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Autor(en): **Oliver, Bob**

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## The representation ring of a compact Lie group revisited

Bob Oliver

**Abstract.** We describe a new construction of the induction homomorphism for representation rings of compact Lie groups: a homomorphism first defined by Graeme Segal. The idea is to first define the induction homomorphism for class functions, and then show that this map sends characters to characters. This requires a detection theorem — a class function of  $G$  is a character if its restrictions to certain subgroups of  $G$  are characters — which in turn requires a review of the representation theory for nonconnected compact Lie groups.

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In his 1968 paper, Segal [Seg] used elliptic operators to construct induction homomorphisms  $R(H) \rightarrow R(G)$  for an arbitrary pair  $H \subseteq G$  of compact Lie groups, and then applied this to prove (among other things) a detection result for when a class function on  $G$  is a character. In this paper, we give new proofs of these results, but in the reverse order. We begin in Section 1 by showing that a class function on  $G$  is a character if its restrictions to all finite subgroups of  $G$  are characters. Then, in Section 2, we first define induction homomorphisms  $Cl(H) \rightarrow Cl(G)$  for class functions, and afterwards apply the results of Section 1 to show that they send characters to characters and hence define induction maps between the representation rings. This gives a construction of the induction homomorphisms which is more elementary than that of Segal (though also less elegant), in that it only assumes the standard theory of representations of a compact connected Lie group.

It is the results in Section 3 which, while more technical, provided the original motivation for this work. Let  $\mathcal{S}_p(G)$  be the family of all  $p$ -toral subgroups of  $G$  (for all primes  $p$ ), where a group is called  $p$ -toral if it is an extension of a torus by a finite  $p$ -group. Let  $R_p(G)$  be the inverse limit of the representation rings  $R(P)$  for all  $P \in \mathcal{S}_p(G)$ , where the limit is taken with respect to restriction and conjugation in  $G$ . This group  $R_p(G)$  was shown in [JO, Theorem 1.8] to be isomorphic to the Grothendieck group  $\mathbb{K}(BG)$  of the monoid of vector bundles over  $BG$ ; and the

“restriction” homomorphism

$$\mathrm{rs}_G : \mathbb{R}(G) \longrightarrow \mathbb{R}_P(G) = \varinjlim_{P \in \mathcal{S}_P(G)} \mathbb{R}(P)$$

is isomorphic to the natural homomorphism  $\mathbb{R}(G) \rightarrow \mathbb{K}(BG)$  which sends a representation  $V$  to the bundle  $(EG \times_G V) \downarrow BG$ .

The main result of Section 3 is a description, for arbitrary  $G$ , of the cokernel of the homomorphism  $\mathrm{rs}_G$ . In particular, we show that it is onto whenever  $G$  is finite or  $\pi_0(G)$  is a  $p$ -group; but that it is not surjective in general. Precise necessary and sufficient conditions for  $\mathrm{rs}_G$  to be onto are given in Theorem 3.10, and several simpler sufficient conditions are given in Corollary 3.11. Note that  $\mathrm{rs}_G$  is surjective if and only if bundles over  $BG$  have the following property: for each  $\xi \downarrow BG$  there exist  $G$ -representations  $V, V'$  such that  $\xi \oplus (EG \times_G V') \cong (EG \times_G V)$  (since by [JO, Theorem 1.8], every bundle over  $BG$  is a summand of a bundle coming from a  $G$ -representation).

In the above discussion, we have for simplicity dealt only with the complex representation rings. But most of the results are shown below for real as well as complex representations.

I would like to thank in particular Stefan Jackowski for his comments and suggestions about this work. Originally, Sections 1 and 3 were intended to go into our joint paper [JO], but then they grew to the point where we decided to publish them separately. I would also like to thank the colleague who, at the 1996 summer research institute in Seattle, showed me the references [Ta] and [Vo] on representation theory for nonconnected compact Lie groups. (After that conference, I asked several people if they were the ones who had done so, but they all denied it.)

