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Degenerations for representations of extended Dynkin quivers

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Abstract. Let A be the path algebra of a quiver of extended Dynkin type $\tilde{\mathbb{A}}_n$, $\tilde{\mathbb{D}}_n$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$. We show that a finite dimensional A-module M degenerates to another A-module N if and only if there are short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ of A-modules such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$ for $1 \leq i \leq s$ and $N = M_{s+1}$ are true for some natural number s.

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1. Introduction and main results

Let A be a finite dimensional associative K-algebra with an identity over an algebraically closed field K of arbitrary characteristic. If $a_1 = 1, \ldots, a_n$ is a basis of A over K, we have the constant structures a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\operatorname{mod}_A(d)$ of d-dimensional unital left A-modules consists of n-tuples $m = (m_1, \ldots, m_n)$ of $d \times d$-matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. The general linear group $\mathrm{Gl}_d(K)$ acts on $\operatorname{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [11]). We shall agree to identify a d-dimensional A-module M with the point of $\operatorname{mod}_A(d)$ corresponding to it. We denote by $\mathcal{O}(M)$ the $\mathrm{Gl}_d(K)$-orbit of a module M in $\operatorname{mod}_A(d)$. Then one says that a module N in $\operatorname{mod}_A(d)$ is a degeneration of a module M in $\operatorname{mod}_A(d)$ if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\operatorname{mod}_A(d)$, and we denote this fact by $M \leq_{\deg} N$. Thus $\leq_{\deg}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. It is not clear how to characterize $\leq_{\deg}$ in terms of representation theory.

There has been a work by S. Abeasis and A. del Fra [1], K. Bongartz [7], [10], [9], Ch. Riedtmann [13], and A. Skowroński and the author [15], [16], [17] connecting $\leq_{\deg}$ with other partial orders $\leq_{\mathrm{ext}}$ and $\leq$ on the isomorphism classes in $\operatorname{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \leq_{\text{ext}} N$: $\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.
- $M \leq N$: $\Leftrightarrow$ $[X, M] \leq [X, N]$ holds for all modules X.

Here and later on we abbreviate $\dim_K \operatorname{Hom}_A(X, Y)$ by $[X, Y]$, and furthermore $\dim_K \operatorname{Ext}^i_A(X, Y)$ by $[X, Y]^i$. Then for modules M and N in $\operatorname{mod}_A(d)$ the following implications hold:

$$M \leq_{\text{ext}} N \Longrightarrow M \leq_{\text{deg}} N \Longrightarrow M \leq N$$

(see [10], [13]). Unfortunately the reverse implications are not true in general, and it would be interesting to find out when they are. K. Bongartz proved in [10] (see also [8]) that it is the case for all representations of Dynkin quivers and the double arrow. Recently, the author proved in [17] that $\leq$ and $\leq_{\text{ext}}$ are also equivalent for all modules over representation-finite blocks of group algebras. Moreover, in [9] K. Bongartz proved that $\leq_{\text{deg}}$ and $\leq$ coincide for all representations of extended Dynkin quivers, and conjectured that possibly $\leq_{\text{ext}}$ and $\leq_{\text{deg}}$ also coincide. The main aim of this paper is to prove the following theorem.

Theorem. *The partial orders $\leq$ and $\leq_{\text{ext}}$ coincide for modules over all tame concealed algebras.*

In particular we get the positive answer to the above question.

Corollary. *The partial orders $\leq$, $\leq_{deg}$ and $\leq_{\text{ext}}$ are equivalent for all representations of extended Dynkin quivers.*

We mention that K. Bongartz described in [8, Theorem 4] the set-theoretic structure of minimal degenerations of modules provided the partial orders $\leq_{\text{ext}}$ and $\leq$ coincide. In a forthcoming paper we shall describe the minimal singularities for representations of extended Dynkin quivers.

The paper is organized as follows. In Section 2 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules which we apply in our investigations. In Section 3 we recall several known facts on tame concealed algebras. In particular we describe some properties of the additive categories of standard stable tubes. Section 4 is devoted to the proof of the Theorem.

For basic background on the topics considered here we refer to [5], [10], [9], [11] and [14]. The results presented in this paper form a part of the author's doctoral dissertation written under supervision of professor A. Skowroński. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 020 08.

2. Preliminary results

2.1. Throughout the paper A denotes a fixed finite dimensional associative K-algebra with an identity over an algebraically closed field K. We denote by $\operatorname{mod} A$ the category of finite dimensional left A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by indecomposable modules, and by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$. By an A-module is meant an object from $\operatorname{mod} A$. Further, we denote by Γ_A the Auslander-Reiten quiver of A and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}\, D$, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable modules. For a module M we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if M and N have the same simple composition factors including the multiplicities. Finally, for a family $\mathcal{F}$ of A-modules, we denote by $\operatorname{add}(\mathcal{F})$ the additive category given by $\mathcal{F}$, that is, the full subcategory of $\operatorname{mod} A$ formed by all modules isomorphic to the direct summands of direct sums of modules from $\mathcal{F}$.

2.2. Following [13], for M, N from $\operatorname{mod} A$, we set $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all A-modules X. The fact that $\leq$ is a partial order on the isomorphism classes of A-modules follows from a result by M. Auslander [3] (see also [7]). Observe that, if M and N have the same dimension and $M \leq N$, then $[M] = [N]$. Moreover, M. Auslander and I. Reiten have shown in [4] that, if M and N are A-modules with $[M] = [N]$, then for all nonprojective indecomposable A-modules X and all noninjective indecomposable modules Y the following formulas hold (see [12]):

$$[X, M] - [M, \tau X] = [X, N] - [N, \tau X]$$
$$[M, Y] - [\tau^- Y, M] = [N, Y] - [\tau^- Y, N]$$

Hence, if $[M] = [N]$, then $M \leq N$ if and only if $[M, X] \leq [N, X]$ for all A-modules X.

2.3. Let M and N be A-modules with $[M] = [N]$ and

$$\Sigma :\ 0 \to D \to E \to F \to 0$$

an exact sequence in $\operatorname{mod} A$. Following [13] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$ and δ_Σ on A-modules X as follows

$$\delta_{M,N}(X) = [N, X] - [M, X]$$
$$\delta'_{M,N}(X) = [X, N] - [X, M]$$
$$\delta_\Sigma(X) = \delta_{E, D\oplus F}(X) = [D \oplus F, X] - [E, X].$$

From the Auslander-Reiten formulas (2.2) we get the following very useful equalities

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \qquad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

for all A-modules X. Observe also that $\delta_{M,N}(I) = 0$ for any injective A-module I, and $\delta'_{M,N}(P) = 0$ for any projective A-module P. In particular, the following conditions are equivalent:

(1) $M \leq N$,
(2) $\delta_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$,
(3) $\delta'_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$.

2.4. For an A-module M and an indecomposable A-module Z, we denote by $\mu(M, Z)$ the multiplicity of Z as a direct summand of M. For a nonprojective indecomposable A-module U, we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U): \ 0 \to \tau U \to E(U) \to U \to 0,$$

and, for an injective indecomposable A-module I, we set $E(I) = I/\mathrm{soc}(I)$, $\tau^- I = 0$.

