties of the Casson invariant, such as surgery formulae. In this paper we solve the problem of defining a (fully) perturbative $SU(3)$ generalization, $\Lambda_{SU(3)}(X)$, of the Casson invariant, and begin the study of its properties. Some of these recall well-known facts about the $SU(2)$-Casson invariant:

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(1) An integrality property: $4 \cdot \Lambda_{SU(3)}(X) \in \mathbb{Z}$.

(2) In the cases computed here, for $1/k$-surgery on some torus knots, the invariants are given by quadratic polynomials in k, for k positive (resp. negative) while in the $SU(2)$ case they are linear.

(3) It is preserved under the change of orientation, just as the $SU(2)$- invariant is reversed.

On the other hand, it differs intriguingly from the $SU(2)$-invariant in that the polynomials giving the values for $1/k$-surgery on the torus knots in (2) for k positive are not the same as those for k negative.

Our investigation has benefited greatly from the excellent series of recent articles of Boden–Herald and of Boden–Herald–Kirk–Klassen [BH 1,2], [BHKK]. In [BH 1] a different gauge-theoretic generalization, $\lambda_{SU(3)}(X)$, of the Casson invariant to $SU(3)$ was introduced using – and allowing – only small perturbations; it is thus not fully perturbative. Among the important properties Boden–Herald obtained for their invariant are: $\lambda_{SU(3)}$ is independent of orientation, $\lambda_{SU(3)}(X) = \lambda_{SU(3)}(-X)$, and has a connect sum formula $\lambda_{SU(3)}(X_1 \# X_2) = \lambda_{SU(3)}(X_1) + \lambda_{SU(3)}(X_2) + 4\lambda_{SU(2)}(X_1)\lambda_{SU(2)}(X_2)$ (see [BH 1,2]). In the paper [BHKK], there are impressive calculations of this invariant for $1/k$-surgery on some torus knots, with the result that the values are given by various rational functions in k, cubic polynomials divided by linear polynomials in their cases. As is already evident from their calculations, in special cases $\lambda_{SU(3)}$ takes values which are fractions with varying denominators; moreover, this would follow more generally from a conjecture on Chern–Simons invariants. Thus this contrasts with the integrability property of the invariant considered here.

Some years ago, in [CLM], we proposed a program for defining a generalized $SU(n)$-Casson invariant based on a Lagrangian intersection number of perturbed subvarieties in the $SU(n)$-representations of $\pi_1(X)$. That program proposed using in such a definition, correction terms obtained from combinations of tangential and normal Maslov indices along the singular strata of reducible representations. In part to understand these correction terms, we studied the relation between Maslov index and spectral flow in [CLM 1,2] and the different definitions of $SU(2)$-Casson invariants for rational homology spheres in [CLM 3]. The present effort could be viewed as a modification and completion of the program of [CLM] for $SU(3)$. The new ingredient in the definition is a further term which involves the boundary maps of the mod-2 Floer chain complex [F]. It is this extra term which makes the invariant well-defined and, as in Theorem (3.4), fully perturbative as we had wished.

We now provide a precise comparison of these two invariants $\lambda_{SU(3)}(X)$ and $\Lambda_{SU(3)}(X)$. Recall that the $SU(2)$-Casson invariant, $\lambda_{SU(2)}(X)$, was reformulated by Taubes [T] in a gauge-theoretic setting, as the sum:

$$\lambda_{SU(2)}(X) = (-1) \sum_{[A] \in \mathcal{M}^*_{SU(2),h}} (-1)^{SF(\theta,A,h;su(2))} \tag{1.1}$$

where $[A]$ runs through all gauge equivalent classes of h-perturbed $SU(2)$-connections.[1] The sign $(-1)^{SF(\theta,A,h;su(2))}$ is specified by the spectral flow $SF(\theta,A,h;su(2))$ associated to a path of connections from the trivial connection θ to A. Throughout this paper, we fix a trivial connection θ in the space $\mathcal{A}$ of $SU(3)$-connections. To compute spectral flow, we have to choose a lifting of $[A] \in \mathcal{B}$ to $A \in \mathcal{A}$ and then connect up by a path from θ to A. From the path lifting property, this amounts to the choice of a path from $[\theta]$ to $[A]$ in $\mathcal{B}$. Hence, this last spectral flow $SF(\theta, A, h; su(2))$ depends on the choice of such a path; however the ambiguity equals to 0 (mod 8) and disappears when we form the sign $(-1)^{SF(\theta,A,h;su(2))}$.

In the work of Boden–Herald [BH 1], as briefly reviewed in Section 2 below, the invariant $\lambda_{SU(3)}(X)$ is given by

$$\begin{aligned} \lambda_{SU(3)}(X) &= \lambda'_{SU(3)}(X) + \lambda''_{SU(3)}(X) \\ \lambda'_{SU(3)}(X) &= \sum_{[A]\in\mathcal{M}^*_{SU(3),h}} (-1)^{SF(\Theta,A,h;su(3))} \\ \lambda''_{SU(3)}(X) &= \sum_{[A]\in\mathcal{M}^*_{S(U(2)\times U(1)),h}} (-1)^{SF(\theta,A,h;su(2))}[SF(\theta,A,h;\mathbb{C}^2) - 2cs(\hat{A}) + 1] \end{aligned} \tag{1.2}$$

after making a "small" perturbation h. The correction term $\lambda''_{SU(3)}$ is introduced because the number $\lambda'_{SU(3)}$ of h-perturbed flat, irreducible, $SU(3)$- connections in $\mathcal{M}^*_{SU(3),h}$ depends on the choice of perturbations. Given two perturbations h_0, h_1, we can connect them up by a family of small perturbations $h_t, 0 \le t \le 1$. Along this path, there would exist a cobordism joining points in $\mathcal{M}^*_{SU(3),h_0}$ and $\mathcal{M}^*_{SU(3),h_1}$ but for the phenomena of irreducible $SU(3)$-connections sinking into or emerging from the $SU(2)$-stratum. Whenever this occurs, a corresponding integer jump occurs in the normal spectral flow $SF(\theta, A, h; \mathbb{C}^2)$. Thus the discrepancy in $\lambda'_{SU(3)}(X)$ is compensated by the sum $\Sigma(-1)^{SF(\theta,A,h;su(2))}[SF(\theta, A, h; \mathbb{C}^2)]$.

However, the above spectral flow $SF(\theta, A, h; \mathbb{C}^2)$ depends on the choice of paths from the trivial connection θ to A. By definition, a "small" perturbation has the property that the h-perturbed flat, irreducible, $SU(2)$-connections $[A] \in \mathcal{M}^*_{SU(2),h}$ is within ϵ-distance of a unique component $\hat{A}$ in the space $\mathcal{M}^*_{SU(2)}$ of flat connections. In particular, we have a well-defined path class α from A to an element in the component $\hat{A}$. Given such a component $\hat{A}$, we can also choose a path β connecting an element in $\hat{A}$ to the trivial connection θ and using this path we can calculate the Chern–Simons invariant $cs(\hat{A})$. On the other hand, the composite $\beta \circ \alpha$ provides a way to connect up A with θ, and hence a spectral flow invariant $SF(\theta, A, h; \mathbb{C}^2)$. Although both $cs(\hat{A})$ and $SF(\theta, A, h; \mathbb{C}^2)$ depend on the choice

[1] The first minus sign (-1) is explained in [KK] also page 5 of [BH 1].

of the path β, the ambiguities cancel each other and the combination yields a well-defined term $[SF(\theta, A, h; \mathbb{C}^2) - 2cs(\hat{A}) + 1]$ in (1.2).

Now the present, perturbative $SU(3)$-Casson invariant, $\Lambda_{SU(3)}(X)$, is given by the formula:

$$\begin{aligned}
\Lambda_{SU(3)}(X) &= \Lambda'_{SU(3)} + \Lambda''_{SU(3)}(X) - \frac{1}{4}\mathrm{Floer}(X, h) \qquad (1.3)\\
\Lambda'_{SU(3)}(X) &= \sum_{[A]\in \mathcal{M}^*_{SU(3),h}} (-1)^{SF(\Theta,A,h;su(3))}\\
\Lambda''_{SU(3)}(X) &= \sum_{[A]\in \mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))}[SF(\theta, A, h; \mathbb{C}^2)\\
&\quad - \frac{1}{4} SF(\theta, A, h; s(u(1) \times u(2))) + 5/8]\\
\mathrm{Floer}(X, h) &= \sum_{p=0}^{7}(-1)^p \dim_{Z/2}(\text{Image d:} \quad FC_{p+1}(X, h) \to FC_p(X, h)).
\end{aligned}$$

Here the first term $\Lambda'_{SU(3)}(X)$ is the same as $\lambda'_{SU(3)}(X)$. In the second term $\Lambda''_{SU(3)}(X)$, the normal spectral flow $SF(\theta, A, h; \mathbb{C}^2)$ is the same as that in $\lambda''_{SU(3)}(X)$ while the Chern–Simons term $cs(\hat{A})$ is replaced by 1/4 of the tangential spectral flow, $\frac{1}{4}SF(\theta, A, h; s(u(1)\times u(2)))$. The combination $[SF(\theta, A, h; \mathbb{C}^2) - \frac{1}{4}SF(\theta, A, h; s(u(1) \times u(2)))]$ was shown in [CLM] to be independent of the choice of paths connecting θ to $[A]$ and has the advantage of being free from the restrictive assumption of small perturbations. Unfortunately, the tangential spectral flow $SF(\theta, A, h; s(u(1) \times u(2)))$ also creates a problem of its own. For a family of perturbations h_t, a pair $(A_t(1), A_t(2))$ of h_t-perturbed flat, irreducible, $SU(2)$-connections can be created or destroyed through their collision at a birth-death point (the analogue of Whitney disk cancellation in the context of finite dimensional handle decompositions). Whenever this happens, the terms in the sum $\Sigma SF(\theta, A, h; s(u(1)\times u(2)))$ corresponding to $(A_t(1), A_t(2))$ will cause a jump and so $\Lambda'_{SU(3)} + \Lambda''_{SU(3)}(X)$ is not a well-defined invariant.

Analogues of such problems of jumps have been studied in parametrized Morse theory, but here we have to adjust this to the infinite dimensional gauge space with the Chern–Simons functional as the Morse function. Although the Floer homology $FH_*(X)$ with $Z/2$-coefficients[2] is well-defined, its Floer chain groups $FC_*(X, h)$ varies precisely because of the existence of these birth-death points. Indeed, a fixed integer jump occurs in $\mathrm{Floer}(X, h_t)$ when h_t goes through such a birth or death point. Hence $\mathrm{Floer}(X, h)$ can be used as a correction term for the discrepancy in

[2] We can also work with Floer homology in integer or other coefficients.

$\Sigma SF(\theta, A, h; s(u(1)\times u(2)))$. Detailed analysis of $\Lambda'_{SU(3)(X)}, \Lambda''_{SU(3)(X)}$, $\text{Floer}(X, h)$ as well as the proof that $\Lambda_{SU(3)}(X)$ is well-defined (Theorem 3.4) can be found in Section 3.

Despite the differences between $\lambda_{SU(3)}$ and $\Lambda_{SU(3)}$, they also share some properties. For example, they are independent of orientation (see Proposition 4.5 for $\Lambda_{SU(3)}$) and have connect sum formulae. Due to the Floer correction term, the formula for $\Lambda_{SU(3)}$ is more complicated than its counterpart in [BH 2], as it involves the Floer chain complex of the connected sum which is a subtle aspect of Floer homology theory (see [Fu], [Li]). The proof of this connect sum formula for $\Lambda_{SU(3)}$ is in Section 4.

In Section 5, we provide explicit calculations of our invariant for the Brieskorn spheres $\Sigma(2, q, 2qk \pm 1), q = 3, 5, 7, 9$, which can also be obtained from $\mp 1/k$ surgery on $(2, q)$-torus knots. Our results are parallel to those in [BHKK] where $\lambda_{SU(3)}(\Sigma(2, q, 2qk \pm 1))$ in the same range are computed. However we have to calculate the spectral flow $SF(\theta, A, h; s(u(1) \times u(2)))$ for all flat, irreducible $SU(2)$-connections $[A]$. In [FS], Fintushel–Stern calculated these spectral flows and their results are tailor-made for us (see Theorem 5.1).[3]

In general, the perturbative $SU(3)$-Casson invariant $\Lambda_{SU(3)}$ does not take on integer value. However, it is a easy matter to find a remedy for this situation. In Section 6, we provide a slight modification of $\Lambda_{SU(3)}(X)$ by adding the terms $\frac{1}{4}\sum_{i=0}^{7}(-1)^i i \ \dim(FH_i) - \frac{5}{8}\sum_{i=0}^{7}(-1)^i \dim(FH_i)$ from Floer homology FH_i. Then the result is an integer-valued invariant $\Lambda_{SU(3)}(X)_{alt.}$ of integral homology 3-sphere X (see (6.1)). The proof of this, as well as the general criterion for integrality of $\Lambda_{SU(3)}(X)$, is in Section 6.