## Section 1. Detection of characters

The main results of this section are Propositions 1.2 and 1.5: on detecting characters among class functions. They follow from Proposition 1.4, which describes the representation theory of nonconnected compact Lie groups. The first part of Proposition 1.4 — the bijection between irreducible  $G$ -representations and certain irreducible representations of  $N_G(T, C)$  — was proven by Takeuchi [Ta, Theorem 4], and is also stated in [Vo, Theorem 1.17]. Since their notation is very different from that used here, we have found it simplest to keep our proof, rather than just refer to [Ta]. Note that the group which we call  $N = N_G(T, C)$  is denoted  $T$  in [Ta] and [Vo] (and called the Cartan subgroup in [Vo]).

Throughout this section,  $G$  denotes a fixed compact Lie group, and  $G_0$  is its identity connected component. Fix a maximal torus  $T \subseteq G$ , let  $W_G = N_G(T)/T$  denote its Weyl group, and let  $\mathfrak{t} \subseteq \mathfrak{g}$  denote the Lie algebras of  $T$  and  $G$ . For any Weyl chamber  $C \subseteq \mathfrak{t}$ , define

$$N_G(T, C) = \{g \in N_G(T) \mid \mathrm{Ad}(g)(C) = C\},$$

and

$$N_G(T, \pm C) = \{g \in N_G(T) \mid \text{Ad}(g)(\pm C) = (\pm C)\}.$$

Here,  $\text{Ad}(g)$  denotes the adjoint (conjugation) action of  $g$  on  $\mathfrak{t}$  and  $\mathfrak{g}$ . We will see in Proposition 1.1 that  $N_G(T, C)$  has exactly one connected component for each connected component of  $G$ , and that every element of  $G$  is conjugate to an element of  $N_G(T, C)$ . Then, in Proposition 1.2 below, we show that a (continuous) class function  $f \in \text{Cl}(G)$  is a character of  $G$  if and only if  $f|_{N_G(T, C)}$  is a character, and that  $f$  is a real character of  $G$  (i.e., the character of a virtual  $\mathbb{R}G$ -representation) if and only if  $f|_{N_G(T, \pm C)}$  is a real character. At the same time, we construct (Proposition 1.4) a one-to-one correspondence between the irreducible representations of  $G$ , and those irreducible representations of  $N_G(T, C)$  whose weights lie in the dual Weyl chamber  $C^*$ . This generalizes the standard relationship, for a connected compact Lie group  $G$ , between the irreducible representations of  $G$  and those of  $T$ .

Afterwards, the detection result is extended to show that an element  $f \in \text{Cl}(G)$  is a character (real character) if and only if  $f|_H$  is a character (real character) of  $H$  for each finite subgroup  $H \subseteq G$ . The classical theorem of Brauer for detecting characters on finite groups can then be applied to further restrict the class of finite subgroups of  $G$  which have to be considered.

We first recall the definition and basic properties of the Weyl chambers of a compact connected Lie group  $G$ . The set of irreducible representations (or irreducible characters) of  $T$  will be identified here with  $T^* \stackrel{\text{def}}{=} \text{Hom}(T, S^1)$ ; which will in turn be regarded as a lattice in  $\mathfrak{t}^* = \text{Hom}(\mathfrak{t}, \mathbb{R})$ .

The *roots* of  $G$  (or of  $G_0$ ) are the characters of the nontrivial irreducible summands of the adjoint representation of  $T$  on  $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ . They occur in pairs  $\pm\theta$ . Let  $R \subseteq T^* \subseteq \mathfrak{t}^*$  denote the set of roots of  $G$ . Any element  $x_0 \in \mathfrak{t}$  such that  $\theta(x_0) \neq 0$  for all  $\theta \in R$  determines a choice of positive roots

$$R_+ = \{\theta \in R \mid \theta(x_0) > 0\}.$$

And this in turn determines a Weyl chamber

$$C = \{x \in \mathfrak{t} \mid \theta(x) \geq 0 \ \forall \theta \in R_+\} \subseteq \mathfrak{t}$$

and a dual Weyl chamber

$$C^* = \{x \in \mathfrak{t}^* \mid \langle \theta, x \rangle \geq 0 \ \forall \theta \in R_+\} \subseteq \mathfrak{t}^*.$$

Here, in the definition of  $C^*$ ,  $\langle -, - \rangle$  denotes any  $G$ -invariant inner product on  $\mathfrak{g}^*$ . Note that  $C^*$  is independent of the choice of inner product, since a  $G$ -invariant inner product is uniquely defined up to scalar on each simple component of  $G$ .