We shall need the following lemma.

Lemma 2.5. *Let M, N be A-modules with $[M] = [N]$ and U an indecomposable A-module. Then*

$$\mu(N,U) - \mu(M,U) = \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta_{M,N}(\tau U).$$

Proof. If U is nonprojective, then the Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \mathrm{Hom}_A(M, \tau U) \to \mathrm{Hom}_A(M, E(U)) \to \mathrm{rad}(M, U) \to 0,$$

and hence we get

$$[M, \tau U \oplus U] - [M, E(U)] = [M, U] - \dim_K \mathrm{rad}(M, U) = \mu(M, U).$$

Similarly, we have

$$[N, \tau U \oplus U] - [N, E(U)] = \mu(N, U).$$

Then we obtain the equalities

$$\begin{aligned} \mu(N,U) - \mu(M,U) &= ([N, \tau U \oplus U] - [M, \tau U \oplus U]) - (N, [E(U)] - [M, E(U)]) \\ &= \delta_{M,N}(\tau U) + \delta_{M,N}(U) - \delta_{M,N}(E(U)). \end{aligned}$$

Assume now that U is projective. Then $\mathrm{Hom}_A(M, \mathrm{rad}\, U) \simeq \mathrm{rad}(M, U)$, and so

$$[M, U] - [M, \mathrm{rad}\, U] = \mu(M, U).$$

Similarly, we have

$$[N, U] - [N, \operatorname{rad} U] = \mu(N, U).$$

Therefore, we get

$$\begin{aligned}\mu(N,U) - \mu(M,U) &= ([N,U] - [M,U]) - ([N, \operatorname{rad} U] - [M, \operatorname{rad} U]) \\ &= \delta_{M,N}(U) - \delta_{M,N}(\operatorname{rad} U) \\ &= \delta_{M,N}(U) - \delta_{M,N}(E(U)) + \delta_{M,N}(\tau U).\end{aligned}$$

2.6. A component Γ of Γ_A, without oriented cycles and such that any τ-orbit contains a projective module is called *preprojective*. Also any module $X \in \operatorname{add}(\Gamma)$ is called *preprojective*. There is a partial order $\preceq$ on the set of vertices of a preprojective component Γ with $U \preceq V$ if there exists a path in Γ leading from U to V. Preinjective components and preinjective modules are defined dually.
2.7. Let M and N be A-modules with $M < N$. A short nonsplittable exact sequence

$$\Sigma : 0 \to L_1 \to M' \to L_2 \to 0$$

is said to be *admissible for* (M, N) if $M = M' \oplus V$ for some A-module V and $[L_1 \oplus L_2 \oplus V, X] \leq [N, X]$ for any A-module X (equivalently, $\delta_\Sigma \leq \delta_{M,N}$ or $\delta'_\Sigma \leq \delta'_{M,N}$).

We shall need the following fact.

Proposition. *Let M and N be A-modules with $[M] = [N]$, and assume that M is preprojective and $M < N$ holds. Then there exists an admissible sequence $0 \to L_1 \to M \to L_2 \to 0$ for (M, N).*

Proof. We can repeat the proof of Theorem 4.1 in [10], since Bongartz has used the fact that N is preprojective only to prove that M is preprojective.

3. Some properties of modules over tame concealed algebras

Here and later on A denotes a fixed tame concealed algebra [14].
3.1. We recall those aspects of the representation theory of tame concealed algebras that we will need later (see [14], [10]). We have a decomposition of Γ_A into the preprojective part $\mathcal{P}$, the preinjective part $\mathcal{I}$ and the regular one $\mathcal{R}$, where $\mathcal{R}$ is a sum of stable tubes $\mathcal{T}_\mu$ of ranks $r_\mu \geq 1$, for $\mu \in \mathbb{P}^1(K) = K \cup \{\infty\}$. For any A-module X we can write $X = X_P \oplus X_R \oplus X_I$, where $X_P \in \operatorname{add}(\mathcal{P})$, $X_I \in \operatorname{add}(\mathcal{I})$ and $X_R = \bigoplus_{\mu \in \mathbb{P}^1(K)} X_\mu$ with $X_\mu \in \operatorname{add}(\mathcal{T}_\mu)$. All connected components of Γ_A are standard (see [14] for definition). A tube of rank 1 is called *homogeneous* and $\mathcal{T}_\mu$ is not homogeneous for at most three $\mu \in \mathbb{P}^1(K)$. For any $X, Y \in \Gamma_A$, if $[X, Y] > 0$

and X and Y do not belong to the same connected component of Γ_A, then X is preprojective or Y is preinjective. The abelian category $\mathrm{add}(\mathcal{T}_\mu)$ is serial and closed under extensions, so we may speak about simple regular modules, composition series in $\mathrm{add}(\mathcal{T}_\mu)$, and so on. A tube $\mathcal{T}_\mu$ has r_μ simple regular modules, which are conjugate under τ. If a tube $\mathcal{T}_\mu$ is homogeneous ($r_\mu = 1$), then we denote a unique simple regular module in $\mathcal{T}_\mu$ by E_μ. For any simple regular module E in $\mathcal{T}_\mu$ we denote by

$$\cdots \to \varphi^3 E \to \varphi^2 E \to \varphi E \to \varphi^0 E = E$$

a unique infinite sectional path in $\mathcal{T}_\mu$ of epimorphisms and by

$$E = \psi^0 E \to \psi E \to \psi^2 E \to \psi^3 E \to \cdots$$

a unique infinite sectional path in $\mathcal{T}_\mu$ of monomorphisms. Then every indecomposable module in $\mathcal{T}_\mu$ is of the form $\varphi^j E$ and $\psi^j E'$ for some $j \geq 0$ and simple regular modules E, E' in $\mathcal{T}_\mu$. In an obvious way we define functions

$$\varphi^k, \psi^k : \mathcal{T}_\mu \to \mathcal{T}_\mu \cup \{0\}$$

for any integer k, such that for any simple regular module E in $\mathcal{T}_\mu$ and $l \geq 0$ we have:

- $\varphi^k(\varphi^l E) = \varphi^{k+l} E$ if $k + l \geq 0$, and $\varphi^k(\varphi^l E) = 0$ otherwise;
- $\psi^k(\psi^l E) = \psi^{k+l} E$ if $k + l \geq 0$, and $\psi^k(\psi^l E) = 0$ otherwise.

Observe that for any integer k and $X \in \mathcal{T}_\mu$ we have $\tau X = \psi^- \varphi X$, $\tau^- X = \varphi^- \psi X$ and $\varphi^{kr} X = \psi^{kr} X$, where $r = r_\mu$.

There is a positive, sincere vector $\underline{h}$ in $K_0(A)$, such that

$$[\varphi^{kr_\mu - 1} E] = [\psi^{kr_\mu - 1} E] = k \cdot \underline{h}$$

for any simple regular module E in $\mathcal{T}_\mu$ and $k \geq 1$.