As mentioned above, Casson's $SU(2)$-invariant was first defined using a Heegaard decomposition and intersection of perturbed Lagrangians in the representation varieties. We briefly discuss how the representation-theoretic analogue of the present gauge-theoretic treatment of $\Lambda_{SU(3)}$ would proceed, as this was the context envisioned in [CLM]: Using a Heegaard decomposition, we can write X as a union $X_1 \cup X_2$ of two handle bodies X_1, X_2 glued along a Riemann surface Σ. Then the moduli space $\mathcal{M}_{SU(3)}(X)$ of flat $SU(3)$-connections can be identified with the intersection of the Lagrangian subspaces $R_{SU(2)}(X_i) = \text{Hom}(\pi_1(X_i), SU(3))/SU(3)$ inside $R_{SU(3)}(\Sigma) = \text{Hom}(\pi_1(\Sigma), SU(3))/SU(3)$. After a suitable Hamiltonian perturbation, the Maslov indices at the reducibles are defined and a Floer correction introduced. Then the symplectic definition of $\Lambda_{SU(3)}$ is the same as in (1.3). Indeed, to define $\text{Floer}(X, h)$, it is natural to consider a symplectic Floer homology theory based on the intersection of $R_{SU(2)}(X_i)$ in the $SU(2)$-stratum $R_{SU(2)}(\Sigma)$. In this direction, there are the works of Lee–Li [LL], which treats the singular

[3] Extending the results of [BHKK] and [FS] used here, we will present general calculations of rho invariants of Seifert fiber spaces in [CLM 4].

nature of $R_{SU(2)}(\Sigma)$, and of Sullivan [S] which addresses the change of Floer chain complexes in the smooth context under perturbations.

Finally, the general methodology introduced here to define fully perturbative invariants by using Floer(X, h) may appear complicated in that this term has a "tertiary" character, being the correction to the Maslov index correction term along singularities. But this method opens up for $\Lambda_{SU(3)}$, and perhaps more generally for other invariants in both the gauge theoretic settings and in the settings of intersections of Lagrangian subvarieties of Symplectic varieties, the possibility of intriguing relations with still unknown Floer theories. In particular, as $8 \cdot \Lambda_{SU(3)}$[4] is an integer invariant, it suggests the existence of a $SU(3)$-Floer homology with $8 \cdot \Lambda_{SU(3)}$ as its Euler characteristic.

2. Review of the work of Boden and Herald

Let X be an oriented, integral homology 3-sphere and let $\mathcal{A}$ be the space of smooth, $SU(3)$-connections on the trivial product bundle $P = X \times SU(3)$. This last space $\mathcal{A}$ is an infinite dimensional affine space and, in fact, by fixing a trivial product connection θ on P, we can identify $\mathcal{A}$ with the space $\Omega^1(X, AdP) = \Omega^1(X, su(3))$ of $su(3)$-valued 1-form on X.

Let $\mathcal{G} = \mathrm{Map}(X, SU(3)) = C^\infty(X, SU(3))$ denote the gauge group of $SU(3)$-bundle automorphisms $g : P \to P$ of P. Then as these gauge transformations change the bundle structure and hence the connections $A \to g \cdot A = gAg^{-1} + gdg^{-1}$, they give rise to an action of $\mathcal{G}$ on $\mathcal{A}$ with $\mathcal{B} = \mathcal{A}/\mathcal{G}$ as quotient. This action is not free, and according to the isotropy subgroups there is a natural Whitney stratification on $\mathcal{A}$ and also on the orbit space $\mathcal{B} = \mathcal{A}/\mathcal{G}$. An $SU(3)$-connection A in $\mathcal{A}$ is said to be irreducible if its isotropy subgroup consists of constant maps to $\mathbf{Z}(SU(3)) = \mathbf{Z}/3$. Altogether these irreducibles form the top stratum $\mathcal{A}^*$ and its quotient $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ has a structure of a pre-Banach manifold.

Below the top stratum, there are strata whose isotropy subgroups are respectively $U(1), S(U(1) \times U(1) \times U(1))$, $S(U(1) \times U(2))$ and $SU(3)$, they correspond to the situation where the underlying 3-dimensional complex vector bundles and connections are decomposed respectively into:

(2.1)

(a) A sum $L \oplus Q$ of line bundle L and a 2-plane bundle Q with structure group $S(U(1) \times U(2))$.

(b) A sum $L_1 \oplus L_2 \oplus L_3$ of three line bundles L_1, L_2, L_3 which are all different and with structure group $S(U(1) \times U(1) \times U(1))$.

(c) A sum $L_1 \oplus L_1 \oplus L_2$, $L_1 \neq L2$ of three line bundles, two of which are the same and with structure group $S(U(1) \times U(1))$.

[4] Although $4 \cdot \Lambda_{SU(3)}$ is an integer, it is more natural to consider $8 \cdot \Lambda_{SU(3)}$ as an Euler characteristic.

(d) A sum $L_1 \oplus L_1 \oplus L_1$ of three isomorphic line bundles with structure group $\mathbf{Z}/3$.

If we consider only the subspace $\mathcal{A}_{\text{flat}}$ of flat $SU(3)$-connections, then the relevant strata are those of isotropy subgroup $\mathbf{Z}/3$, $U(1)$ and $SU(3)$, i.e. the irreducibles together with (2.1)(a) and (d). The reason is that, for an integral homology sphere M, there exist no nontrivial $U(1)$-representations $\pi_1(M) \to U(1)$ and hence every flat connection is gauge equivalent to the trivial connection.

Now, over $\mathcal{A}$ there is the Chern–Simons functional $cs : \mathcal{A} \to \mathbf{R}$ given by

$$cs(A) = \frac{1}{8\pi^2} \int_X tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{2.2}$$

With respect to a gauge transformation $g \in \mathcal{G}$, we have

$$cs(g \cdot A) = cs(A) + \deg(g) \tag{2.3}$$

where $\deg g$ is the image, under $g^* : H^3(SU(3)) \to H^3(X) = \mathbf{Z}$, of a canonical generator in $H^3(SU(3))$. Because of (2.3), there is an induced map

$$cs : \mathcal{B} \to \mathbf{R}/\mathbf{Z}$$

on the quotient spaces. As is well-known [T], the gradient of cs is given by

$$\nabla cs(A) = -\frac{1}{4\pi^2} * F_A, \tag{2.4}$$

and so the set of critical points of cs coincides with the moduli space

$$\begin{aligned} \mathcal{M}_{SU(3)}(X) &= \mathcal{A}_{\text{flat}}/\mathcal{G} \\ &= \{[A] \in \mathcal{B} \mid *F_A = 0\} \end{aligned} \tag{2.5}$$

of gauge equivalent classes of flat $SU(3)$-connections on X.

By taking the intersection with the strata on $\mathcal{B}$, we obtain an induced stratification on $\mathcal{M}_{SU(3)}(X)$. In fact, because of (2.1), we can give an explicit description of all these strata. First of all, we have the top stratum of irreducible, flat, $SU(3)$-connections, denoted by $\mathcal{M}^*_{SU(3)}$. Then we have the stratum consisting of $SU(3)$-connections which are the sum of an irreducible, flat, $SU(2)$-connection and a trivial product, $U(1)$-connection. Since this last stratum is isomorphic to the moduli space of irreducible, flat, $SU(2)$-connections, we will denote it by $\mathcal{M}^*_{SU(2)}$. Finally, there is the stratum $[\theta]$ consisting of the single, isolated, trivial $SU(3)$-connection.

To obtain a well-defined invariant, Boden and Herald perturb the Chern–Simons functional so that the resulting critical points are finite number of regular points, i.e. points cut out transversely by the equation [BH 1]. Following the idea of

Floer and others [F] in $SU(2)$-gauge theory, they consider the space $\mathcal{F}$ of admissible perturbations consisting of a collection of n solid tori $\gamma_i : S^1 \times D^2 \to X, 1 \leq i \leq n$, and invariant functions $\tau_i : SU(3) \to \mathbb{R}$ and compactly supported 2-form η on D^2 with $\int_{D^2} \eta = 1$. Then, for each element in $\mathcal{F}$, the perturbation is given by adding to the Chern–Simons functional the following:

$$h(A) = \sum_{i=1}^{n} \int_{D^2} \tau_i(hol_i(x, A))\eta(x)dx$$

where $hol_i(x, A)$ is the holonomy of the connection A around the loop $\gamma_i(S^1 \times x)$.

Note that h is invariant under gauge transformation and so $A \to cs(A) + h(A)$ descends to a function on $\mathcal{B}$. After taking the differential, we obtain a section of $\mathcal{A} \times \Omega^1(X; su(3))$

$$\begin{aligned} \zeta_h : \mathcal{A} &\longrightarrow \Omega^1(X; su(3)) \\ A &\longrightarrow \frac{-1}{4\pi^2} * F_A + \nabla h. \end{aligned}$$

A connection is said to be h-perturbed flat if it satisfies the equation $\frac{-1}{4\pi^2} * F_A + \nabla h = 0$. The set of all gauge equivalent classes of such connections forms a moduli space, called the perturbed moduli space, $\mathcal{M}_{SU(3),h}(X) = \zeta_h^{-1}(0)/\mathcal{G}$. It has many properties of $\mathcal{M}_{SU(3)}$: For example, it is compact (Proposition 2.9 of [BH 1]). In Theorem 3.13 of [BH 1], it is shown that, inside the space $\mathcal{F}(\epsilon_0)$ of small ($\|h\| \leq \epsilon_0$), admissible perturbations, there exists a Baire set $\mathcal{F}(\epsilon_0)'$ of perturbations under which $\mathcal{M}_{SU(3),h}(X)$ is regular. Moreover, for any two perturbations h_{-1}, h_1, in $\mathcal{F}(\epsilon_0)'$, there exits a path h_t of small perturbations connecting h_{-1}, h_1 such that the parametrized moduli space $W = \{(A, t) \in \mathcal{A} \times [-1, 1] \mid \zeta_{h_t}(A) = 0\}$ is also regular.

The precise definition of small perturbation $h \in \mathcal{F}(\epsilon_0)$ is in Proposition 3.7 of [BH 1]. Basically, ϵ_0 is chosen so that

If $\|h\| \leq \epsilon_0$ and A is h-perturbed flat, then there exists $\hat{A} \in \mathcal{A}_{flat}$ with $\|A - \hat{A}_0\| \leq \epsilon_0$. (2.6)

If A, A' are flat and lie in different components of $\mathcal{A}_{flat}$ then $\|A - A'\| \geq 2\epsilon_0$. (2.7)

Since $\mathcal{A}_{flat}$ is disjoint from those strata with isotropy subgroups $S(U(1) \times U(1) \times U(1)), S(U(1) \times U(2))$ we can choose ϵ_0 so small that by (2.6) the perturbed moduli space

$$\mathcal{M}_{SU(3),h}(X) = \mathcal{M}^*_{SU(3),h}(X) \cup \mathcal{M}^*_{S(U(1)\times U(1))}(X) \cup [\theta],$$

in other words, a h-perturbed flat $SU(3)$-connection is either irreducible or with isotropy subgroup $U(1)$ or $SU(3)$.

Another consequence of (2.6), (2.7) is that associated to a h-perturbed flat connection A, there is a unique component $\hat{A}$ of flat connections which is within ϵ_0-distance. From this, there is a well defined invariant,

$$SF(\theta, A, h; s(u(1) \times u(2))) - 2cs(\hat{A}),$$

where the ambiguity of the path-dependent spectral flow $SF(\theta, A, h; s(u(1) \times u(2)))$ is cancelled by the corresponding choice in $cs(\hat{A})$, as explained in Section 1.

We will need several closely related spectral flows whose definitions can all be traced back to the linearized operator of ζ_h :

$$*d_{A,h} = *d_A - 4\pi^2 \cdot \text{Hess } h(A) : \Omega^1(X; su(3)) \to \Omega^1(X; su(3))$$

where Hess $h(A)$ is the Hessian of h. In terms of $*d_{A,h}$, there is the self adjoint, Fredholm operator $K(A, h; su(3))$ given by

$$\begin{aligned} K(A, h, su(3)) :& (\Omega^0 \oplus \Omega^1)(X; su(3)) \longrightarrow (\Omega^0 \oplus \Omega^1)(X; su(3)) \\ & (\xi, a) \longrightarrow (*d_A a, d_a \xi + *d_{A,h}(a)). \end{aligned} \tag{2.8}$$

Similarly, for a connection $A \in \mathcal{A}$ with isotropy subgroup $U(1)$, the structure group of A can be reduced to $S(U(1) \times U(2))$. Hence, we can form the operator $K(A, h, s(u(1) \times u(2))$ by taking the tensor product of the self-adjoint operator in (2.8) with the adjoint representation $s(u(1) \times u(2))$.