**Proposition 1.1.** *Fix a maximal torus  $T \subseteq G$  and a Weyl chamber  $C \subseteq \mathfrak{t}$ , and set  $N = N_G(T, C)$ . Then  $N \cap G_0 = T$ ,  $N \cdot G_0 = G$ , and hence  $N/T \cong G/G_0$ . Also, any element of  $G$  is conjugate to an element of  $N$ .*

*Proof.* Recall that the Weyl group  $W_{G_0} = N_{G_0}(T)/T$  of  $G_0$  permutes the Weyl chambers of  $T$  simply and transitively (cf. [Ad, Lemma 5.13]). Hence each coset of  $N_{G_0}(T)$  in  $N_G(T)$  contains exactly one connected component of  $N = N_G(T, C)$ ; and so  $N \cap G_0 = T$ ,  $N \cdot G_0 = G$ , and  $N/T \cong G/G_0$ .

By [Bo, §5.3, Theorem 1(b)], any automorphism of  $G_0$  leaves invariant some maximal torus and some Weyl chamber in  $G_0$ . Hence, any element  $g \in G$  is contained in  $N_G(T', C')$  for some maximal torus  $T'$  and some Weyl chamber  $C' \subseteq T'$ ; and  $T'$  and  $T$  are conjugate in  $G_0$  (cf. [Ad, Corollary 4.23]). Since  $N_{G_0}(T)/T$  permutes the Weyl chambers for  $T$  transitively, there is  $a \in G_0$  such that  $T = aT'a^{-1}$  and  $C = aC'a^{-1}$ ; and  $aga^{-1} \in N = N_G(T, C)$ . □

When dealing with real representations, we need to distinguish between the different types of irreducible representations and characters. As usual, we say that a  $G$ -representation  $V$  (over  $\mathbb{C}$ ) has *real type* if it has the form  $V \cong \mathbb{C} \otimes_{\mathbb{R}} V'$  for some  $\mathbb{R}G$ -representation  $V'$ ; and that  $V$  has *quaternion type* if it is the restriction of an  $\mathbb{H}G$ -representation. If  $V$  is irreducible and its character is real-valued, then  $V$  has real or quaternion type, but not both [Ad, Proposition 3.56]. By a real character will be meant the character of a virtual representation of real type (i.e., the difference of two representations of real type).

**Proposition 1.2.** *Fix a maximal torus  $T \subseteq G$  and a Weyl chamber  $C \subseteq \mathfrak{t}$ . Then a continuous class function  $f : G \rightarrow \mathbb{C}$  is a character of  $G$  if and only if  $f|_{N_G(T, C)}$  is a character of  $N_G(T, C)$ . And a continuous class function  $f : G \rightarrow \mathbb{R}$  is a real character of  $G$  if and only if  $f|_{N_G(T, \pm C)}$  is a real character.*

The proof of Proposition 1.2 will be given after that of Proposition 1.4 below. We first note some elementary conditions for  $f$  to be a character or a real character. In the following lemma, we write as usual  $\langle \varphi, \psi \rangle = \int_G \varphi(g) \overline{\psi(g)}$  for any pair of continuous functions  $\varphi, \psi : G \rightarrow \mathbb{C}$  (where the integral is the Haar integral on  $G$  with measure 1).