3.2 The global dimension of A is at most 2. All preprojective and regular modules have projective dimension at most 1, and dually all preinjective and regular modules have injective dimension at most 1. The bilinear form on $K_0(A) = \mathbb{Z}^n$ which extends the equality

$$< [M], [N] >= [M, N] - [M, N]^1 + [M, N]^2$$

and the associated quadratic form $\chi : K_0(A) \to \mathbb{Z}$, $\chi(\underline{y}) =< \underline{y}, \underline{y} >$, will play an important role. If M has no non-zero preinjective direct summand or N has no non-zero preprojective direct summand, then

$$< [M], [N] >= [M, N] - [M, N]^1.$$

The quadratic form χ is positive semidefinite and controls the category mod A (see [14]). This means that the following conditions are satisfied:

(1) For any $X \in \Gamma_A$, $\chi([X]) \in \{0, 1\}$.
(2) For any connected, positive vector $\underline{y}$ with $\chi(\underline{y}) = 1$, there is precisely one $X \in \Gamma_A$ with $[X] = \underline{y}$.
(3) For any connected, positive vector $\underline{y}$ with $\chi(\underline{y}) = 0$, there is an infinite family of pairwise nonisomorphic modules $X \in \Gamma_A$ with $[X] = \underline{y}$.

Moreover, $\chi(\underline{h}) = 0$ and $< \underline{h}, \underline{y} >= - < \underline{y}, \underline{h} >$ for any $\underline{y} \in K_0(A)$. Finally, we define a linear function $\partial : K_0(A) \to \mathbb{Z}$, called the *defect*, as follows

$$\partial \underline{y} =< \underline{h}, \underline{y} >= - < \underline{y}, \underline{h} > .$$

The main property of ∂ is that the value $\partial[X]$ is negative for any $X \in \mathcal{P}$, positive for any $X \in \mathcal{I}$, and zero for any $X \in \mathcal{R}$.

Lemma 3.3. *If $M \leq N$, then $\partial[M_P] - \partial[N_P] = \partial[N_I] - \partial[M_I] \geq 0$.*

Proof. Since $[M] = [N]$, then

$$\partial[M_P] + \partial[M_R] + \partial[M_I] = \partial[N_P] + \partial[N_R] + \partial[N_I].$$

The equalities $\partial[M_R] = \partial[N_R] = 0$ imply $\partial[M_P] - \partial[N_P] = \partial[N_I] - \partial[M_I]$. Take a homogeneous tube $\mathcal{T}_\mu$ with $(M \oplus N)_\mu = 0$. Then

$$\begin{aligned} 0 \leq &[N, E_\mu] - [M, E_\mu] = [N_P, E_\mu] - [M_P, E_\mu] \\ &= < [N_P], [E_\mu] > - < [M_P], [E_\mu] >=< [N_P], \underline{h} > - < [M_P], \underline{h} > \\ &= \partial[M_P] - \partial[N_P]. \end{aligned}$$

3.4. Fix a tube $\mathcal{T}_\mu$, $\mu \in \mathbb{P}^1(K)$, and a module $X \in \mathrm{add}(\mathcal{T}_\mu)$. Let $H(X) \geq 0$ be the minimal number such that for any indecomposable direct summand $\varphi^j E$ of X, where E is a simple regular module in $\mathcal{T}_\mu$, we have $j < H(X)$ (so $H(X)$ is the maximal quasi-length of an indecomposable direct summand of X). For any simple regular module E in $\mathcal{T}_\mu$ we denote by $\ell_E(X)$ the multiplicity of E as a composition factor of a composition series of X in the category $\mathrm{add}(\mathcal{T}_\mu)$. If $E_1, \dots, E_r$ $(r = r_\mu)$ denote all simple regular modules in $\mathcal{T}_\mu$, then

$$[X] = \ell_{E_1}(X)[E_1] + \ell_{E_2}(X)[E_2] + \cdots + \ell_{E_r}(X)[E_r].$$

Moreover, the following lemma holds (see Lemma 5.1 in [15]).

Lemma 3.5. *Let X be a module in $\mathrm{add}(\mathcal{T}_\mu)$ and E be any simple regular module in $\mathcal{T}_\mu$. Then for any $k \geq H(X) - 1$ we have*

$$[X, \psi^k E] = \ell_E(X) = [\varphi^k E, X].$$

As a consequence of the above lemma we obtain

Lemma 3.6. *Let i, j be integers with $j \geq 0$ and E be any simple regular module in $\mathcal{T}_\mu$. Then*

(i) $[\varphi^s\psi^t E, \psi^{r-1}E] = 1$ *for all* $s \geq 0$, $0 \leq t < r$, *and* $[X, \psi^{r-1}E] = 0$ *for the remaining indecomposable modules* $X \in \mathcal{T}_\mu$.
(ii) $[\varphi^s\psi^t E, \psi^{r-1}\varphi^j E] - [\varphi^s\psi^t E, \psi^-\varphi^j E] = 1$ *for all* $s \geq j$, $0 \leq t < r$, *and* $[X, \psi^{r-1}\varphi^j E] - [X, \psi^-\varphi^j E] = 0$ *for the remaining indecomposable modules* $X \in \mathcal{T}_\mu$.
(iii) *If* $j \geq r$, *then* $[\psi^j E, \psi^j E] > 1$.
(iv) $[E, \psi^j E] = 1$ *and* $[E', \psi^j E] = 0$ *for all simple regular modules* $E' \neq E$ *in* $\mathcal{T}_\mu$.

Applying Lemmas 4.3 and 4.6 in [15], we obtain the following result (see also Corollary 2.2 in [2]).

Lemma 3.7. *Let* $X \in \mathcal{T}_\mu$, $s, t \geq 0$ *be integers, and* M, N *be* A*-modules with* $[M] = [N]$. *Then*

(i) *There exists a nonsplittable exact sequence*

$$\Sigma : 0 \to \varphi^s X \to \varphi^s\psi^{t+1}X \oplus \varphi^- X \to \varphi^-\psi^{t+1}X \to 0.$$

Moreover, if $s < r$ *or* $t < r$, *then* $\delta_\Sigma(\varphi^i\psi^j X) = 1$ *for all* $0 \leq i \leq s$, $0 \leq j \leq t$, *and* $\delta_\Sigma(Y) = 0$ *for the remaining indecomposable* A*-modules.*

(ii)

$$\sum_{0\leq i\leq s}\sum_{0\leq j\leq t} \mu(N, \varphi^{-i}\psi^j X) - \mu(M, \varphi^{-i}\psi^j X)$$
$$= \delta_{M,N}(\psi^-\varphi^{s+1}X) - \delta_{M,N}(\psi^- X) - \delta_{M,N}(\varphi^{s+1}\psi^t X) + \delta_{M,N}(\psi^t X).$$

Lemma 3.8. *Let* M, N *be* A*-modules with* $M \leq N$ *and* $\partial[M_P] = \partial[N_P]$. *Then*

(i) $[M_P] \geq [N_P]$.
(ii) *For any indecomposable simple regular module* E *in a tube* $\mathcal{T}_\mu$ *we have*

$$\ell_E(M_\mu) \leq \ell_E(N_\mu).$$

(iii) *For any tube* $\mathcal{T}_\mu$, $[M_\mu] \leq [N_\mu]$ *holds.*

Proof. (i) Let I be any indecomposable injective A-module. We shall show that $[M_P, I] \geq [N_P, I]$. For all but finitely many $k > 0$, the vector $k \cdot \underline{h} - [I]$ is positive and connected. Moreover,

$$\chi(k \cdot \underline{h} - [I]) =< k \cdot \underline{h} - [I], k \cdot \underline{h} - [I] >=< [I], [I] >= \chi([I]) = 1.$$