All the above are real, self-adjoint operators, and so when we discuss their spectral flow we count the number of real eigenspaces crossing a $(-\epsilon/-\epsilon)$-reference line. However, for a $S(U(1) \times U(1))$-connection A, we also have the complex operator $K(A, h; \mathbb{C}^2)$ obtained by coupling the self-adjoint operator with the regular representation $\mathbb{C}^2$ of $S(U(1) \times U(1))$. Following the convention in [BHKK], the spectral flows for these operators refers to the number of complex eigenspaces crossing the $(-\epsilon/-\epsilon)$- reference line.

In the background of all these, there is also the deformation complex:

$$\begin{aligned} &\Omega^0(X; su(3)) \xrightarrow{d_A} \Omega^1(X; su(3)) \xrightarrow{*d_{A,h}} \\ &\Omega^1(X; su(3)) \xrightarrow{d_A^*} \Omega^0(X; su(3)) \end{aligned} \tag{2.9}$$

associated to a h-perturbed flat, $SU(3)$-connection A. In [BH 1], it is shown that this is a Fredholm, elliptic complex with $H^0(X; su(3)) = \text{Ker} d_A$ and

$$H^1_{(A,h)}(X; su(3)) = \text{Ker}(*d_{(A,h)})/Im d_A.$$

In particular, for a h-perturbed flat $SU(3)$-connection A, we have

$$\text{Ker} K(A, h; su(3)) = H^0(X; su(3)) \oplus H^1_{(A,h)}(X; su(3)), \tag{2.10}$$

and when A is irreducible $H^0(X; su(3)) = 0$ and the vanishing of the kernel of $K(A, h, su(3))$ is the same as the vanishing of $H^1_{(A,h)}(X; su(3))$.

Given a path $\{A_t \mid 0 \leq t \leq 1\}$ of connections from the trivial $SU(3)$-connection, denoted by Θ, to the connection $A = A_1$, we have the family of self-adjoint, Fredholm operators $K(A_t, h; su(2))$ and hence its spectral flow $SF(\Theta, A, h; su(3))$. Although the latter depends on the choice of paths, it only enters into our discussion through the expression $(-1)^{SF(\Theta,A,h;su(3))}$ for the sign. Since the ambiguity due to the choice of paths of $SF(\Theta, A, h; su(3))$ is 12 (see Prop 4.3 of [BH 1]), this last sign is well-defined.

Similarly for a path $\{A_t \mid 0 \leq t \leq 1\}$ of $S(U(1) \times U(2))$-connections from the trivial representation, here denoted by θ, we have the spectral flows $SF(\theta, A, h; s(u(1) \times u(2)))$ and $SF(\theta, A, h; \mathbb{C}^2)$ for the two families of self-adjoint operators $K(A_t, h; s(u(1) \times u(2)))$ and $K(A_t, h; \mathbb{C}^2)$. The ambiguities due to the choice of paths for $K(A_t, h; s(u(1) \times u(2)))$ are 8 and for $K(A_t, h; \mathbb{C}^2)$ are 2. Once again we suppress this dependence because they come into our application either as $(-1)^{SF(\theta,A,h;s(u(1)\times u(2)))}$ or as $SF(\theta, A, h; s(u(1) \times u(2))) - cs(\hat{A})$. Here, in the second case, the ambiguities have been compensated by the Chern–Simons term.

With a choice of small perturbation h which makes $\mathcal{M}_{SU(3),h}(X)$ regular and with the convention of spectral flows as explained above, Boden and Herald define their invariant $\lambda_{SU(3)}(X)$ by the formula (1.2). The following is their main theorem (Theorem 1 of [BH 1]).

Theorem 2.11. *Suppose X is an integral homology 3-sphere. For generic small perturbation $h, \mathcal{M}^*_{SU(3),h}(X)$ and $\mathcal{M}^*_{S(U(2)\times U(1)),h}(X)$ are smooth, compact, 0-dimensional manifolds. Choose a representative A for each orbit $[A] \in \mathcal{M}^*_{SU(3),h}(X)$ and in case $[A] \in \mathcal{M}^*_{S(U(2)\times U(1)),h}(X)$ choose also a flat connection $\hat{A}$ close to A. Define $\lambda_{SU(3)}(X)$ as in (1.2). Then for h sufficiently small, $\lambda_{SU(3)}(X)$ is independent of h and the Riemannian metric and hence is a well-defined topological invariant of X.*

3. Correction term via Floer chain complex

Recall that the reason for introducing the Chern–Simons term $cs(\hat{A})$ is to make the expression $[SF(\theta, A, h; \mathbb{C}^2) - 2cs(\hat{A})]$ well defined, independent of the choice of path. However, there are other devices which can achieve the same goal.

Lemma (3.1). *If we use the same path $\{A_t \mid 0 \leq t \leq 1\}, A_0 = \theta, A_1 = A$ in computing the spectral flows $SF(\theta, A, h; \mathbb{C}^2), SF(\theta, A, h; s(u(1) \times u(2)))$, then the difference $\left[SF\left(\theta, A, h; \mathbb{C}^2\right) - \frac{1}{4}\left(SF\left(\theta, A, h; s\left(u(1) \times u(2)\right)\right)\right)\right]$ is well-defined, independent of the choice of paths $\{A_t \mid 0 \leq t \leq 1\}$.*

Proof. The ambiguities in $SF\left(\theta, A, h; \mathbb{C}^2\right)$ and $SF\left(\theta, A, h; s(u(1) \times u(2)\right)$ are the result of the nontrivial nature of the fundamental group of the gauge space $\pi_1(\mathcal{B}(S(U(1)\times U(2))) = \pi_0(\mathrm{Map}(X,U(2)) = \mathbb{Z}$. A straightforward computation shows that they are 8 for $SF(\theta,A,h;s(u(1)\times u(2)))$ and 2 for $SF\big(\theta,A,h;\mathbb{C}^2\big)$. Hence, they cancel out in taking the difference $SF(\theta,A,h;\mathbb{C}^2)) - \frac{1}{4}(SF(\theta,A,h;s(u(1)\times u(2)))$.

In view of (3.1),we can replace $\lambda''_{SU(3)}(X)$ in (1.2) by the expression:

$$\Lambda''_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}(X)} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))}\Big[SF\big(\theta,A,h;\mathbb{C}^2\big) - \frac{1}{4}SF(\theta,A,h;s(u(1)\times u(2))) + (5/8)\Big]. \tag{3.2}$$

This has the advantage that we can free ourselves from the restriction of using only small perturbations.

On the other hand, in this correction scheme, a new phenomenon can occur. Namely, during a parametrized family of perturbations h_t a pair of h_t-perturbed connections $A_t(1), A_t(2)$ from different components of $\mathcal{M}^*_{S(U(1)\times U(2))}$ can annihilate each other, as in the birth-death point situation in parametrized Morse theory. In fact, as we will see such an annihilation will cause a jump in the sum (3.2) and to compensate for this we have to introduce a tertiary correction term from the Floer chain complex.

From now on, we consider the space of admissible perturbations $h\in\mathcal{F}$ without the assumption of being small, i.e. (2.6), (2.7) but subject to the admissibility condition introduced below.

Note that the choice of Wilson's loops $\gamma_i : S^1\times D^2 \to X$ and the invariant functions $\tau_i : SU(3)\to\mathbb{R}$ are the same as in those in Floer's work. In particular, when we restrict to the stratum $\mathcal{A}_{S(U(1)\times U(2))}$, we obtain the analogue of Floer's theory. Namely, we have a chain complex $FC_*(X,h)$ over $\mathbb{Z}/2$, which has the elements of $\mathcal{M}^*_{S(U(1)\times U(2)),h}(X)$ as generators and is indexed by the Floer degree. This Floer degree for a h-perturbed flat connection A is given by $SF\left(K(A_t,h,s(u(1)\times u(2))\right)$ mod 8 where A_t is any path of connections from the trivial connection θ to A.

Hence associated to h, we have the integer

$$\mathrm{Floer}\,(X,h) = \sum_{p=0}^{7}(-1)^p \dim_{\mathbb{Z}/2}\left\{\text{image of } d : FC_{p+1}(X,h)\to FC_p(X,h)\right\}$$

where the chain complex is a slight extension of Floer's treatment for $SU(2)$ to $S(U(1)\times U(2))$. The associated Floer homology is the same since by concentrating on small perturbations near $\mathcal{A}_{SU(2)}$, we can deform $FC_*(X,h)$ back to the $SU(2)$

situation. Note that the integer Floer (X, h) is sensitive to the perturbation h and is precisely a device which can account for the birth-death points between different perturbations. With the Floer correction term as explained above, the perturbative $SU(3)$-Casson invariant $\Lambda_{SU(3)}(X)$ of an integral homology 3-sphere X is defined by the formula (1.3).

Remark (3.3). In the definition of our invariant $\Lambda_{SU(3)}$, we have to use admissible perturbations h with the property that the h-perturbed flat, reducible connections can have isotropy subgroups of the type $\mathbb{Z}/3, U(1)$ or $SU(3)$. Note this is the case with the flat $SU(3)$-connections before any perturbations $h = 0$, and also holds for sufficiently small perturbations $h \leq \epsilon$ nearby. For example, in the treatment [F] [T], it is required that near the trivial representation there is no contributions by requiring the perturbation small near this point. We will also maintain this condition for our perturbations. In addition, for any two such perturbations h_0, h_1, we require that they can be connected up by a path $h_t, 0 \leq t \leq 1$, with the same property. Thus we have to delete from the space of admissible perturbations $\mathcal{F}$ those which have perturbed flat connections with bad isotropy subgroups $S(U(1) \times U(2)), S(U(1) \times U(1) \times U(1))$, and then, in the resulting space, consider the connected component $\mathcal{F}^o$ which contains the trivial perturbation. For the rest of this paper, we will work with this perturbation space $\mathcal{F}^o$ and its subspace $\mathcal{F}'$ which satisfies the regularity condition. Note that it is open and since any two perturbation can be connected up to the trivial one they can also be connected up to each other.

Theorem 3.4. *The number $\Lambda_{SU(3)}(X)$ is independent of the Riemannian metric on X and the admissible perturbation $h \in \mathcal{F}'$, and hence gives a well-defined, topological invariant of the integral homology 3-sphere X.*

Proof. For the most part, we follow the argument of Boden and Herald in [BH 1] in establishing the well-definedness of $\lambda_{SU(3)}(X)$. First of all, as in Theorem 3.13 of [BH 1], there exists a Baire set $\mathcal{F}'$ of admissible perturbations (not necessarily small) such that for $h \in \mathcal{F}'$, an h-perturbed flat connection A has isotropy subgroup $\mathbb{Z}_3$ (irreducible case) or $U(1)$ (reducible case). In the irreducible case, $\mathrm{Ker}(K(A, h; su(3))) = 0$ and in the reducible case $\mathrm{Ker}(K(A, h; \mathbb{C}^2)) = \mathrm{Ker}(K(A, h; s(u(1) \times u(2))) = 0$. These are referred to as the regularity conditions because under these conditions the moduli spaces $\mathcal{M}^*_{SU(2),h}(X)$ and $\mathcal{M}^*_{S(U(1)\times U(2)),h}(X)$ are smooth, 0-dimensional oriented compact manifolds. In particular, they consist of finitely many points (up to gauge equivalence) and using the data associated to them we can compute the sum $\Lambda_{SU(3)}(X) = \Lambda'_{SU(3)}(X) + \Lambda''_{SU(3)}(X) - \frac{1}{4}$ Floer (X, h) as in (1.3).

Now for two such perturbations h_0, h_1, we can connect them up by a path of admissible perturbations $\rho = \{h(t) \mid 0 \le t \le 1\}$ such that the parametrized moduli space W_ρ of $h(t)$-perturbed flat connections is regular. More precisely, $W_\rho = W_\rho^* \cup W_\rho^r$ with W_ρ^* a space of irreducible $SU(3)$-connections and W_ρ^r a space of $S(U(1) \times U(2))$-connections. Both W_ρ^* and W_ρ^r are properly embedded, smooth, oriented 1-manifold with boundary, where the boundary of W_ρ^* is the union $\mathcal{M}^*_{SU(3),h_0} \cup \mathcal{M}^*_{SU(3),h_1} \cup F$ with F a finite set of points in W_ρ^r; and the boundary of W_ρ^r is $\mathcal{M}^*_{S(U(1)\times U(2)),h_0} \cup \mathcal{M}^*_{S(U(1)\times U(2)),h_1}$.

Note that W_ρ^r may contain circle components. However, the regularity together with compactness condition for parametrized family implies that they are finite in number because each gives rise to critical points with respect to the projection in the t-direction and there are finitely many such critical points. Thus by partitioning $[0,1]$ into small intervals $[t(i), t(i+1)], 0 = t(0) < t(1) < \cdots < t(n) = 1$ in a suitable fashion, we can break down these circles as a union of arcs whose intersection with the closure $\bar{W}_\rho^*$ lie in the interior of these arcs. As $\Lambda_{SU(3),h_1} - \Lambda_{SU(3),h_0} = \sum_{i=0}^{n-1}[\Lambda_{SU(3),h_{t(i+1)}} - \Lambda_{SU(3),h_{t(i)}}]$ is additive, we can concentrate on the parametrized families over these small intervals $[t(i), t(i+1)]$. In short, we can assume that no circle components exist in W_ρ^r.