**Lemma 1.3.** (a) *A class function  $f \in \text{Cl}(G)$  is a character of  $G$  if and only if  $\langle f, \chi \rangle \in \mathbb{Z}$  for each character  $\chi$  of  $G$ .*

(b) *A class function  $f : G \rightarrow \mathbb{R}$  is a real character of  $G$  if and only if  $f$  is a character, and  $\langle f, \chi_V \rangle \in 2\mathbb{Z}$  for each  $G$ -representation  $V$  of quaternion type.*

(c) *A class function  $f : G \rightarrow \mathbb{R}$  is a real character of  $G$  if  $f$  is a character, and  $f|_H$  is a real character of  $H$  for some  $H \triangleleft G$  of finite odd index.*

*Proof.* For any pair  $W, V$  of complex  $G$ -representations,

$$\langle \chi_W, \chi_V \rangle = \dim_{\mathbb{C}}((W^* \otimes_{\mathbb{C}} V)^G) = \dim_{\mathbb{C}}(\text{Hom}_{\mathbb{C}G}(W, V)) \in \mathbb{Z}.$$

(Recall that  $\chi_{W^*}(g) = \overline{\chi_W(g)}$  for all  $g \in G$ .) Also, if  $W$  has real type and  $V$

has quaternion type, then  $\text{Hom}_{\mathbb{C}G}(W, V)$  is a quaternion vector space, and so its complex dimension is even. This proves the “only if” parts of points (a) and (b).

Conversely, assume that  $f \in \text{Cl}(G)$  is such that  $\langle f, \chi \rangle \in \mathbb{Z}$  for each character  $\chi$  of  $G$ . Since the irreducible characters form an orthonormal set, we know that  $\langle f, f \rangle \geq \sum_{i=1}^k \langle f, \chi_i \rangle^2$  for any set  $\chi_1, \dots, \chi_k$  of distinct irreducible characters. Since each  $\langle f, \chi \rangle \in \mathbb{Z}$ , this shows that  $\langle f, \chi \rangle = 0$  for all but finitely many irreducible characters  $\chi$ ; and so  $f = \sum_{\chi} \langle f, \chi \rangle \cdot \chi$  is a character of  $G$  by the Peter-Weyl theorem (cf. [Ad, Theorem 3.47]).

We now consider conditions for a real valued character to be a real character; or equivalently for a self-adjoint representation to be of real type. An irreducible  $G$ -representation (over  $\mathbb{C}$ ) is of *complex type* if its character is not real valued; i.e., if  $V \not\cong V^*$ . It follows from [Ad, Theorem 3.57] that a  $G$ -representation  $V$  (over  $\mathbb{C}$ ) is of real type if and only if it is a sum of irreducible representations of real type, of representations  $\mathbb{C} \otimes_{\mathbb{R}} W \cong W \oplus W^*$  for  $W$  irreducible of complex type, and of representations  $\mathbb{C} \otimes_{\mathbb{R}} W \cong W \oplus W$  for  $W$  irreducible of quaternion type. If  $v = \sum_{i=1}^k n_i [V_i] \in \text{R}(G)$  has real valued character, where the  $V_i$  are distinct irreducible  $G$ -representations, then  $\sum_{i=1}^k n_i [V_i] = \sum_{i=1}^k n_i [(V_i)^*]$ , and so each pair  $V_i, (V_i)^*$  occurs with the same multiplicity. Hence  $v$  has real type if  $2|n_i$  for each  $i$  such that  $V_i$  has quaternion type. Since  $n_i = \langle \chi_v, \chi_{V_i} \rangle$ , this proves point (b).