Thus for all but finitely many $k > 0$ there is an indecomposable A-module X_k with $[X_k] = k \cdot \underline{h} - [I]$. Of course

$$\partial[X_k] =< \underline{h}, k \cdot \underline{h} - [I] >= - < \underline{h}, [I] >= -\partial[I] < 0,$$

which implies that X_k is preprojective. Take $k > 0$ such that there exists a preprojective A-module X_k with $[X_k] = k\underline{h} - [I]$ and $[M_P \oplus N_P, X_k]^1 = 0$. Then

$$\begin{aligned}[M_P, I] =& < [M_P], [I] >= -k\partial[M_P] - < [M_P], [X_k] >= -k\partial[M_P] - [M_P, X_k] \\ &\geq -k\partial[N_P] - [N_P, X_k] = -k\partial[N_P] - < [N_P], [X_k] >=< [N_P], [I] > \\ &= [N_P, I].\end{aligned}$$

Hence, $[M_P] \geq [N_P]$.

(ii) Let $r = r_\mu$ and s be a natural number such that $sr \geq H(M_\mu \oplus N_\mu)$. Then

$$\begin{aligned}0 \leq& [N, \psi^{sr-1}E] - [M, \psi^{sr-1}E] = [N_P, \psi^{sr-1}E] - [M_P, \psi^{sr-1}E] + [N_\mu, \psi^{sr-1}E] \\ &- [M_\mu, \psi^{sr-1}E] =< [N_P], s \cdot \underline{h} > - < [M_P], s \cdot \underline{h} > + \ell_E(N_\mu) - \ell_E(M_\mu) \\ =& - s(\partial[N_P] - \partial[M_P]) + \ell_E(N_\mu) - \ell_E(M_\mu) = \ell_E(N_\mu) - \ell_E(M_\mu),\end{aligned}$$

by Lemma 3.5.

(iii) follows from (ii), since for any $X \in \mathrm{add}(\mathcal{T}_\mu)$ we have

$$[X] = \ell_{E_1}(X)[E_1] + \ldots + \ell_{E_r}(X)[E_r],$$

where $r = r_\mu$ and $E_1, \ldots, E_r$ denote all simple regular modules in $\mathcal{T}_\mu$.

Lemma 3.9. *Let Γ' be a disjoint union of some tubes in Γ_A and $\Gamma'' = \Gamma_A \setminus \Gamma'$. Then for any $X \in \mathrm{add}(\Gamma'')$ and $R_1, R_2 \in \mathrm{add}(\Gamma')$ with $[R_1] = [R_2]$ we have*

$$[X, R_1] = [X, R_2] \qquad \textit{and} \qquad [R_1, X] = [R_2, X].$$

Proof. By duality, it is enough to prove the first equality. We may assume that X is indecomposable and preprojective, because $[X, R_1] = [X, R_2] = 0$ for any regular or preinjective A-module $X \in \mathrm{add}(\Gamma'')$. Hence, we get

$$[X, R_1] - [X, R_1]^1 =< [X], [R_1] >=< [X], [R_2] >= [X, R_2] - [X, R_2]^1.$$

Since $[X, R_1]^1 = [X, R_2]^1 = 0$ for any preprojective A-module X, we obtain the required equality $[X, R_1] = [X, R_2]$.

4. Proof of the Theorem

We shall divide our proof of the Theorem into several steps. We use the notations introduced in Sections 2 and 3.

Proposition 4.1. *Let M and $N = N_0 \oplus N_1$ be A-modules without any common indecomposable direct summands. Assume that $M < N$ and N_0 is a preprojective indecomposable A-module with $[N_0, N] = [N_0, M]$. If there is no admissible sequence of the form $0 \to N_0 \to M \to C \to 0$ for (M, N), then there exist a homogeneous tube $\mathcal{T}_\nu$ in Γ_A, for which $(M \oplus N)_\nu = 0$, and a nonsplittable exact sequence*

$$0 \to L \to M \to E_\nu \to 0,$$

such that $[L \oplus E_\nu, X] \leq [N, X]$ for any indecomposable A-module $X \notin \mathcal{T}_\nu$.

Proof. By Theorem 2.4 in [10] N_0 embeds into M and the closure $\overline{\mathcal{Q}}$ of the quotients of M by N_0 contains N_1. Let $t = \dim_K M + 1$ and $\Gamma' \cup \mathcal{T}_{\mu_1} \cup \cdots \cup \mathcal{T}_{\mu_t}$ be the disjoin union of all homogeneous tubes which do not contain any indecomposable direct summand of $M \oplus N$. We set $\Gamma'' = \Gamma_A \setminus \Gamma'$. Then Γ'' is the disjoint union of finitely many connected components of Γ_A, and for any natural number d, there is only a finite number of isomorphism classes of d-dimensional modules from $\operatorname{add}(\Gamma'')$. We decompose the set $\mathcal{Q}$ into a finite union of pairwise disjoint subsets $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_r$ such that two modules $U_1 \oplus U_2$ and $V_1 \oplus V_2$ from $\mathcal{Q}$ with $U_1, V_1 \in \operatorname{add}(\Gamma'')$, $U_2, V_2 \in \operatorname{add}(\Gamma')$, belong to the same $\mathcal{D}_i$, $1 \leq i \leq r$, if and only if $U_1 \simeq V_1$. Since $\overline{\mathcal{Q}} = \overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2} \cup \cdots \cup \overline{\mathcal{D}_r}$, the module N_1 belongs to $\overline{\mathcal{D}_i}$ for some $1 \leq i \leq r$. Take any $V \oplus R \in \mathcal{D}_i$ with $V \in \operatorname{add}(\Gamma'')$ and $R \in \operatorname{add}(\Gamma')$. Then any module from $\mathcal{D}_i$ is, up to isomorphism, of the form $V \oplus R'$ for some $R' \in \operatorname{add}(\Gamma')$ with $[R'] = [R]$. Consequently, for any indecomposable module $X \in \operatorname{add}(\Gamma'')$ we have $[R', X] = [R, X]$, by Lemma 3.9. Applying upper semicontinuity of the function $(Z \to \dim_K \operatorname{Hom}_A(Z, X))$, we conclude that the set

$$\mathcal{S}_X = \{Z \in \overline{\mathcal{D}_i};\ [Z, X] \geq [V \oplus R, X] = [V \oplus R', X]\}$$

is closed (see [11],[13]), for any $X \in \operatorname{add}(\Gamma'')$. Since $\mathcal{D}_i$ is a subset of $\mathcal{S}_X$, we obtain that $[N_1, X] \geq [V \oplus R, X]$ for any $X \in \operatorname{add}(\Gamma'')$. Take a tube $\mathcal{T}_{\mu_c} \subset \Gamma''$, for some $1 \leq c \leq t$, such that any direct summand of $V \oplus N_1$ does not belong to $\mathcal{T}_{\mu_c}$. It is possible, because $\dim_K V < t$.