In view of the above discussion, let $S(0,1)$ denote the union of curves in W_ρ^r that pass from $t = 0$ to $t = 1$, $S(0,0)$ denote those that pass from $t = 0$ to $t = 0$, and $S(1,1)$ from $t = 1$ to $t = 1$. To simplify our notation, we list them as parametrized curves:

$$\begin{aligned} S(0,1) &= \{\gamma(j,u) \mid 0 \le u \le 1,\ j = 1, \cdots, N\} \\ S(0,0) &= \{\gamma'(j',u) \mid 0 \le u \le 1,\ j' = 1, \cdots, N'\} \\ S(1,1) &= \{\gamma''(j'',u) \mid 0 \le u \le 1,\ j'' = 1, \cdots, N''\}. \end{aligned}$$

As we move along a curve $\{\gamma(j,u) \mid 0 \le u \le 1\}$ in $S(0,1)$, Taubes [T] shows that the "tangential" signs $(-1)^{SF(\theta,A,h;s(u(1)\times u(2))}$ at the two ends agree. Denote this common value by $s_{n(j)} = s_{n(\gamma(j,0))} = s_{n(\gamma(j,1))}$. On the other hand, by [BH 1] there are precisely $s_{n(j)}[SF(K(\gamma(j,u),h;\mathbb{C}^2) \mid 0 \le u \le 1)]$ many h-perturbed flat, irreducible $SU(3)$ connections sinking into or emitting from this curve, each of which is counted with sign $(-1)^{SF(K(A,h;su(3))}$. Hence we have Sum(01) $= \sum_{j=1}^{N} s_{n(j)}[SF(K(\gamma(j,u),h;\mathbb{C}^2) \mid 0 \le u \le 1)]$.

Similarly, for a curve $\gamma'(j',u) \mid 0 \le u \le 1$ in $S(0,0)$ it follows from [T] that the "tangential signs" $(-1)^{SF(\theta,A,h;s(u(1)\times u(2))}$ disagree. So we orient the curve in such a way that it traces from sign -1 to sign $+1$. Then, in [BH 1], it is shown that there are $-[SF(K(\gamma'(j',u),h;\mathbb{C}^2) \mid 0 \le u \le 1)]$ many h-perturbed flat, irreducible, $SU(3)$-connections sinking into (or emitting from if negative) points on this curve, counted with the signs, $(-1)^{SF(K(A,h;su(3))}$. In toto, they give Sum(00) $= \sum_{j'=1}^{N'} -\big[SF(K(\gamma'(j',u),h;\mathbb{C}^2) \mid\le u \le 1)\big]$.

The analysis for a curve $\gamma''(j'',u) \mid 0 \le u \le 1$ in $S(1,1)$ is the same. From [T], the tangential signs at the two ends disagree and we orient the curve so that it travels from -1 to $+1$. From [BH 1], during its history, there are precisely $+[SF(K(\gamma''(j'',u),h;\mathbb{C}^2) \mid 0 \le u \le 1)]$ many h-perturbed flat, irreducible, $SU(3)$-connections sinking into (or emitting from) points on this curve, counted with their signs, $(-1)^{SF(K(A,h;su(3))}$. These give the sum: $\mathrm{Sum}(11) = \sum_{j''=1}^{N''} -[SF(K(\gamma''(j'',u),h;\mathbb{C}^2) \mid 0 \le u \le 1)]$.

Note that an irreducible $SU(3)$-connection in $\mathcal{M}^*_{SU(3),h_0}(X)$ at $t=0$ can either travel all the way to $\mathcal{M}^*_{SU(3),h_1}(X)$ at $t=1$ or be destroyed (likewise created) along the paths in $S(0,1), S(0,0), S(1,1)$. In the first case, by [T], the contribution of the two end points cancel each other in the difference $\Lambda'_{SU(3),h_0} - \Lambda'_{SU(3),h_1}$ while in the second case it enters as a term in $-\mathrm{Sum}(01), \mathrm{Sum}(00), -\mathrm{Sum}(11)$ (respectively for points created). Thus we have the formula

$$\Lambda'_{SU(3),h_0} - \Lambda'_{SU(3),h_1} = -\mathrm{Sum}(01) + \mathrm{Sum}(00) - \mathrm{Sum}(11). \tag{3.5}$$

To prove (3.4), we subtract the term $\Lambda''_{SU(3),h_0} - \Lambda''_{SU(3),h_1}$ from the two sides of (3.5) to get:

$$\begin{aligned} &\left[\Lambda'_{SU(3),h_0} - \Lambda''_{SU(3),h_0}\right] - \left[\Lambda'_{SU(3),h_1} - \Lambda''_{SU(3),h_1}\right]. \\ &\quad = -\mathrm{Sum}(01) + \mathrm{Sum}(00) - \mathrm{Sum}(11) + \left[\Lambda''_{SU(3),h_0} - \Lambda''_{SU(3),h_1}\right] \end{aligned} \tag{3.6}$$

The idea is to rewrite the right-hand side so that it can be identified with the difference of Floer correction terms. Note that, for a path $\{\gamma(u) \mid 0 \le u \le 1\}$ of $S(U(1) \times U(2))$-connections, the difference of the two spectral flows

$$\begin{aligned} &\left[SF\left(\theta,\gamma(1),h;\mathbb{C}^2\right) - \frac{1}{4}SF\left(\theta,\gamma(1),h;s\left(U(1)\times U(2)\right)\right)\right] \\ -&\left[SF\left(\theta,\gamma(0),h;\mathbb{C}^2\right) - \frac{1}{4}SF\left(\theta,\gamma(0),h;s\left(U(1)\times U(2)\right)\right)\right] \end{aligned}$$

can be simplified into

$$SF\left[K(\gamma(u),h;\mathbb{C}^2) \mid 0 \le u \le 1\right] - \frac{1}{4}SF\left[K(\gamma(u),h;s(u(1)\times u(2))) \mid 0 \le u \le 1\right]$$

by the additivity of spectral flows. We will apply this device to the terms in $\Lambda''_{SU(3),h_0}(X) - \Lambda''_{SU(3),h_1}(X)$ which correspond to pairs of points, connected up by paths in $S(01), S(00), S(11)$.

For example, along a curve $\gamma(j,u)$ in $S(01)$ the signs $s_{(\gamma(j,u))}$, at the two ends $u = 0,1$ are the same, and so in the difference $\Lambda''_{SU(3),h_0}(X) - \Lambda''_{SU(3),h_1}(X)$ we have

$$\begin{aligned}
&s_{(\gamma(j,1))}\left[SF\big(\theta,\gamma(j,1),\ h;\mathbb{C}^2\big)-\frac{1}{4}\,SF\big(\theta,\gamma(j,1),\ h;s(u(1)\times u(2))\big)\right]\\
&-s_{(\gamma(j,0))}\left[SF\left(\theta,\gamma(j,0),h;\mathbb{C}^2\right)-\frac{1}{4}\,SF\big(\theta,\gamma(j,0),\ h;s(u(1)\times u(2))\big)\right]\\
&=s_{(\gamma(j,0))}\Bigg[SF\left(K\left(\gamma(j,u),\ h;\mathbb{C}\right)\mid 0\le u\le 1\right)\\
&\qquad -\frac{1}{4}\,SF\left(K\left(\gamma(j,u),\ h;s(u(1)\times u(2)\right)\mid 0\le u\le 1)\right)\Bigg].
\end{aligned}$$

Note that the first sum cancels the contribution to the sum $S(01)$ by the same curve $\gamma(j,u)$.

Similarly, along a curve $\gamma'(j,u)$ in $S(00)$, we have the following contribution to $\Lambda''_{SU(3),h_0}(X)-\Lambda''_{SU(3),h_1}(X)$:

$$\begin{aligned}
&-s_{(\gamma'(j',1))}\left[SF\left(\theta,\gamma'(j',1),h;\mathbb{C}^2\right)-\frac{1}{4}SF(\theta,\gamma'(j',1),h;s(u(1)\times u(2)))\right]\\
&-s_{(\gamma'(j',0))}\left[SF\left(\theta,\gamma'(j',0),h;\mathbb{C}^2\right)-\frac{1}{4}SF(\theta,\gamma'(j',0),h;s(u(1)\times u(2)))\right]\\
&=-\Bigg[SF\left(K\left(\gamma'(j',u),\ h;\mathbb{C}^2\right)\mid 0\le u\le 1\right)\\
&\qquad -\frac{1}{4}\,SF\left(K\left(\gamma'(j',u),\ h,s(u(1)\times u(2))\right)\mid 0\le u\le 1\right)\Bigg].
\end{aligned}$$

In the last line, the first term cancels the corresponding contribution to Sum (00) in (3.6) by the curve. The same works for a curve $\gamma''(j'',u)$ in $S(11)$ and provides us with the contribution to $\Lambda''_{SU(3),h_0}(X)-\Lambda''_{SU(3),h_1}(X)$:

$$\begin{aligned}
&s_{(\gamma''(j'',1))}\left[SF\left(\theta,\gamma''(j'',1),h;\mathbb{C}^2\right)-\frac{1}{4}SF\left(\theta,\gamma''(j'',1),h;s(u(1)\times u(2))\right)\right]\\
&+s_{(\gamma''(j'',0))}\left[SF(\theta,\gamma''(j'',0),h;\mathbb{C}^2)-\frac{1}{4}SF(\theta,\gamma''(j'',0),h;s(u(1)\times u(2)))\right]\\
&=+\Bigg[SF\left(K\left(\gamma''(j'',u),\ h;\mathbb{C}^2\right)\mid 0\le u\le 1\right)\\
&\qquad -\frac{1}{4}\,SF\left(K\left(\gamma''(j'',u),\ h;s(u(1)\times u(2))\right)\mid 0\le u\le 1\right)\Bigg].
\end{aligned}$$

Once again, this last term cancels the contribution to $-$Sum (11) in (3.6) by the same curve.

Thus we can rewrite (3.6) as follows:

$$\left[\Lambda'_{SU(3),h_0}(X)+\Lambda''_{SU(3),h_0}(X)\right]-\left[\Lambda'_{SU(3),h_0}(X)+\Lambda''_{SU(3),h_1}(X)\right] = \frac{1}{4}\left[-\mathrm{Sum}'(01)+\mathrm{Sum}'(00)-\mathrm{Sum}'(11)\right]. \tag{3.7}$$

Here the sums $\mathrm{Sum}'(01), \mathrm{Sum}'(00)$, and $\mathrm{Sum}'(11)$, are obtained from the corresponding sums, $\mathrm{Sum}(01), \mathrm{Sum}(00)$, and $\mathrm{Sum}(11)$ by replacing the spectral flow of the normal operator $K\left(A_t, h; \mathbb{C}^2\right)$ by the corresponding tangential operator $K\left(A_t, h; s(u(1)\times u(2)\right)$ over the same path of connections A_t.

To complete the proof of (3.4), it remains to show that the sum on the right-hand side of (3.7) is $\frac{1}{4}[\text{Floer }(X,h_0)-\text{Floer }(X,h_1)]$. First of all, we can make sure that our perturbations are in the Floer homology setting: Not only are the critical points nondegenerate but the gradient flow equations along the flow lines are also nondegenerate, so that Floer homology is defined. Then we observe that $\mathrm{Sum}'(01)=0$ because by regularity the kernel of the operator $K\left(\gamma(j,u),\, h; s(u(1)\times u(2)\right)$ is zero for every u, $0\le u\le 1$. On the other hand, the spectral flows in $\mathrm{Sum}'(00)$ and $\mathrm{Sum}'(11)$ are not always zero as the kernels of $K\left(\gamma'(j',u),\, h; s(u(1)\times u(2))\right)$ and $K\left(\gamma''(j'',u),\, h, s(u(1)\times u(2))\right)$ may have jumps at critical points of $t\left(\gamma'(j',u)\right)$ and $t\left(\gamma''(j'',u)\right)$. The situation can be explained in terms of deformations of Floer chain complexes. In the language of parametrized Morse theory, a Floer chain complex can be deformed from one to another by a sequence of four moves [S]:

Move 1. (isotopy) The chain complex is unchanged.

Move 2. (handle slide) The chain groups are unchanged but one of the differentials are changed by composing with an elementary matrix.

Move 3. (birth point) Two new generators e, f are added in dimension p, $p+1$ with $\partial f=e$, direct sum the earlier chain complex.

Move 4. (death point) the reverse of move 3.

Furthermore, in the above Moves, the generators, other than those pairs from birth-death points, move smoothly with constant Floer index and zero tangential spectral flows. While in a neighborhood of a birth point in Move 3, we have pairs of generators with consecutive Floer indices $p, p+1$. These pairs of generators trace out a curve $\{\gamma(t), 0\le t\le 1\}$, and the tangential spectral flow $SF(K(\gamma(t),h;s(u(1)\times u(2)) \mid 0\le t\le 1\}$ along this curve equals 1 as it starts from index p and ends at index $p+1$. In the case of the death point, this is just the opposite.