It remains to prove point (c): that an element  $v \in \text{R}(G)$  with real valued character has real type if  $v|H$  has real type for some normal subgroup  $H \triangleleft G$  of finite odd index; we may assume that  $v$  is the class of an actual  $\mathbb{C}[G]$ -representation  $V$ . Since all irreducible  $\mathbb{C}[G/H]$ -representations, aside from the trivial one, have complex type (cf. [Ser, Exercise 13.12]), we can write  $\mathbb{C}[G/H] \cong \mathbb{C} \oplus W \oplus W^*$  for some representation  $W$ . Since by assumption,  $V|H$  has real type and  $V^* \cong V$ , the isomorphism

$$\text{Ind}_H^G(V|H) \cong \mathbb{C}[G/H] \otimes_{\mathbb{C}} V \cong V \oplus (W \otimes_{\mathbb{C}} V) \oplus (W \otimes_{\mathbb{C}} V)^* \cong V \oplus \mathbb{C} \otimes_{\mathbb{R}} (W \otimes_{\mathbb{C}} V)$$

shows that  $V$  has real type.  $\square$

By a *weight* of the compact Lie group  $G$  is meant an element of the lattice  $T^* \subseteq \mathfrak{t}^*$ , regarded as an irreducible character of  $T$ . If  $V$  is any representation of  $G$ , then the set of “weights of  $V$ ” is defined to be the set of characters of irreducible components of  $V|T$ . Consider the partial ordering of the weights of  $G$ , where  $\phi_1 \leq \phi_2$  if  $\phi_1$  is contained in the convex hull of the  $W_G$ -orbit of  $\phi_2$  (cf. [Ad, Definition 6.23]). One of the basic theorems of representation theory says that if  $G$  is *connected*, then any irreducible  $G$ -representation  $V$  has a unique  $W_G$ -orbit of highest (maximal) weights, each of which occurs with multiplicity one. Furthermore, distinct irreducible representations have distinct orbits of higher weights, and every weight of  $G$  can be realized as the highest weight of some irreducible

$G$ -representation. Thus, the irreducible representations of any connected  $G$  are in one-to-one correspondence with the  $W_G$ -orbits of weights of  $G$ . And since any given dual Weyl chamber  $C^* \subseteq \mathfrak{t}^*$  contains exactly one element in each  $W_G$  orbit in  $\mathfrak{t}^*$  (cf. [Ad, Corollary 5.16]), the irreducible representations of  $G$  are in one-to-one correspondence with the weights in  $C^*$ . For more detail, see, e.g., [Ad, Theorem 6.33] or [BtD, Section VI.2].

Now assume that  $G$  is not connected. If  $V$  is an irreducible  $G$ -representation, and if  $V_0$  is any irreducible component of  $V|_{G_0}$ , then  $V$  is an irreducible summand of  $\text{Ind}_{G_0}^G(V_0)$ . Hence each irreducible summand of  $V|_{G_0}$  is obtained from  $V_0$  by conjugation by some element of  $\pi_0(G)$ ; and there is still a uniquely defined  $W_G$ -orbit of highest weights for  $V$ . In this case, however, the highest weights can occur with multiplicity greater than one; and there can be several irreducible  $G$ -representations with the same orbit of highest weights.

In the next proposition,  $\text{Irr}(G)$  will denote the set of irreducible representations of  $G$ . Also, if  $N = N_G(T, C)$  (for any maximal torus  $T \subseteq G$  and any Weyl chamber  $C \subseteq \mathfrak{t}$ ), then  $\text{Irr}(N, C^*)$  denotes the set of irreducible representations of  $N$  whose weights all lie in the dual Weyl chamber  $C^*$  of  $C$ . For any  $V \in \text{Irr}(G)$ ,  $\text{mx}_{C^*}(V) \subseteq C^* \cap T^*$  denotes the set of those maximal weights of the irreducible summands of  $V|_{G_0}$  which lie in  $C^*$ . And for any  $N$ -invariant set of weights  $\Phi \subseteq T^*$ ,  $V\langle\Phi\rangle$  denotes the sum of all irreducible summands of  $V|_T$  with weights in  $\Phi$ , regarded as an  $N$ -representation.