Assume that $R = 0$. Then by Lemma 3.9, for any $\mathcal{T}_\lambda \subset \Gamma'$ and $j \geq 0$, we have

$$[N_1, \varphi^j E_\lambda] = [N_1, \varphi^j E_{\mu_c}] \geq [V, \varphi^j E_{\mu_c}] = [V, \varphi^j E_\lambda].$$

This leads to a contradiction, since the sequence $0 \to N_0 \to M \to V \to 0$ is admissible for (M, N). So, there is a tube $\mathcal{T}_\nu \subset \Gamma'$ such that $V \oplus R = I \oplus \varphi^j E_\nu$ for some A-module I and $j \geq 0$. Then, for an epimorphism $p : \varphi^j E_\nu \to E_\nu$ we obtain the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & 0 & & & I \oplus \varphi^{j-1} E_\nu & & \\
 & \downarrow & & & \downarrow & & \\
0 \to & N_0 & \longrightarrow & M & \longrightarrow & I \oplus \varphi^j E_\nu & \to 0 \\
 & \downarrow & & \| & & \downarrow {\scriptstyle (0,p)} & \\
0 \to & L & \longrightarrow & M & \longrightarrow & E_\nu & \to 0 \\
 & \downarrow & & & & \downarrow & \\
 & I \oplus \varphi^{j-1} E_\nu & & & & 0 & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}$$

Hence, for any $\mathcal{T}_\lambda \subset (\Gamma' \setminus \mathcal{T}_\nu)$ and $k \geq 0$, applying Lemma 3.9, we get

$$\begin{aligned}
[N, \varphi^k E_\lambda] &= [N, \varphi^k E_{\mu_c}] \geq [N_0 \oplus V \oplus R, \varphi^k E_{\mu_c}] \\
&= [N_0 \oplus I \oplus \varphi^j E_\nu, \varphi^k E_{\mu_c}] \\
&= [N_0 \oplus I \oplus \varphi^{j-1} E_\nu \oplus E_\nu, \varphi^k E_{\mu_c}] \\
&\geq [L \oplus E_\nu, \varphi^k E_{\mu_c}] = [L \oplus E_\nu, \varphi^k E_\lambda].
\end{aligned}$$

This leads to $[L \oplus E_\nu, X] \leq [N, X]$ for any $X \in \Gamma_A \setminus \mathcal{T}_\nu$.

Proposition 4.2. *Let M and N be A-modules without any common indecomposable direct summand and such that $M < N$ and $M_P \oplus N_P$ is nonzero. Let $r = r_\mu$ and E be any simple regular module in $\mathcal{T}_\mu$ for some $\mu \in \mathbb{P}^1(K)$. If there is no admissible sequence for (M, N), then*

- *(i)* $\partial[M_P] = \partial[N_P]$.
- *(ii)* $\delta'_{M,N}(\varphi^s \psi^t E) = 0$ *holds for some* $s \geq 0$ *and* $0 \leq t < r$.
- *(iii) For any* $j \geq 1$ *such that* $\psi^- \varphi^j E$ *is a direct summand of* M*, the equality* $\delta'_{M,N}(\varphi^s \psi^t E) = 0$ *holds for some* $s \geq j$ *and* $0 \leq t < r$.
- *(iv) There are infinitely many modules* X *in* $\mathcal{T}_\mu$ *with* $\delta'_{M,N}(X) = 0$.
- *(v) There are infinitely many modules* X *in* $\mathcal{T}_\mu$ *with* $\delta_{M,N}(X) = 0$.

Proof. (i) If $\delta_{M,N}(X) = 0$ for all indecomposable preprojective A-modules, then, by Lemma 2.5, $\mu(M_P, X) = \mu(N_P, X)$ for any indecomposable preprojective A-module, and consequently $M_P = N_P = 0$, which gives a contradiction. Let N_0 be a minimal, with respect to $\preceq$, indecomposable preprojective A-module with $\delta_{M,N}(N_0) > 0$. Then by Lemma 2.5 we get

$$\mu(N, N_0) - \mu(M, N_0) = \delta_{M,N}(N_0) > 0,$$

because $X \prec N_0$ for any indecomposable direct summand X of $E(N_0) \oplus \tau N_0$. This implies that $N = N_0 \oplus N_1$ for some A-module N_1. Of course, $\delta'_{M,N}(N_0) = \delta_{M,N}(\tau N_0) = 0$ and consequently $[N_0, N] = [N_0, M]$. By Proposition 4.1, there is a nonsplittable exact sequence

$$0 \to L \to M \to E_\nu \to 0$$

such that $\mathcal{T}_\nu$ is a homogeneous tube for which $(M \oplus N)_\nu = 0$ and $[L \oplus E_\nu, X] \leq [N, X]$ for any indecomposable A-module $X \notin \mathcal{T}_\nu$. Observe that $L_R \oplus L_I = M_R \oplus M_I$. Then we get a nonsplittable exact sequence

$$\Sigma : 0 \to L_P \to M_P \to E_\nu \to 0$$

such that $\delta_\Sigma(X) \leq \delta_{M,N}(X)$ for any indecomposable A-module $X \notin \mathcal{T}_\nu$. Thus there is $t \geq 0$ such that $\delta_\Sigma(\varphi^t E_\nu) > \delta_{M,N}(\varphi^t E_\nu)$, because Σ is not admissible for (M, N). We set $F = E_\nu$. Since $\tau^- \varphi^t F = \varphi^t F$, we get

$$\delta_\Sigma(\varphi^t F) = \delta'_\Sigma(\varphi^t F) = [\varphi^t F, L_P \oplus F] - [\varphi^t F, M_P] = [\varphi^t F, F] = 1$$

and

$$\begin{aligned} \delta_{M,N}(\varphi^t F) =& [N, \varphi^t F] - [M, \varphi^t F] = [N_P, \varphi^t F] - [M_P, \varphi^t F] = < [N_P], [\varphi^t F] > \\ & - < [M_P], [\varphi^t F] > = < [N_P], (t+1) \cdot \underline{h} > - < [M_P], (t+1) \cdot \underline{h} > \\ =& (t+1)(\partial[M_P] - \partial[N_P]). \end{aligned}$$

This leads to $\partial[M_P] - \partial[N_P] < 1$ and, by Lemma 3.3, we have $\partial[M_P] = \partial[N_P]$.