Hence in Moves 1, 2, the expression $[-\mathrm{Sum}'(01)+\mathrm{Sum}'(00)-\mathrm{Sum}'(11)]$ is unchanged. In Move 3, this sum is increased by $(-1)^{p+1}(p+1)+(-1)^p p=(-1)^p$, and in Move 4, it is decreased by $(-1)^p$. We now show that the Floer correction term Floer (X,h) changes in the same way.

Let $C_i(1), B_i(1), Z_i(1)$ be the i^{th}-chains, i^{th}-boundaries, i^{th}-cycles associated to the mod 2 Floer chain complex before making any move. Let $C_i(2), B_i(2), Z_i(2)$

be the corresponding $\mathbb{Z}_2$-vector spaces after one of the above moves. In Move 1, the dimension of all these are unchanged since the Floer chain complexes before and after are identical.

For the second Move, the only changes are in the differentials from $(p+1)$- to p-chains and from p- to $(p-1)$-chains, and so $\dim B_i(1) = \dim B_i(2)$, for $i \neq p,\ p-1$. As for $p, p-1$ terms, we have

$$\begin{aligned}\dim B_p(1) &= \dim C_{p+1}(1) - \dim Z_{p+1}(1)\\ &= \dim C_{p+1}(1) - \dim FH_{p+1} - \dim B_{p+1}(1).\end{aligned}$$

Since the last terms are the same for the chain complex after the move, it follows that $\dim B_p(1) = \dim B_p(2)$. Similarly, we have $\dim B_{p-1}(1) = \dim Z_{p-1}(1) - \dim FH_{p-1}$. As the latter are the same for both complexes, we have $\dim B_{p-1}(1) = \dim B_{p-1}(2)$. Consequently, in Move 2 the Floer correction term Floer (X, h) is unchanged.

Consider the third Move where the dimension of $C_p(1)$, $C_{p+1}(1)$ are increased by $+1$ in going to $C_p(2)$, $C_{p+1}(2)$. Again $\dim B_i(1) = \dim B_i(2)$ for $i \neq p+1, p, p-1$. As in the above but with degree shifting by 1, we have

$$\begin{aligned}\dim B_{p+1}(1) &= \dim C_{p+2}(1) - \dim Z_{p+2}(1)\\ &= \dim C_{p+2}(1) - \dim FH_{p+2} - \dim B_{p+2}(1).\end{aligned}$$

Since these agree before and after, we have $\dim B_{p+1}(1) = \dim B_{p+1}(2)$. Using this last equality, it also follows that

$$\begin{aligned}\dim B_p(1) &= \dim C_{p+1}(1) - \dim Z_{p+1}(1)\\ &= \dim C_{p+1}(1) - \dim FH_{p+1} - \dim B_{p+1}(1)\\ &= \left[\dim C_{p+1}(2) - 1\right] - \dim FH_{p+1} - \dim B_{p+1}(2)\\ &= \dim B_{p+1}(2) - 1.\end{aligned}$$

Finally, by working from the lower degree end, we can deduce the formula $\dim B_{p-1}(1) = \dim Z_{p-1}(1) - \dim FH_{p-1}$. As these last terms are the same for the chain complex after the Move, we have

$$\dim B_{p-1}(1) = \dim B_{p-1}(2).$$

Consequently, we can conclude that the Floer correction term Floer (X, h) is changed by $(-1)^p$ in Move 3.

Similarly, in Move 4, the Floer correction term Floer (X, h) is changed by $-(-1)^p$. Since the argument is the same as above, we will omit the details.

Thus we may conclude that for a generic homotopy of perturbations the change in $[-\mathrm{Sum}'(01) + \mathrm{Sum}'(00) - \mathrm{Sum}'(11)]$ is the same as the change in $\mathrm{Floer}(X, h)$.

This completes the proof that our invariant $\Lambda_{SU(3)}(X)$ is independent of all the choices.

Remark (3.8). As we will see in Section 4, the reason for (5/8) in the formula of $\Lambda''_{SU(3)}(X)$ is a normalization factor to make sure that our invariant has the property: $\Lambda_{SU(3)}(-X) = \Lambda_{SU(3)}(X)$. From Definition (1.3) it is clear that $8 \cdot \Lambda_{SU(3)}$ is an integer; however, $4 \cdot \Lambda_{SU(3)}$ is already an integer because $\Sigma(-1)^{SF(\theta,A,h;s(u(1)\times u(2)))}$ is divisible by 2.

4. Properties of $\Lambda_{\mathbf{SU(3)}}(\mathbf{X})$

The $SU(3)$-Casson invariants $\lambda_{SU(3)}(X)$ and $\Lambda_{SU(3)}(X)$ are clearly different; nonetheless, they share many properties. For example, if all the irreducible, flat, $SU(2)$-connections of X are cut out transversely, i.e. $H^1(X; s(u(1)\times u(2))) = H^1(X, \mathbb{C}^2) = 0$, then no perturbation along $\mathcal{M}^*_{s(u(1)\times u(2))}(X)$ stratum is necessary. In this case, according to Theorem 5.10 of [BHKK], the correction term $\lambda''_{SU(2)}(X)$ is given by

$$\lambda''_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}^*_{SU(2)}(X)} (-1)^{SF(\theta,A,h;SU(2))} \left[\frac{1}{2}\,\rho\left(K(A;\mathbb{C}^2)\right)\right] \tag{4.1}$$

where $\rho\left(K(A;\mathbb{C}^2)\right)$ is the ρ-invariant of the self-dual operator coupled to the regular representation of $SU(2)$. A similar result holds for $\Lambda''_{SU(3)}(X)$.

Proposition (4.2). *Suppose X is a homology 3-sphere with the property that every irreducible flat $SU(2)$-connection A has $H^1(X; su(2)_A) = 0$ and $H^1(X;\mathbb{C}^2_A) = 0$. Then there exist admissible perturbations h which are zero on a neighborhood of $\mathcal{M}_{SU(2)}(X)$ and with respect to such perturbations:*

$$\Lambda''_{SU(3)}(X) = \sum_{[A]\in\mathcal{M}^*_{SU(2)}(X)} (-1)^{SF(\theta,A;SU(2))} \left[\frac{1}{2}\,\rho\left(K(A;\mathbb{C}^2)\right) - \frac{1}{8}\,\rho\left(K(A,su(2))\right)\right].$$

Proof. To calculate the spectral flows in $\Lambda''_{SU(3)}(X)$, we choose a path of connections $\{A(t) \mid 0 \le t \le 1\}$ joining the trivial connection $A(0) = \theta$ with an element $A(1) = A$ in the unperturbed moduli space $\mathcal{M}^*_{SU(2)}(X)$. Since $\mathcal{A}_{SU(2)}$ is connected, we can choose the path lying inside $\mathcal{A}_{SU(2)}$. Note that along this path the coefficients $s(u(1)\times u(2))$ is decomposed into the sum $su(2)\oplus\mathbb{R}$. In particular, the kernel of the operator $K(A(t);\mathbb{R})$ from the second factor is constant and hence gives no contribution to spectral flow, i.e. $SF\left[K\left(A(t);\mathbb{R}\right) \mid 0 \le t \le 1\right] = 0$. It follows that

$$\begin{aligned} &SF\left[K\left(A(t); s\left(u(1)\times u(2)\right)\right) \mid 0 \le t \le 1\right] \\ &= SF\left[K\left(A(t); su(2)\right) \mid 0 \le t \le 1\right]. \end{aligned}$$

From (5.4) and (6.5) of [BHKK], we have the following:

$$\begin{aligned} &SF[K(A(t);\mathbb{C}^2) \mid 0 \le t \le 1] \\ &= 2cs(A) + \frac{1}{2}\big[\rho(K(A(1);\mathbb{C}^2)) - \rho(K(A(0);\mathbb{C}^2))\big] \\ &\quad + \frac{1}{2}\big[\dim \mathrm{Ker}(K(A(1);\mathbb{C}^2)) - \dim \mathrm{Ker}(K(A(0);\mathbb{C}^2))\big], \\ &SF\big[K(A(t), su(2)) \mid 0 \le t \le 1\big] \\ &= 8cs(A) + \frac{1}{2}\big[\rho(K(A(1); su(2))) - \rho(K(A(0); su(2)))\big] \\ &\quad + \frac{1}{2}\big[\dim \mathrm{Ker}(K(A(1); su(2))) - \dim \mathrm{Ker}(K(A(0); su(2)))\big]. \end{aligned} \tag{4.3}$$

After substitution of (4.3) into $\Lambda''_{SU(2)}(X)$, all the terms except for the ρ-invariants cancel out and the result is the formula in (4.2).

Corollary (4.4). *For the Brieskorn homology 3-sphere $\Sigma(2,q,r)$, the difference of the two $SU(3)$ Casson invariants $(\lambda_{SU(3)} - \Lambda_{SU(3)})(\Sigma(2,q,r))$ is given by*

$$\sum_{[A]\in\mathcal{M}^*_{SU(2)}(X)} (-1)^{SF(\theta,A;SU(2))}\left[\frac{1}{8}\rho\left(K(A, su(2))\right)\right].$$

Proof. It has been shown in [BHKK] that for $\Sigma(2,q,r)$ satisfies the transversality condition in (4.2). In addition, its Floer chain complex is concentrated on odd degrees and so Floer $(\Sigma(2,q,r),0) = 0$. Our assertion follows immediately from comparing formulas in (4.4) and (4.3).

In (5.3) of [BH 1], it has been established that the invariant $\lambda_{SU(3)}(X)$ is independent of orientation. We now show that this is also true for the perturbative $SU(3)$-Casson invariant $\Lambda_{SU(3)}(X)$.

Proposition (4.5). $\Lambda_{SU(3)}(-X) = \Lambda_{SU(3)}(X)$.

Proof. We first consider the effect of reversing the orientation $X \to -X$ on the Floer chain complex $FC_*(X)$. As in the usual Morse theory, the effect of changing X to $-X$ is accomplished by replacing the perturbed Chern–Simons functional by its negative and so replaces $C_p(X)$ by its dual $C^{-3-p}(X) = \mathrm{Hom}\big(C_{-3-p}(X), \mathbb{Z}/2\big)$. Thus, we have

$$\begin{aligned}\dim B_p(-X) &= \dim\ \mathrm{Image}\ \big[d : C_{p+1}(-X) \to C_p(-X)\big] \\ &= \dim\ \mathrm{Image}\ \Big[d^* : C^{-3-(p+1)}(X) \to C^{-3-p}(X)\Big] \\ &= \dim\ \mathrm{Image}\ \big[d : C_{-3-p}(X) \to C_{-4-p}(X)\big] \\ &= \dim B_{-4-p}(X),\end{aligned}$$

and so Floer $(X, h) = \mathrm{Floer}(-X, -h)$.

On the other hand, the spectral flows change via:

$$\begin{aligned}SF_{-X}\left(\Theta, A, -h; su(3)\right) &= -SF_X\left(\Theta, A, h; su(3)\right) - 8 \\ SF_{-X}\left(\theta, A, -h; s(u(1)\times u(2))\right) &= -SF_X\left(\theta, A, h; s(u(1)\times u(2))\right) - 3 \\ SF_{-X}\left(\theta, A, -h; \mathbb{C}^2\right) &= -SF_X\left(\theta, A, h; \mathbb{C}^2\right) - 2.\end{aligned}$$

Thus, changing the orientation leaves the signs of the $SU(3)$-irreducibles $[A] \in \mathcal{M}^*_{SU(3)}(X)$ unchanged as $(-1)^{-p-8} = (-1)^p$. On the other hand, for a h-perturbed flat, $S\left(U(1)\times U(2)\right)$-connection $A \in \mathcal{M}^*_{S(U(1)\times U(2))}(X)$, we have

$$\begin{aligned}&(-1)^{SF_{-X}\left(\theta,A,-h;s\left(u(1)\times u(2)\right)\right)}\Big[SF_{-X}\big(\theta, A, -h; \mathbb{C}^2\big) \\ &\qquad - \frac{1}{4}\big(SF_{-X}\big(\theta, A, -h; \mathbb{C}^2\big)\big) + \frac{5}{8}\Big] \\ &= -(-1)^{SF_X\left(\theta,A,-h;s\left(u(1)\times u(2)\right)\right)}\Big[-SF_X\big(\theta, A, -h; \mathbb{C}^2\big) - 2 \\ &\qquad - \frac{1}{4}\big(-SF_X\big(\theta, A, -h; s\big(u(1)\times u(2)\big)\big) - 3\big) + \frac{5}{8}\Big] \\ &= -(-1)^{SF_X\left(\theta,A,-h;s\left(u(1)\times u(2)\right)\right)}\Big[SF_X\big(\theta, A, -h; \mathbb{C}^2\big) \\ &\qquad - \frac{1}{4}\big(SF_X\big(\theta, A, -h; s\big(u(1)\times u(2)\big)\big)\big) + \frac{5}{8}\Big].\end{aligned}$$

Consequently, our invariant $\Lambda_{SU(3)}(X)$ is unchanged when we reverse the orientation of X.