**Proposition 1.4.** *Fix a maximal torus  $T \subseteq G$  and a Weyl chamber  $C \subseteq \mathfrak{t}$ , and set  $N = N_G(T, C)$ . For any irreducible  $G$ -representation  $V$ , the subspace  $V\langle\text{mx}_{C^*}(V)\rangle$  is always an irreducible summand of  $V|_N$  having multiplicity one. This induces a bijection*

$$\beta_G : \text{Irr}(G) \xrightarrow{\cong} \text{Irr}(N, C^*) \quad \text{defined by} \quad \beta_G([V]) = [V\langle\text{mx}_{C^*}(V)\rangle], \quad (1)$$

and an isomorphism

$$\bar{\beta}_G : \text{R}(G) \xrightarrow{\cong} \text{R}(N, C^*) \quad \text{defined by} \quad \bar{\beta}_G([V]) = [V\langle C^*\rangle]. \quad (2)$$

*Proof.* Fix an irreducible  $G_0$ -representation  $V_0$ , and let  $\phi$  be the maximal weight of  $V_0$  lying in  $C^*$ . Set  $\Phi = (N/T)\cdot\phi \subseteq C^*$ , the  $N/T$ -orbit of  $\phi$ . Let  $(V_0) \subseteq \text{Irr}(G_0)$  denote the  $G/G_0$ -orbit of  $V_0$ , and let  $\text{Irr}(G, (V_0))$  denote the set of all irreducible  $G$ -representations with support in  $(V_0)$ ; i.e., the set of those irreducible  $G$ -representations  $V$  such that all irreducible summands of  $V|_{G_0}$  lie in  $(V_0)$ .

Let  $V_\phi$  denote the (1-dimensional) irreducible representation with weight (character)  $\phi$ ; regarded as a subspace of  $V_0$ . Since  $G/G_0 \cong N/T$ , the uniqueness of maximal weights in  $C^*$  shows that each irreducible component of  $(\text{Ind}_{G_0}^G(V_0))|_{G_0}$  contains exactly one weight in  $\Phi = (N/T)\cdot\phi$  (and with multiplicity one). Thus,

$$V_0\langle\Phi\rangle = V_\phi \quad \text{and} \quad \text{Ind}_{G_0}^G(V_0)\langle\Phi\rangle = \text{Ind}_T^N(V_\phi). \quad (3)$$

So for any  $G$ -representation  $V'$  with support in  $(V_0)$  (i.e., for any  $[V'] \in \text{Irr}(G, (V_0))$ ), there is a commutative diagram

$$\begin{array}{ccc}
 \text{Hom}_{G_0}(V_0, V') & \xrightarrow[r_1]{\cong} & \text{Hom}_T(V_\phi, V'\langle\Phi\rangle) \\
 F_1 \downarrow \cong & & F_2 \downarrow \cong \\
 \text{Hom}_G(\text{Ind}_{G_0}^G(V_0), V') & \xrightarrow{r_2} & \text{Hom}_N(\text{Ind}_T^N(V_\phi), V'\langle\Phi\rangle)
 \end{array} \tag{4}$$

where  $F_1$  and  $F_2$  are the Frobenius reciprocity isomorphisms, and  $r_1$  and  $r_2$  are defined by restriction to summands with weights in  $\Phi$ . The one-to-one correspondence between irreducible  $G_0$ -representations and highest weights contained in  $C^*$  shows that  $r_1$  is an isomorphism, and thus that  $r_2$  is also an isomorphism.

Now assume that  $V$  and  $V'$  are two irreducible  $G$ -representations with support in  $(V_0)$ . By Frobenius reciprocity again ( $\text{Hom}_G(\text{Ind}_{G_0}^G(V_0), V) \cong \text{Hom}_{G_0}(V_0, V) \neq 0$ ),  $V$  is a summand of  $\text{Ind}_{G_0}^G(V_0)$ . So by (3) and (4), for any  $[V], [V'] \in \text{Irr}(G, (V_0))$ ,

$$\text{Hom}_N(V\langle\Phi\rangle, V'\langle\Phi\rangle) \cong \text{Hom}_G(V, V') \cong \begin{cases} \mathbb{C} & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

This shows that  $V\langle\Phi\rangle$  is  $N$ -irreducible for any  $[V] \in \text{Irr}(G, (V_0))$ , and that  $V\langle\Phi\rangle \cong V'\langle\Phi\rangle$  if and only if  $V \cong V'$ . And finally, any irreducible  $N$ -representation with support in  $\Phi$  is a summand of  $\text{Ind}_T^N(V_\phi) \cong \text{Ind}_{G_0}^G(V_0)\langle\Phi\rangle$ , and hence has the form  $V\langle\Phi\rangle$  for some  $V \in \text{Irr}(G_0, (V_0))$ .