(ii) Since $M_P \leq_{\text{ext}} L_P \oplus E_\nu$, then $M_P \leq L_P \oplus E_\nu$. Let X be any indecomposable A-module. If $X \notin \mathcal{P} \cup \mathcal{T}_\mu$, then $[X, M_P] = [X, L_P \oplus \psi^{r-1} E] = 0$. If $X \in \mathcal{T}_\mu$, then $0 = [X, M_P] \leq [X, L_P \oplus \psi^{r-1} E]$. Since $[E_\nu] = \underline{h} = [\psi^{r-1} E]$, applying Lemma 3.9 for any preprojective module X, we obtain

$$\begin{aligned} 0 \leq & [X, L_P \oplus \psi^{r-1} E] - [X, M_P] = [X, L_P \oplus E_\nu] - [X, M_P] \\ =& [X, L \oplus E_\nu] - [X, M] \leq [X, N] - [X, M]. \end{aligned}$$

Thus $M_P \leq L_P \oplus \psi^{r-1} E$ and

$$[X, L_P \oplus \psi^{r-1} E] - [X, M_P] \leq [X, N] - [X, M]$$

for any indecomposable A-module $X \notin \mathcal{T}_\mu$. By Proposition 2.7, there is an admissible sequence

$$\Sigma_0 : 0 \to L_1 \to M_P \to L_2 \to 0$$

for $(M_P, L_P \oplus \psi^{r-1} E)$. Hence, $[X, L_1 \oplus L_2] \leq [X, L_P \oplus \psi^{r-1} E] = 0$ for any indecomposable module $X \notin \mathcal{P} \cup \mathcal{T}_\mu$. This implies that $L_1 \oplus L_2 \in \operatorname{add}(\mathcal{P} \cup \mathcal{T}_\mu)$. Since the sequence Σ_0 is not admissible for (M, N), we get

$$[X, \psi^{r-1} E] = [X, L_P \oplus \psi^{r-1} E] - [X, M_P] > [X, N] - [X, M]$$

for some indecomposable module $X \in \mathcal{T}_\mu$. By Lemma 3.6(i), $[\varphi^s\psi^t E, \psi^{r-1}E] = 1$ for all $s \geq 0$, $0 \leq t < r$ and $[X, \psi^{r-1}E] = 0$ for the remaining modules $X \in \mathcal{T}_\mu$. Hence, $\delta'_{M,N}(X) = [X,N] - [X,M] = 0$ for some $X = \varphi^s\psi^t E$, $s \geq 0$ and $0 \leq t < r$.

(iii) Assume that $\psi^-\varphi^j E$ is a direct summand of M for some $j \geq 1$. Take the admissible sequence

$$\Sigma_0 : 0 \to L_1 \to M_P \to L_2 \to 0$$

for $(M_P, L_P \oplus \psi^{r-1}E)$, considered in (ii). We can write $L_2 = L'_2 \oplus Y$ such that $L_1 \oplus L'_2$ is preprojective and $Y \in \operatorname{add}(\mathcal{T}_\mu)$. If $Y = 0$, then $[X, L_1 \oplus L_2] - [X, M_P] = 0$ for any $X \in \mathcal{T}_\mu$, and moreover Σ_0 is an admissible sequence for (M,N). Hence $Y \neq 0$, and consequently

$$[X,Y] = [X, L_1 \oplus L'_2 \oplus Y] - [X, M_P] \leq [X, L_P \oplus \psi^{r-1}E] - [X, M_P] = [X, \psi^{r-1}E]$$

for any X in $\mathcal{T}_\mu$. Applying Lemma 3.6(iv) we get $[E,Y] \leq [E, \psi^{r-1}E] = 1$ and $[E', Y] \leq [E', \psi^{r-1}E] = 0$, for all simple regular modules $E' \neq E$ in $\mathcal{T}_\mu$, and consequently Y is indecomposable and $Y = \psi^k E$ for some $k \geq 0$. Since $[Y,Y] \leq [Y, \psi^{r-1}E] \leq 1$, we obtain $k < r$, by Lemma 3.6. Let

$$e : L'_2 \oplus \varphi^j\psi^k E \to L'_2 \oplus \psi^k E = L_2$$

be a natural epimorphism. Then the pull back of Σ_0 under e is a sequence of the form

$$\Sigma_j : 0 \to L_1 \to M_P \oplus \psi^-\varphi^j E \to L'_2 \oplus \varphi^j\psi^k E \to 0,$$

because $\ker e$ is isomorphic to $\psi^-\varphi^j E$ and $Ext^1(M_P, \psi^-\varphi^j E) = 0$. Observe that $M_P \oplus \psi^-\varphi^j E$ is a direct summand of M and $\delta'_{\Sigma_j} \leq \delta'_{\Sigma_0}$. This implies that $\delta'_{\Sigma_j}(X) \leq \delta'_{M,N}(X)$ for any indecomposable A-module $X \notin \mathcal{T}_\mu$. Since the sequence Σ_j is not admissible for (M,N), we get $\delta'_{\Sigma_j}(X) > \delta'_{M,N}(X)$ for some $X \in \mathcal{T}_\mu$. Then

$$\delta'_{\Sigma_j}(X) = [X, \varphi^j\psi^k E] - [X, \psi^-\varphi^j E] \leq [X, \varphi^j\psi^{r-1}E] - [X, \psi^-\varphi^j E],$$

because $\varphi^j\psi^k E$ may be treated as a submodule of $\varphi^j\psi^{r-1}E$. Applying Lemma 3.6(ii) we get that $[\varphi^s\psi^t E, \varphi^j\psi^{r-1}E] - [\varphi^s\psi^t E, \psi^-\varphi^j E] = 1$ for all $s \geq j, 0 \leq t < r$, and $[Y, \varphi^j\psi^{r-1}E] - [Y, \psi^-\varphi^j E] = 0$ for the remaining indecomposable modules $Y \in \mathcal{T}_\mu$. Thus, $X = \varphi^s\psi^t E$ and $\delta'_{M,N}(X) = 0$ for some $s \geq j$ and $0 \leq t < r$.

(iv) Suppose that the required claim is not true. Take a maximal $s \geq 0$ and a simple regular module E' in $\mathcal{T}_\mu$ such that $\delta'_{M,N}(\varphi^s E') = 0$. Applying (ii) for the simple regular module $\tau^- E'$, we infer that there are numbers $s' \geq 0$ and $0 \leq t' < r$ with $\delta'_{M,N}(\varphi^{s'}\psi^{t'}\tau^- E') = \delta'_{M,N}(\varphi^{s'-1}\psi^{t'+1}E') = 0$. Take a pair (s', t') with maximal number s'. Since $\delta'_{M,N}(\varphi^{s'}\psi^{t'}\tau^- E') = \varphi^{s'+t'}(\tau^{-t'-1}E')$, then $s' \leq s' + t' \leq s$, by maximality of s. Thus, $\delta'_{M,N}(\varphi^k\psi^l\tau^- E') > 0$ for all