In [BH 2], Boden and Herald showed that their $SU(3)$-Casson invariant satisfy the connect sum formula:

$$\begin{aligned}\lambda_{SU(3)}\left(X_1\#X_2\right) = \lambda_{SU(3)}(X_1) + \lambda_{SU(3)}(X_2) \\ + 4\lambda_{SU(2)}(X_1)\cdot\lambda_{SU(2)}(X_2)\end{aligned} \tag{4.6}$$

where $\lambda_{SU(2)}(X_i)$ is the normalized $SU(2)$-Casson invariant (see [W]). For the proof, they consider the connected sum $X_1 \# X_2$ as obtained from removing two flat 3-balls B_1, B_2 from X_1, X_2 and gluing along the boundaries $X_1 - B_1, X_2 - B_2$ by an isometry. Then they choose system of loops in X_1, X_2 away from these balls B_1, B_2, and based on these loops they choose small perturbations h_i of the flatness equation on $\mathcal{A}(X_i)$. The advantage of this construction is that they can form the sum $h_1 \# h_2$ perturbation on $\mathcal{A}(X)$ such that all the $h_1 \# h_2$-perturbation flat connections are obtained from gluing two h_i-perturbed flat connections from X_i. However, the moduli space $\mathcal{M}_{SU(3),h_1\#h_2}(X)$ obtained in this manner is not necessarily regular. Hence, they have to choose an additional perturbation h of $h_1 \# h_2$ to get a regular moduli space $\mathcal{M}_{SU(3),h}(X)$ for which they can compute $\lambda_{SU(3)}(X)$ (see [BH 2] for details).

To conclude this section, we obtain a similar connect sum formula for $\Lambda_{SU(3)}(X)$.

First let us recall the theory of Floer homology of the connected sum $X_1 \# X_2$, as developed in [Li] [Fu]. It has been shown that, when we restrict our attention to the $S[U(1) \times U(2)]$-stratum, the above perturbation $h_1 \# h_2$ of the Chern Simons functional is a Bott–Morse function whose critical point submanifolds are either isolated points or copies of $SO(3)$. From this point of view, the additional perturbation h amounts to breaking up these critical submanifolds $SO(3)$ into isolated critical points.

With the finite dimensional Bott–Morse theory as a guide, Fukaya obtained [Fu] a filtered chain complex of the following form:

$$\begin{aligned} Fu_*(X_1 \# X_2) =& (C_*(X_1) \otimes C_*(X_2)) \otimes C_*^{gt}(SO(3)) \\ & \oplus C_*(X_1) \oplus C_*(X_2) \end{aligned} \tag{4.7}$$

where $C_*^{gt}(SO(3))$ is a subchain complex of the singular chain complex of $SO(3)$ consisting of smooth chains satisfying certain transversality condition. On one hand, $Fu_*(X_1 \# X_2)$ is chain homotopy equivalent to the Floer chain complex $FC_*(X_1 \# X_2, h)$ after the perturbation h. On the other hand, associated to the filtration on $Fu_*(X_1 \# X_2)$, there is a spectral sequence which has the following $E^2_{p,q}$-terms

$$E^2_{p,q} = H_p(C_*(X_1) \otimes C_*(X_2), H_q(SO(3); \mathbb{Z}))$$

and converges to the homology of $Fu_*(X_1 \# X_2)$. Note the coefficients in the above spectral sequence are $H_q(SO(3); \mathbb{Z})$ because, as established in [Fu], $H_q(C_*^{gt}(SO(3)); \mathbb{Z}) = H_q(SO(3); \mathbb{Z})$.

Now Fukaya's theory can be extended to the $\mathbb{Z}/2$- coefficients instead of $\mathbb{Z}$ and in particular we have the mod-2 chain complex:

$$\begin{aligned} Fu_*(X_1 \# X_2) \otimes \mathbb{Z}/2 =& (C_*(X_1) \otimes C_*(X_2)) \otimes C_*^{gt}(SO(3)) \otimes \mathbb{Z}/2 \\ & \oplus C_*(X_1) \otimes \mathbb{Z}/2 \oplus C_*(X_2) \otimes \mathbb{Z}/2 \end{aligned}$$

In fact, as $\mathbb{Z}/2$ is also a field, there exists a subcomplex $C_*(SO(3))_{\min}$ in $C_*^{gt}(SO(3)) \otimes \mathbb{Z}/2$ such that

$$C_j(SO(3))_{\min} = H_j(SO(3); \mathbb{Z}/2) = \mathbb{Z}/2$$

and has trivial boundary maps $\partial = 0$. Then, in the mod-2 Fukaya chain complex, we can replace the factor $C_*^{gt}(SO(3)) \otimes \mathbb{Z}/2$ by $C_*(SO(3))_{\min}$ to get the subchain complex:

$$Fu_*(X_1\#X_2)_{\min} = (C_*(X_1) \otimes C_*(X_2)) \otimes C_*(SO(3))_{\min} \oplus C_*(X_1) \otimes \mathbb{Z}/2 \oplus C_*(X_2) \otimes \mathbb{Z}/2$$

in $Fu_*(X_1\#X_2) \otimes \mathbb{Z}/2$. Note that $Fu_*(X_1\#X_2)_{\min}$ is chain homotopy equivalent to the original mod-2 chain complex. As it is finite dimensional, we can proceed as in (1.3) to form the sum

$$\sum_{p=0}^{7} (-1)^p \dim_{Z/2}(\text{Image d}\colon Fu_{p+1}(X_1\#X_2)_{\min} \to Fu_p(X_1\#X_2)_{\min})$$

which will be denoted by $\text{Floer}(X_1\#X_2, h_{\min})$ in the following.

Theorem (4.8). *Let X_1, X_2 be integral homology 3-spheres and $X_1\#X_2$ be their connected sum and let* $\text{Floer}(X_1\#X_2, h_{\min})$ *be defined as above. Then,*

$$\Lambda_{SU(3)}(X_1\#X_2) = \Lambda_{SU(3)}(X_1) + \Lambda_{SU(3)}(X_2) + \frac{9}{2}\Lambda_{SU(2)}(X_1)\Lambda_{SU(2)}(X_2) - \frac{1}{4}\left[\text{Floer}(X_1\#X_2, h_{\min}) - \text{Floer}(X_1, h_1) - \text{Floer}(X_2, h_2)\right].$$

Proof. As in [BH 2], we choose small perturbations h_1, h_2 for the self-dual equations of $\mathcal{A}_1, \mathcal{A}_2$ such that $\mathcal{M}^*_{SU(3),h_i}(X_i) = \{A_{ij} \mid j = 1, \cdots m_i\}$ and $\mathcal{M}^*_{SU(2),h_i}(X_i) = \{B_{ij} \mid j = 1, \cdots m_i\}$ consist of respectively isolated, h_i-perturbed flat $SU(3)$-,$SU(2)$- connections. Then with respect to $h_1\#h_2$, the perturbed flat connections in $\mathcal{A}(X_1\#X_2)$ are given by the glued connections $C_1\#C_2$ where C_1, C_2 ranges over the orbits of $\{\theta_1, A_{1j}, B_{1k}\} \times \{\theta_2.A_{2j}, B_{2k}\}$. In particular, when the pair has isotropy subgroups Γ_1, Γ_2, then the glued connections ranges over a connected component isomorphic to the double coset space $\Gamma_1 \backslash SU(3)/\Gamma_2$.

As explained before, it requires a further perturbation h to achieve regularity. In [BH 2], there is an explicit description of all the resulting h-perturbed flat connections and their spectral flows as follows.

The pairs $A_{1j}\#\theta_2$ are single points and remain so after h-perturbation. They are irreducible $SU(3)$-connections with

$$SF_{X_1\#X_2}(\Theta, A_{1j}\#\theta_2, h; su(3)) = SF_{X_1}(\Theta, A_{1j}, h; su(3)).$$

The pairs $B_{1k}\#\theta_2$ are also single points and represent irreducible $SU(2)$-connections with the same normal and tangential spectral flows as the corresponding spectral flows of B_{1k}. In particular, the signed correction term for B_{1k} in

$\Lambda''_{SU(3)}(X_1\#X_2)$ is the same as the corresponding term for B_{1k} in $\Lambda''_{SU(3)}(X_1)$. The same holds for the pair $\theta_1\#A_{2j}, \theta_1\#B_{2k}$. It follows that the contribution for these four type of points to $\Lambda_{SU(3)}(X_1\#X_2)$ is the sum

$$\left(\Lambda_{SU(3)}(X_1)+\frac{1}{4}\text{Floer}(X_1,h_1)\right)+\left(\Lambda_{SU(3)}(X_2)+\frac{1}{4}\text{Floer}(X_2,h_2)\right).$$

Next we consider the pairs $A_{1j}\#A_{2k}$, each of which yields a component of $SU(3)$-irreducible connections isomorphic to $PSU(3)$. Further perturbation by h has the effect of introducing a Morse function f to this component with its critical points $Q_{i,i'}$ as h-perturbed flat connections associated to this component. The tangential spectral flow $SF_{X_1\#X_2}(\Theta, Q_{i,i'}, h; su(3))$ of $Q_{i.i'}$ is given by

$$SF_{X_1}(\Theta_1, A_{1j}, h_1; su(3)) + SF_{X_2}(\Theta_2, A_{2k}, h_2; su(3)) + \text{ index of } f \text{ at } Q_{i,i'}.$$

As we add up the signs $(-1)^{SF_{X_1\#X_2}(\Theta,Q_{i,i'},h,su(3))}$ in computing our invariant and as the Euler number of $PSU(3)$ is zero, the total contribution of these points to our invariant $\Lambda_{SU(3)}(X_1\#X_2)$ is zero.

In a similar manner, the pairs $A_{1j}\#B_{2k}$ yield a component of $SU(3)$-connections isomorphic to $SU(3)/U(1)$. Since the Euler number of the latter is zero, the same analysis shows that these pairs give no contribution to $\Lambda_{SU(3)}(X_1\#X_2)$. Similarly the pairs $B_{1k}\#A_{2j}$, again give no contribution.

There remain the pairs $B_{1k}\#B_{2k'}$, each of which gives rise to a copy of

$$\Gamma_1\backslash SU(3)/\Gamma_2, \Gamma_1=\Gamma_2=S(U(1)\times U(2)).$$

However, because of the relative position of the two isotropy subgroups Γ_1, Γ_2 there are two types of gluing with the result of irreducible $SU(2)$'s and irreducible $SU(3)$'s. First we consider the situation of irreducible $SU(3)$'s. By making an equivariant Morse function perturbation, each pair $B_{1k}\#B_{2k'}$ gives four irreducible $SU(3)$- orbits $P_{k,k',t}, t=0,1,2,3$ with identical sign $(-1)^{a_1+a_2}$. Consequently, these four points give $4(-1)^{a_1+a_2}$ and the sum of all of them is $4\lambda_{SU(2)}(X_1)\cdot\lambda_{SU(2)}(X_2)$.

Now in the situation of irreducible $SU(2)$'s, the double coset forms a copy of $SO(3)=RP^3$. Suppose we are in an situation that we can use a Morse function perturbation, to break up these components into four points $\{Q_{k,k',t}, t=0,1,2,3\}$ indexed by the Morse index t. Then the tangential spectral flow $SF_{X_1\#X_2}(\theta, Q_{k,k',t}, h; s(u(1)\times u(2)))$ of $Q_{k,k',t}$ is the sum a_1+a_2+t, where a_1, a_2 are respectively the tangential spectral flows $SF_{X_1}(\theta_1, B_{1k}, h_1; s(u(1)\times u(2))$, $SF_{X_2}(\theta_2, B_{2k'}, h_2; s(u(1)\times u(2))$ of $B_{1k}, B_{1k'}$. As for the normal spectral flows $SF_{X_1\#X_2}(\theta, Q_{k,k',t}, h; \mathbb{C}^2)$, they are the sum b_1+b_2 for all four points with b_i the normal spectral flows $SF_{X_i}(\theta_i, B_{ik}, h_i; \mathbb{C}^2)$. Hence the normal contribution to $\Lambda''_{SU(3)}(X_1\#X_2)$ by these four points is $(-1)^{a_1+a_2}(1-1+1-1)=0$, or in other words the total contribution is zero. As for the tangential contribution to $\Lambda''_{SU(3)}(X_1 \# X_2)$, we have

$$(-1/4)(-1)^{a_1+a_2}[(a_1+a_2)-(a_1+a_2+1)+(a_1+a_2+2)-(a_1+a_2+3)] \\ =(1/2)(-1)^{a_1+a_2}.$$

Therefore the total contribution of the tangential spectral flow of these irreducible $SU(2)$ representations is $(1/2)\lambda_{SU(2)}(X_1)\cdot\lambda_{SU(2)}(X_2)$ as $\lambda_{SU(2)}(X_i)=-\Sigma(-1)^{a_i}$. Note that the constant term $(5/8)$ has no effect because it is counted with the tangential signs and so gives $(1-1+1-1)=0$. In addition, there is also the Floer homology contribution due to these four points. It is not difficult to see that this last contribution in $\text{Floer}(X,h)$ is the same as the contribution of the corresponding terms in $\{k\}\times\{k'\}\times C_*(SO(3))_{\text{min}}$. Thus when the critical points of $SO(3)$ are minimal, all the terms in the connect sum formula can be accounted for.

In general, the additional perturbation h could create more than 4 critical points along an $SO(3)$-component. Nonetheless, their tangential and normal spectral flow can be computed the same way as before. For example, the total contribution of the normal spectral flows equals to zero because the Euler number of $SO(3)$ is zero. As for the tangential spectral flows, according to the Morse index $0,1,2,3$, each of these critical points gives, respectively, a contribution of the form:

$$(-1)^{a_1+a_2}(a_1+a_2),-(-1)^{a_1+a_2}(a_1+a_2-1), \\ (-1)^{a_1+a_2}(a_1+a_2-2),(-1)^{a_1+a_2}(a_1+a_2-3).$$

As we go through two critical points with consecutive Morse indices, the terms $\pm(a_1+a_2)$ in the above cancel and leaves us with $\pm(-1)^{a_1+a_2}$ which depends only on the Floer index. Therefore the total tangential spectral flow can be written as a sum of these terms as we go through pair of basis elements in $C_*(X_1)\otimes C_*(X_2)\otimes C_*(SO(3),\mathbb{Z}/2)$. Here for each $SO(3)$ component we have a chain complex $C_*(SO(3),\mathbb{Z}/2)$ of $SO(3)$ generated by the critical points of the Morse function perturbation along this component.

Note these chain complexes $C_*(SO(3),\mathbb{Z}/2)$ contain $C_*(SO(3))_{\text{min}}$ as a subcomplex. In fact, using the chain homotopy equivalence in [Fu], we obtain a subchain complex in $C_*(X_1)\otimes C_*(X_2)\otimes C_*(SO(3),\mathbb{Z}/2)$ which is isomorphic to $Fu_*(X_1\#X_2)_{\text{min}}$. Furthermore, as the quotient chain complex is acyclic, by a standard argument in homological algebra we can decompose the above chain complex into the following sum:

$$C_*(X_1)\otimes C_*(X_2)\otimes C_*(SO(3))_{\text{min}}\oplus[C_*(X_1)\otimes C_*(X_2)\otimes C_*(SO(3))]_{\text{acyclic}}$$

According to the above decomposition, the total tangential spectral flow is the sum of two corresponding terms. As in calculation before, the sum over the first summand gives us $\frac{1}{2}\lambda_{SU(2)}(X_1)\lambda_{SU(2)}(X_2)$. Also in the Floer correction term, this summand gives us the contribution Floer$(X_1\#X_2, h_{\min})$. Since the second summand is acyclic, the chain complex is chain homotopic to the trivial one. As $K_1(\mathbb{Z}/2) = 0$, from the general consideration of simple homotopy theory it can be shown that such chain homotopy equivalence can be realized by a combination of algebraic operations known as elementary base change, elementary collapsing and expansion. As in the proof of (3.4), the base change operation does not change anything but the last two operations will change the Floer correction term by the same amount as its change in the sum of the tangential spectral flows. From this it follows that the sum of the tangential spectral flows over the basis elements of the second summand is the same as $-$Floer$(X_1\#X_2, h)$ + Floer$(X_1\#X_2, h_{\min})$. Adding up all these, we have the connect sum formula as claimed.

5. Calculation of $SU(3)$-invariant for $(2,q)$-torus knots

Given a knot $T \subset S^3$, we have an integral homology 3-sphere $X(T, 1/k)$ given by $1/k$-surgery of T. In turn, these homology 3-spheres provide a sequence of $SU(3)$-invariants $\Lambda_{SU(3)}(X(T, 1/k)), k = \pm 1, \pm 2 \cdots$ of the knot T. A natural question is the relation of these knot invariants to other known knot invariants. As a first step, we consider in this section the $(2,q)$-torus knot $T(2,q)$ and make explicit calculations of these $SU(3)$-invariants.

Theorem 5.1. *Let X_K denote the integral homology 3-sphere given by $1/K$-surgery on the $(2,q)$ torus knot $T(2,q)$. Then for $q = 3,5,7,9$, the $SU(3)$-Casson invariants $\Lambda_{SU(3)}(X_K)$ are as listed in the following table* (5.2).

(2,q)-torus knot	$\Lambda_{SU(3)}(X_K), K > 0$	$\Lambda_{SU(3)}(X_K), K < 0$
(2,3)	$\frac{1}{4}(10K - 9)K$	$\frac{1}{4}(10K - 11)K$
(2,5)	$\frac{1}{4}(126K - 79)K$	$\frac{1}{4}(126K - 85)K$
(2,7)	$\frac{1}{4}(540K - 230)K$	$\frac{1}{4}(540K - 242)K$
(2,9)	$\frac{1}{4}(1540K - 514)K$	$\frac{1}{4}(1540K - 534)K$

Table (5.2)

Proof. As is well-known, the $1/k$-surgery on a $(2,q)$ torus knot yields a Brieskorn sphere: $-\Sigma(2, q, 2qk - 1)$ for $K = k > 0$ and $\Sigma(2, 2q, 2qk + 1)$ for $K = -k, k > 0$, with the natural orientation from singularity theory. The invariant $\lambda_{SU(3)} = \lambda'_{SU(3)} + \lambda''_{SU(3)}$ of $\Sigma(2, q, 2qk \pm 1)$ have been studied in great detail in [B], [BHKK]. To simplify the notation, we write:

(5.3)

(a) $A(q,K) = \lambda'_{SU(3)} = \Sigma sign(B_j)$, sum over the irreducible $SU(3)$-representations B_j of $\pi_1(X_K)$.

(b) $B(q,K) = \lambda''_{SU(3)} = (\epsilon/2)\Sigma\rho_{X_K}(A_j;\mathbb{C}^2)$, sum over the irreducible $SU(2)$-representations A_j of $\pi_1(X_K)$.

(c) $C(q,K) = (-\epsilon/8)\Sigma\rho_{X_K}(A_j;su(2))$, sum over the irreducible $SU(2)$-representations A_j of $\pi_1(X_K)$.

(d) $D(q,K) = \text{Floer}(X_K)$, the Floer correction term.

Here ϵ is -1 for $K = k > 0$ and $+1$ for $K = -k < 0$. In terms of $A(q,K)$, $B(q,K)$, $C(q,K)$, $D(q,K)$, we have

$$\lambda_{SU(3)}(X_K) = A(q,K) + B(q,K)$$
$$\Lambda_{SU(3)}(X_K) = A(q,K) + B(q,K) + C(q,K) + D(q,K).$$

In [B], Boden has shown that the irreducible $SU(3)$-representations B_j of $\pi_1(X_K)$ all satisfy the regularity condition, i.e. cut out transversely by equation, and contribute with $Sign(B_j) = (-1)^{SF(\Theta,B_j;su(3))} = 1$. Thus no further perturbation is necessary, $h = 0$, and $A(q,K)$ is the number of irreducible $SU(3)$-representations of $\pi_1(X_K)$, listed in the first column of Table (5.4) below.

$A(q,K)$	$B(q,K)$	$C(q,K)$ for $K > 0$	$C(q,K)$ for $K < 0$
$3K^2 - K$	$\frac{K(-24K^2-84K+13)}{6(6K-1)}$	$\frac{K(12K^2+84K-11)}{12(6K-1)}$	$\frac{K(12K^2 48K-5)}{12(6K-1)}$
$33K^2 - 9K$	$\frac{K(-200K^2-1620K+151)}{10(10K-1)}$	$\frac{K(100K^2+1120K-87)}{20(10K-1)}$	$\frac{K(100K^2+48K-57)}{20(10K-1)}$
$138K^2 - 26K$	$\frac{K(-784K^2-9128K+606)}{14(14K-1)}$	$\frac{K(392K^2+5992K-330)}{28(14K-1)}$	$\frac{K(392K^2+4816K-246)}{28(14K-1)}$
$390K^2 - 58K$	$\frac{K(-2160K^2-33192K+1714)}{18(18K-1)}$	$\frac{K(1080K^2+20880K-890)}{36(18K-1)}$	$\frac{K(1080K^2+17640K-710)}{36(18K-1)}$

Table (5.4)

The aforementioned work of Boden can be regarded as an extension of the results on $SU(2)$-representations of $\pi_1(X_K)$, all of which satisfy the regularity condition. As in [B2], [FS], there are $(q^2-1)k/4$ of these $SU(2)$ representations $A_1, \cdots, A_{(q^2-1)k/4}$ which have odd spectral flow $SF(\theta, A_j; su(2))$ for $K =$

$k > 0$ and even for $K = -k < 0$. It follows that the Floer chain complex has zero boundary map in all these cases. In particular, our Floer correction term $D(q, K) = \text{Floer}(X_K) = 0$ for all K.

In [BHKK], the terms $B(q, K)$ are computed and are listed in the second column of (5.4). As $A(q, K), B(q, K), D(q, K)$ are all known, our job is to calculate the remaining $C(q, K)$ in the third and fourth columns in (5.4). Once this is achieved, the proof of (5.1) is immediate by adding these columns together.

Here the horizontal rows are the values of $A(q, K), B(q, K), C(q, K), K > 0, C(q, K), K < 0$ for $q = 3, 5, 7, 9$ respectively. Recall that $C(q, K)$ is the sum of ρ-invariants of the adjoint representation $Ad(A_j)$ where A_j runs through all the irreducible $SU(2)$-representations of $\pi_1(\Sigma(2, q, 2qk \pm 1))$. Our first step is to tabulate these representations in a convenient manner. There are two cases, $K > 0$ and $K < 0$, which have to be treated separately.

The Case $\Sigma(2, q, 2qk - 1)$, $K > 0$

As is well-known, the Brieskorn sphere $\Sigma(2, q, 2qk - 1) = \Sigma(a_1, a_2, a_3)$, $a_1 = 2, a_2 = q, a_3 = 2qk - 1$ is a Seifert 3-manifold with its Seifert invariant given by $(b_0, b_1, b_2, b_3) = (-1, 1, m, k), m = (q-1)/2$. As a Seifert manifold, its fundamental group has the following presentation:

$$\begin{aligned} &\text{generators} : x_1, x_2, x_3 \\ &\text{relations} : x_1 x_2 x_3 = h \text{ central}, x_1^2 = h^{-1}, x_2^q = h^{-m}, x_3^{2qk-1} = h^{-k}. \end{aligned} \tag{5.5}$$

The central element h plays an important role for an irreducible representation $f : \pi_1(\Sigma(2, q, 2qk - 1)) \to SU(2)$ because by Schur's lemma $f(h) = \pm 1$. If $f(h) = 1$, then $f(x_1)$ is also central and the representation becomes abelian. As $H_1(\Sigma(2, q, 2qk - 1)) = 0$, this implies f is the trivial representation. Hence we can omit this case and concentrate on $f(h) = -I$

Let $X_i = f(x_i)$. Then from (5.5) we have the following condition:

$$X_1^2 = -I, X_2^q = (-I)^m, X_3^{2qk-1} = (-I)^k, X_1 X_2 X_3 = -I. \tag{5.6}$$

Consider an element $g \in SU(3)$ as a unit quaternion, written uniquely in the form $g = \cos\theta + \sin\theta[i\ \cos(\pi t) + j\ \sin(\pi t)], 0 \leq \theta < \pi$. Then the first three equations in (5.6) imply that

$$\begin{aligned} &\text{trace}(X_1) = 2\cos(L_1\pi), \quad \text{trace}(X_2) = 2\cos(L_2\pi/q), \\ &\text{trace}(X_3) = 2\cos(L_3\pi/(2qk-1)) \end{aligned}$$

where L_1, L_2, L_3 are integers with $L_1 = 1, 0 < L_2 < q, 0 < L_3 < (2qk - 1), L_2 = m \pmod 2, L_3 = k \pmod{(2)}$.

In fact, by conjugation, we may assume that the pair (X_1, X_2) takes the form $X_1 = i$, $X_2 = \cos(L_2\pi/q) + \sin(L_2\pi/q)[i\ \cos(\pi t) + j\ \sin(\pi t)]$, with $0 < t < 1$. Such a choice of (X_1, X_2) uniquely determines the representation because $X_3 =$

$-X_2^{-1}X_1^{-1} = i\ \cos(L_2\pi/q) + \sin(L_2\pi/q)[\cos(\pi t) - k\ \sin(\pi t)]$. Substitution of this into $X_3^{2qk-1} = (-I)^k$ gives the constraint: $\sin(L_2\pi/q)\cos(\pi t) = \cos(L_3\pi/(2qk-1))$ on t. To solve this equation, we observe that as t varies over $(0,1)$ the left-hand side ranges monotonically inside the interval $[-1,1]$. Hence it is not difficult to work out the permissible values of L_3 for a fixed L_2. For example, with $q=3$, we have $L_2 = 1$ and as $\sin(\pi/3) = \cos(\pi/6), -\sin(\pi/3) = \cos(5\pi/6)$ the above constraint yields: $\pi/6 < L_3\pi/(6k-1) < 5\pi/6$ and $L_3 = k \bmod 2$ which is equivalent to $L_3 = (k-2)+2t, t = 1,\ldots,2k$. In this way we work out the following table of all admissible L_1, L_2, L_3 for $q = 3,5,7,9$.