$k > s'$ and $0 \leq l < r$. Applying Lemma 3.7(ii), we get

$$\begin{aligned}\sum_{s' \leq i \leq s}\sum_{0 \leq j \leq t'} \mu(N, \varphi^i \psi^j E') - \mu(M, \varphi^i \psi^j E') &= \delta_{M,N}(\psi^- \varphi^{s+1} E') \\ &- \delta_{M,N}(\psi^- \varphi^{s'} E') - \delta_{M,N}(\varphi^{s+1} \psi^{t'} E') + \delta_{M,N}(\varphi^{s'} \psi^{t'} E') \\ &\leq \delta'_{M,N}(\varphi^s E') - \delta'_{M,N}(\varphi^{s+1} \psi^{t'} \tau^- E') + \delta'_{M,N}(\varphi^{s'-1} \psi^{t'+1} E') \\ &= - \delta'_{M,N}(\varphi^{s+1} \psi^{t'} \tau^- E') < 0,\end{aligned}$$

because $s + 1 > s'$ and $0 \leq t' < r$. Thus $\varphi^i \psi^j E'$ is a direct summand of M for some $s' \leq i \leq s$ and $0 \leq j < r$. Let $E = \tau^{-j-1} E'$. Then $\psi^- \varphi^{i+j+1} E$ is a direct summand of M, and applying (iii), we get numbers $p \geq i + j + 1$ and $0 \leq q < r$ with $\delta'_{M,N}(\varphi^p \psi^q E) = 0$. Observe that $\varphi^p \psi^q E = \varphi^{p-j} \psi^{q+j} \tau^- E'$ and $0 \leq q + j < 2r$. If $q + j < r$, then $\delta'_{M,N}(\varphi^{p-j} \psi^{q+j} \tau^- E') = 0$, because $p - j \geq i + 1 > s'$. This leads to $q + j \geq r$, and $\varphi^{p-j} \psi^{q+j} \tau^- E' = \varphi^{p-j+r} \psi^{q+j-r} \tau^- E'$. But then $\delta'_{M,N}(\varphi^{p-j+r} \psi^{q+j-r} \tau^- E') = 0$, because $p - j + r > s'$ and $0 \leq q + j - r < r$, which is a contradiction.

(v) follows from (iv) and the formula $\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X)$.

Proposition 4.3. *Let M and N be A-modules with $M < N$. Assume that there is a tube $\mathcal{T}_\mu$ in Γ_A such that $\delta_{M,N}(\psi^j E) = 0$ and $\delta_{M,N}(\psi^{j-1} E) > 0$ for some simple regular module E in $\mathcal{T}_\mu$ and $j \geq H(M_\mu \oplus N_\mu) + r$, where $r = r_\mu$. Then there exists an admissible sequence for (M, N).*

Proof. Applying Lemma 3.5 we get

$$\begin{aligned}\delta_{M,N}(\psi^j E) =&[N, \psi^j E] - [M, \psi^j E] = [N_P \oplus N_\mu, \psi^j E] - [M_P \oplus M_\mu, \psi^j E] \\ =& < [N_P], [\psi^j E] > - < [M_P], [\psi^j E] > + \ell_E(N_\mu) - \ell_E(M_\mu),\end{aligned}$$

and similarly

$$\begin{aligned}\delta_{M,N}(\psi^{j-r} E) = &< [N_P], [\psi^{j-r} E] > - < [M_P], [\psi^{j-r} E] > \\ &+ \ell_E(N_\mu) - \ell_E(M_\mu).\end{aligned}$$

This leads to

$$\begin{aligned}\delta_{M,N}(\psi^{j-r} E) &= < [N_P], [\psi^{j-r} E] - [\psi^j E] > - < [M_P], [\psi^{j-r} E] - [\psi^j E] > \\ &= < [N_P], -\underline{h} > - < [M_P], -\underline{h} > = \partial[N_P] - \partial[M_P] = 0.\end{aligned}$$

Take a maximal number k such that $j - r \leq k \leq j - 2$ and $\delta_{M,N}(\psi^k E) = 0$. Then we have $\delta_{M,N}(\psi^t E) > 0$ for any $k < t < j$. If $\delta_{M,N}(\varphi^c \psi^d E) > 0$ for all $-k - 1 \leq c \leq 0$ and $k < d < j$, then we set $Y = 0$, $p = -k - 2$ and $q = k + 1$. Assume now that this is not the case. Take a maximal number c and a number d

such that $-k-1 \le c \le 0$, $k < d < j$ and $\delta_{M,N}(\varphi^c\psi^d E) = 0$. Of course, $c < 0$. Applying Lemma 3.7(ii), we get

$$\sum_{c\le p<0}\sum_{k<q\le d} \mu(N,\varphi^c\psi^d E) - \mu(M,\varphi^c\psi^d E) = \delta_{M,N}(\psi^k E) + \delta_{M,N}(\varphi^c\psi^d E)$$
$$- \delta_{M,N}(\psi^d E) - \delta_{M,N}(\varphi^c\psi^k E) \le -\delta_{M,N}(\psi^d E) < 0,$$

because $k < d < j$. Hence, $Y = \varphi^p\psi^q E$ is a direct summand of M for some $c \le p < 0$ and $k < q \le d$.

We set $V = \psi^q E$ and $W = \varphi^p\psi^j E$. Applying Lemma 3.7(i) for $X = \varphi^{p+1}\psi^q E$, $s = -p-1$, $t = j-q-1$, we get a short exact sequence

$$\Omega : 0 \to V \xrightarrow{\begin{pmatrix} \imath \\ f \end{pmatrix}} \psi^j E \oplus Y \xrightarrow{(f_1,f_2)} W \to 0,$$

where $\imath : V \to \psi^j E$ is a monomorphism. Further, $\delta_\Omega(X) = 1$ for any $X \in \mathcal{Y} = \{\varphi^v\psi^w E;\ p < v \le 0, q \le w < j\}$ and $\delta_\Omega(X) = 0$ for the remaining indecomposable A-modules X, because $t < r$. Thus, $\delta_\Omega \le \delta_{M,N}$, and so $M\oplus V\oplus W \le N\oplus Y\oplus\psi^j E$. Moreover,

$$0 \le [N\oplus Y\oplus\psi^j E, \psi^j E] - [M\oplus V\oplus W, \psi^j E)] \le [N,\psi^j E] - [M,\psi^j E] = 0$$

and $M\oplus V\oplus W \le_{\deg} N\oplus Y\oplus\psi^j E$, by Proposition 3 in [9]. Observe that the set of isomorphism classes of kernels of epimorphisms $M\oplus(V\oplus W) \to \psi^j E$ is finite. Therefore, there is a nonsplittable short exact sequence

$$\Theta : 0 \to L \to M\oplus V\oplus W \xrightarrow{g} \psi^j E \to 0$$

such that $L \le_{\deg} N\oplus Y$, by Theorem 2.4 in [10]. Of course, $M = M'\oplus Y$ for some A-module M'. We may consider the module V as a submodule of $\psi^j E$.

We claim that for any $g' \in \mathrm{Hom}_A(Y\oplus V\oplus W, \psi^j E)$ we have $\mathrm{im}\, g' \subseteq V$. Indeed, since

$$E \subset \psi E \subset \cdots \subset V = \psi^q E \subset \cdots \subset \psi^j E$$

is the unique composition series of $\psi^j E$ in $\mathrm{add}(\mathcal{T}_\mu)$, we get $\mathrm{im}\, g' = \psi^{j'} E$ for some $0 \le j' \le j$. On the other hand, the equality $\mathrm{im}\, g' = \psi^{j'} E$ implies that there is an indecomposable direct summand $\varphi^k\psi^{j'} E$ of $(Y\oplus V\oplus W)$, for some $k \ge 0$. This leads to $j' \le q$, which proves our claim.

Then the epimorphism g is of the form

$$g = (g_1, \imath g_2) : M'\oplus(Y\oplus V\oplus W) \to \psi^j E,$$

for some $g_1 : M' \to \psi^j E$ and $g_2 : Y\oplus V\oplus W \to V$.