(2,q)	L_1	L_2	L_3	t	e
(2,3)	1	1	(k-2)+2t	1 ... 2k	36k+12t-17
(2,5)	1	2	(k-2)+2t	1 ... 4k	100k+20t-29
(2,5)	1	4	(3k-2)+2t	1 ... 2k	160k+20t-33
(2,7)	1	1	(5k-2)+2t	1 ... 2k	196k+28t-37
(2,7)	1	3	(k-2)+2t	1 ... 6k	196k+28t-41
(2,7)	1	5	(3k-2)+2t	1 ... 4k	280k+28t-45
(2,9)	1	2	(5k-2)+2t	1 ... 4k	324k+36t-49
(2,9)	1	4	(k-2)+2t	1 ... 8k	324k+36t-49
(2,9)	1	6	(3k-2)+2t	1 ... 6k	432k+36t-57
(2,9)	1	8	(7k-2)+2t	1 ... 2k	576k+36t-61

Table (5.7)

The case $\Sigma(2,q,2qk+1), K<0$

In this case, the Brieskorn sphere $\Sigma(2,q,2qk+1) = \Sigma(a_1,a_2,a_3), a_1 = 2, a_2 = q, a_3 = 2qk+1$ has its Seifert invariant given by $(b_0,b_1,b_2,b_3) = (1,-1,-m,-k)$. With these minor changes, the argument goes through the same way as before. We will omit the details and just summarize our calculation in the following table.

(2,q)	L_1	L_2	L_3	t	e
(2,3)	1	1	k+2t	1 ... 2k	36k+12t+5
(2,5)	1	2	k+2t	1 ... 4k	100k+20t+9
(2,5)	1	4	3k+2t	1 ... 2k	160k+20t+13
(2,7)	1	1	5k+2t	1 ... 2k	196k+28t+9
(2,7)	1	3	k+2t	1 ... 6k	196k+28t+13
(2,7)	1	5	3k+2t	1 ... 4k	280k+28t+17
(2,9)	1	2	5k+2t	1 ... 4k	324k+36t+13
(2,9)	1	4	k+2t	1 ... 8k	324k+36t+17
(2,9)	1	6	3k+2t	1 ... 6k	432k+36t+21
(2,9)	1	8	7k+2t	1 ... 2k	576k+36t+25

Table (5.8)

Now, as in [FS], the ρ-invariant $\rho_\Sigma(Ad(A_j))$ of the Adjoint representation can be computed by the following formula of trigonometric sum:

$$\rho_{\Sigma,su(2)} = -3 - 2\Bigg(\sum_{i=1}^{3}\sum_{m=1}^{a_i-1}(2/a_i)\cot(\pi am/a_i^2)\cot(\pi m/a_i)\Bigg)\sin^2(\pi em/a_i) \quad (5.9)$$

where $e = \sum_{i=1}^{3} L_i(a/a_i)$ is listed in the last column of (5.7) (5.8). With these data at hand, we can put them into (5.9) and then add up the ρ-invariants to get our formula for $C(q,K)$. In practice, this last step is a little easier. As in Lemma 10.3 of [FS], the sum $\Sigma(2/a_i)\cot(\pi am/a_i^2)\cot(\pi m/a_i)\sin^2(\pi em/a_i)$ in (5.9) is given by $4[\text{Area}\,\Delta(x,y) - \text{Int}\,\Delta(x,y)]$ where $\Delta(x,y)$ is the triangle with vertices $(0,0),(0,x),(x,y)$ and $\text{Area}\,\Delta(x,y)$ is its area and $\text{Int}\,\Delta(x,y)$ is the number of the lattice points in Δ, where boundary points count $1/2$, vertices count $1/4$ and $(0,0)$ is not counted. In our case $(x,y) = (e, ef/a_i)$ where $f > 0$, $-f(a/a_i) = 1 \pmod{a_i}$. After putting these into our data, we see that $\Delta(x,y) - \text{Area}\,\Delta(x,y)$ can be written a sum of greatest integer functions of the form $[\![\text{linear in i/linear in } k]\!]$ where i runs through integers in a fixed interval [0, const.k]. Then, after adding them up, $(2qk-1)C(q,K)$ can be shown to be a cubic polynomial in $K > 0$ (respectively $K < 0$). (This process is similar to the calculation of $B(q,K)$ in [BHKK]). Knowing that this is a cubic polynomial, the proof reduces to a simple matter of linear algebra in deciding the coefficients by going through a finite number of examples. In this way, we obtain the result as tabulated above and complete the proof of (5.1).

Remark (5.10). We conclude with a conjecture. Observe that from our calculation $\Lambda_{SU(3)}(X_K)$ are polynomials of degree 2: $P_+(K,q)$ for $K > 0$ and $P_-(K,q)$ for $K < 0$. Moreover $P_+(K,q) = P_-(K,q) + \frac{1}{4}|K|N(q)$ where $N(3) = 2, N(5) = 6, N(7) = 12, N(9) = 20$. In all the cases computed here $N(q)|K|$ equals the number of irreducible $SU(2)$-representations. On the other hand, $\lambda_{SU(2)}$ is $K(q^2-1)/4$ where $-(q^2-1)/4$ is the second derivatives $\Delta''_{T(2,q)}(1)$ at $+1$ of the normalized Alexander polynomial of the $(2,q)$-torus knot $T(2,q)$. In view of this, a natural **conjecture** is that the $SU(3)$-knot invariants $\Lambda_{SU(3)}(X(T,1/K))$ are polynomials of degree 2 in K: $P_+(K)$ and $P_-(K)$ for $|K|$ large, and their difference are given by the formula $P_+(K) = P_-(K) - |K|\cdot\Delta''_T(1)$.

6. An integer valued variant $\Lambda_{SU(3)}(X)_{alt.}$

In this section we make a slight modification of $\Lambda_{SU(3)}(X)$ to get an integer-valued invariant $\Lambda_{SU(3)}(X)_{alt.}$. The idea is to clear out the fractional parts in the original definition.

Theorem 6.1. *Let X be an integral homology 3-sphere and $\Lambda_{SU(3)}(X)_{alt.}$ be* *defined by the formula:*

$$\Lambda_{SU(3)}(X)_{alt.} = \Lambda_{SU(3)}(X) + \frac{1}{4}\sum_{i=0}^{7}(-1)^i i \cdot \dim(FH_i) - \frac{5}{8}\sum_{i=0}^{7}(-1)^i \dim(FH_i).$$

Then $\Lambda_{SU(3)}(X)$ is a well-defined, integer-valued invariant.

Proof. Recall that the Floer chain complex is $Z/8$-graded with gradings $0, 1, \cdots, 7$. That is, FC_i is identified with $FC_{i-8[\frac{i}{8}]}$ where $[\![\frac{i}{8}]\!]$ is the greatest integer less than or equal to $\frac{i}{8}$.

The short exact sequences of Floer homology $0 \to Z_i \to FC_i \to B_{i-1} \to 0$, $0 \to B_i \to Z_i \to FH_i \to 0$, with FC_i, B_i, Z_i, FH_i being the Floer i-chains, i-boundaries, i-cycles, i-homology, respectively, yield the equality of dimensions of vector spaces, $\dim FC_i = \dim B_{i-1} + \dim B_i + \dim FH_i$. Substituting this into $\sum_{i=1}^{8}(-1)^i i \cdot \dim FC_i$, we obtain

$$\sum_{i=1}^{8}(-1)^i i \cdot (\dim FC_i) =$$
$$\sum_{i=1}^{8}(-1)^i i \cdot \dim B_{i-1} + \sum_{i=1}^{8}(-1)^i i \cdot \dim B_i + \sum_{i=1}^{8}(-1)^i i \cdot \dim FH_i.$$

By replacing i by $j+1$ for $j = 1, \cdots, 7$ in the first sum, the right-hand side may be rewritten as:

$$-\dim B_0 + \sum_{j=1}^{7}(-1)^{j+1}(j+1)\dim B_j + \sum_{i=1}^{8}(-1)^i i \dim B_i + \sum_{i=1}^{8}(-1)^i i \dim FH_i$$
$$= -\dim B_0 - \sum_{j=1}^{7}(-1)^j \cdot \dim B_j + 8 \cdot \dim B_8 + \sum_{i=1}^{8}(-1)^i i \cdot \dim FH_i.$$

Since $\dim FC_8 = \dim B_8 + \dim B_7 + \dim FH_8$, we have

$$\sum_{i=0}^{7}(-1)^i i \cdot (\dim FC_i) = -\sum_{j=0}^{7}(-1)^j(\dim B_j)$$
$$+ 8\ (\dim B_8) - 8(\dim FC_8) + \sum_{i=1}^{8}(-1)^i i \cdot (\dim FH_i)$$
$$= -\left(\sum_{j=0}^{7}(-1)^j(\dim B_j)\right) - 8(\dim B_7) + \sum_{i=0}^{7}(-1)^i i \cdot (\dim FH_i).$$

As $\sum_{j=0}^{7}(-1)^j(\dim B_j)$ is our Floer correction $\mathrm{Floer}(X)$, we have derived the following simple formula:

$$\left[\sum_{i=0}^{7}(-1)^i i\cdot \dim FC_i\right] + \mathrm{Floer}(X) = -8(\dim B_7) + \sum_{i=0}^{7}(-1)^i i \cdot \dim FH_i. \quad (6.2)$$

Now consider the "tangential sum":

$$\sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} SF(\theta,A,h;s(u(1)\times u(2))).$$

Since the Floer grading of $[A]$ is a number in the range $0,\cdots,7$, given by

$$SF(\theta,A,h;s(u(1)\times u(2))) - 8\left[\!\!\left[\frac{SF(\theta,A,h;s(u(1)\times u(2)))}{8}\right]\!\!\right],$$

we can rewrite the above as:

$$\sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} SF(\theta,A,h;s(u(1)\times u(2))) =$$

$$8\sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} \left[\!\!\left[\frac{SF(\theta,A,h;s(u(1)\times u(2)))}{8}\right]\!\!\right]$$

$$+\sum_{i=0}^{7}(-1)^i i\cdot(\dim FC_i).$$

Combining (6.2) and the above, we obtain:

$$\sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} SF(\theta,A,h;s(u(1)\times u(2)))$$

$$+\mathrm{Floer}(X) =$$

$$8\sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} \left[\!\!\left[\frac{SF(\theta,A,h;s(u(1)\times u(2)))}{8}\right]\!\!\right]$$

$$-8\cdot\dim B_7 + \sum_{i=0}^{7}(-1)^i\ i\cdot\dim FH_i$$

Consequently, we see that the invariant $\Lambda_{SU(3)}(X)$ may be written as:

$$\Lambda_{SU(3)}(X) = \Lambda_{SU(3)}(X)_{alt.} - \frac{1}{4}\sum_{i=0}^{7}(-1)^i i\cdot\dim FH_i + \frac{5}{8}\sum_{i=0}^{7}(-1)^i \dim FH_i.$$

Here $\Lambda_{SU(3)}(X)_{alt.}$ is our integer-valued invariant defined by the formula:

$$\Lambda_{SU(3)}(X)_{alt.} = \Lambda'_{SU(3)}(X) + \sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} \Big\{ SF(\theta,A,h;\mathbb{C}^2) - 2\left[\!\left[\frac{SF(\theta,A,h;s(u(1)\times u(2)))}{8} \right]\!\right] \Big\} + 2\ \dim B_7 \tag{6.3}$$

In the above, we have used the equalities:

$$\sum_{[A]\in\mathcal{M}^*_{S(U(1)\times U(2)),h}} (-1)^{SF(\theta,A,h;s(u(1)\times u(2)))} = \sum_{i=0}^{7}(-1)^i(\dim FC_i) = \sum_{i=0}^{7}(-1)^i(\dim FH_i)$$

to rewrite the terms involving 5/8 in term of the Euler number of the Floer complex. This completes the proof.

Corollary 6.4. *The invariant $\Lambda_{SU(3)}(X)$ takes on integer value if and only if*

$$\frac{1}{4}\sum_{i=0}^{7}(-1)^i i\cdot(\dim FH_i) - \frac{5}{8}\sum_{i=0}^{7}(-1)^i(\dim FH_i)$$

is an integer.

Note, in (6.3), we have a simple formula for the invariant $\Lambda_{SU(3)}(X)_{alt.} \bmod 2$. In all cases computed here this mod 2 invariant vanishes. It is a reasonable conjecture that $\Lambda_{SU(3)}(X)_{alt.}$ is even for all integral homology 3-spheres X.

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