Consider the pull back of the sequence

$$0 \to L \to M' \oplus (Y \oplus V \oplus W) \oplus Y \xrightarrow{\begin{pmatrix} g_1 & \imath g_2 & 0 \\ 0 & 0 & 1_Y \end{pmatrix}} \psi^j E \oplus Y \to 0$$

under the monomorphism $\begin{pmatrix} \imath \\ f \end{pmatrix} : V \to \psi^j E \oplus Y$. Then we obtain the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \to & L & \longrightarrow & & Z & \longrightarrow & V & \to 0 \\
 & \| & & & \downarrow & & \downarrow & & \\
0 \to & L & \longrightarrow & & M' \oplus (Y \oplus V \oplus W) \oplus Y & \longrightarrow & \psi^j E \oplus Y & \to 0 \\
 & & & & \downarrow & & \downarrow {\scriptstyle (f_1, f_2)} & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence we get an exact sequence

$$0 \to Z \to M' \oplus (Y \oplus V \oplus W) \oplus Y \xrightarrow{(f_1 g_1, f_1 \imath g_2, f_2)} W \to 0.$$

We may consider the module Z as a submodule of $M' \oplus (Y \oplus V \oplus W) \oplus Y$. Since $f_1 \imath g_2 = -f_2 f g_2$, we obtain a submodule $Z' = \{(0, m, f g_2(m));\ m \in Y \oplus V \oplus W\}$ of Z. It is easy to see that $Z' \simeq Y \oplus V \oplus W$, $Z = Z' \oplus Z_1$ for some A-module Z_1, and there exists an exact sequence of the form

$$\Psi : 0 \to Z_1 \to M' \oplus Y = M \to W \to 0.$$

Observe that, for any A-module X, we have

$$\begin{aligned}
\delta_\Psi(X) =& [Z_1 \oplus W, X] - [M, X] = [Z_1 \oplus W \oplus Y \oplus V, X] - [M \oplus Y \oplus V, X] \\
=& [Z, X] - [M \oplus Y \oplus V, X] \leq [L \oplus V, X] - [M \oplus Y \oplus V, X] \\
=& [L, X] - [M \oplus Y, X] \leq [N \oplus Y, X] - [M \oplus Y, X] = \delta_{M,N}(X),
\end{aligned}$$

because $Z \leq_{\text{ext}} L \oplus V$ and $L \leq_{\text{deg}} N \oplus Y$. Thus the sequence Ψ is admissible for (M, N), and this finishes the proof.

4.4. *Proof of Theorem.* Let M and N be two A-modules such that $M < N$. We shall show that $M <_{\text{ext}} N$. By Lemma 1.2 in [10], we may assume that the relation $M < N$ is minimal.

We claim that there is an admissible exact sequence for (M, N). Suppose that this is not the case. We may assume that M and N have no common indecomposable direct summand. If $M_P = N_P = M_I = N_I = 0$, then by Theorem 1 in [15], or

Section 3 in [9], $M = M_R <_{\text{ext}} N_R = N$. Then by definition of the relation $\leq_{\text{ext}}$, there is an admissible sequence for (M, N), and we get a contradiction. Hence, up to duality, we may assume that $M_P \oplus N_P$ is nonzero. Then by Proposition 4.2(i), $\partial[M_P] = \partial[N_P]$ and applying Lemma 3.8(i) and its dual we obtain

$$[M_P] \geq [N_P] \qquad \text{and} \qquad [M_I] \geq [N_I].$$

Assume that $[M_P] = [N_P]$ and let V be any indecomposable A-module. If V is preprojective, then

$$\delta_{M_P,N_P}(V) = [N_P, V] - [M_P, V] = [N, V] - [M, V] \geq 0,$$

otherwise

$$\delta_{M_P,N_P}(V) = \delta'_{M_P,N_P}(\tau^- V) = [\tau^- V, N_P] - [\tau^- V, M_P] = 0 - 0 = 0.$$

This implies that $M_P < N_P$ and by Corollary 4.2 in [10], $M_P <_{\text{ext}} N_P$. Then, by definition of the relation $\leq_{\text{ext}}$, there is an admissible sequence for (M_P, N_P). Since $\delta_{M_P,N_P} \leq \delta_{M,N}$, this sequence is admissible for (M, N), again a contradiction.

Hence, $[M_P] > [N_P]$, and consequently $\sum[M_\mu] < \sum[N_\mu]$, where the summation runs through all $\mu \in \mathbb{P}^1(K)$. Applying Lemma 3.8(iii), we conclude that there is $\mu \in \mathbb{P}^1(K)$ such that $[M_\mu] < [N_\mu]$. We set $r = r_\mu$ and let $E_1, \ldots, E_r$ be all simple regular modules in $\mathcal{T}_\mu$. Then by Lemma 3.8(ii) there is a simple regular module E in $\mathcal{T}_\mu$ with $\ell_E(M_\mu) < \ell_E(N_\mu)$, because $[X] = \ell_{E_1}(X)[E_1] + \cdots + \ell_{E_r}(X)[E_r]$ for any $X \in \operatorname{add}(\mathcal{T}_\mu)$. Applying Lemma 3.5, we get

$$\begin{aligned}
\delta_{M,N}(\psi^{sr-1}E) =& [N, \psi^{sr-1}E] - [M, \psi^{sr-1}E] = [N_P, \psi^{sr-1}E] \\
& - [M_P, \psi^{sr-1}E] + [N_\mu, \psi^{sr-1}E] - [M_\mu, \psi^{sr-1}E] \\
& = < [N_P], [\psi^{sr-1}E] > - < [M_P], [\psi^{sr-1}E] > + \ell_E(N_\mu) - \ell_E(M_\mu) \\
& > < [N_P], s \cdot \underline{h} > - < [M_P], s \cdot \underline{h} > = -s\partial[N_P] + s\partial[M_P] = 0,
\end{aligned}$$

for any integer s satisfying $sr \geq H(M_\mu \oplus N_\mu)$. Hence $\delta_{M,N}(X) > 0$ for infinitely many X in $\mathcal{T}_\mu$.

Applying Proposition 4.2(v), we infer that there are a simple regular module F in $\mathcal{T}_\mu$ and a number $j > H(M_\mu \oplus N_\mu) + r$ such that $\delta_{M,N}(\psi^j F) = 0$ and either $\delta_{M,N}(\psi^{j-1}F) > 0$ or $\delta_{M,N}(\varphi^- \psi^j F) > 0$. Let $F' = \tau^{-j-1}F$. Then either $\delta_{M,N}(\psi^j F) = 0 < \delta_{M,N}(\psi^{j-1}F)$ or $\delta'_{M,N}(\varphi^j F') = 0 < \delta'_{M,N}(\varphi^{j-1}F')$. Then by Proposition 4.3 or its dual there exists an admissible exact sequence for (M, N). This proves our claim.

Take an admissible sequence $0 \to L_1 \to M' \to L_2 \to 0$ for (M, N). This implies that $M = M' \oplus V$ for some A-module V and we obtain $M <_{\text{ext}} L_1 \oplus L_2 \oplus V \leq N$. Since the relation $M < N$ is minimal, then $N = L_1 \oplus L_2 \oplus V$. This leads to $M <_{\text{ext}} N$, and completes the proof.

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