We have now shown that  $\beta_\Phi : \text{Irr}(G, (V_0)) \xrightarrow{\cong} \text{Irr}(N, \Phi)$ , defined by setting  $\beta_\Phi([V]) = [V\langle\Phi\rangle]$ , is a well defined bijection. Since the restriction to  $G_0$  of any irreducible  $G$ -representation is a sum of representations in just one  $G/G_0$ -orbit of irreducible  $G_0$ -representations,  $\beta_G : \text{Irr}(G) \rightarrow \text{Irr}(N, C^*)$  is the disjoint union of the  $\beta_\Phi$  taken over all  $N/T$ -orbits  $\Phi \subseteq (C^* \cap T^*)$  and hence also a bijection. This proves point (1). At the same time, it shows that the homomorphism  $\bar{\beta}_G : \text{R}(G) \rightarrow \text{R}(N, C^*)$  of (2) is an isomorphism, since its matrix with respect to the bases of irreducible representations is triangular with 1's along the diagonal.  $\square$

We are now ready to prove that a class function is a (real) character if its restriction to  $N_G(T, C)$  ( $N_G(T, \pm C)$ ) is a (real) character.

*Proof of 1.2. Complex case:* Fix a continuous class function  $f : G \rightarrow \mathbb{C}$  such that  $f|N$  is a character of  $N$ . We must show that  $f$  is a character of  $G$ . Let  $v_0 \in \text{R}(N)$  be such that  $\chi_{v_0} = f|N$ , let  $\chi$  be the character of  $\bar{\beta}_G^{-1}(v_0\langle C^*\rangle) \in \text{R}(G)$  (Proposition 1.4(2)), and set  $f' = f - \chi$ . By construction,  $f'|N$  is the character of an element  $v \in \text{R}(N)$  such that  $v\langle C^*\rangle = 0$ . We will show that  $v = 0$ . It then follows that  $f' = 0$  (since every element of  $G$  is conjugate to an element of  $N$ ), and hence that  $f = \chi$  is a character of  $G$ .



Fix any  $\phi \in T^*$ , and let  $N_\phi \subseteq N$  denote the subgroup of elements fixing  $\phi$ . Choose any  $\psi \in \text{interior}(C^*)^N$  ( $N/T$  acts linearly on  $\mathfrak{t}^*$  and leaves the dual Weyl chamber  $C^*$  invariant). Then  $\phi + \mathbb{R}\psi$  is not contained in the wall of any dual Weyl chamber (since  $\psi$  is not); and so there is a dual Weyl chamber  $C_1^*$  such that  $\phi + \epsilon\psi \in \text{interior}(C_1^*)$  for small  $\epsilon > 0$ . Let  $w \in W_G$  be any element such that  $w(C_1^*) = C^*$  ( $W_{G_0}$  permutes the Weyl chambers transitively). Then  $w\phi \in C^*$ , since  $\phi \in C_1^*$ . Also, for any  $a \in N_\phi$ ,  $a(\psi) = \psi$  and  $a(\phi) = \phi$  by assumption, so  $a$  leaves  $C_1^*$  invariant, and hence  $waw^{-1}$  leaves  $C^* = w(C_1^*)$  invariant. Thus  $wN_\phi w^{-1} \subseteq N$ ; and so  $v\langle w\phi \rangle = 0 \in \mathbb{R}(wN_\phi w^{-1})$  since  $v\langle C^* \rangle = 0 \in \mathbb{R}(N)$ . Since  $\chi_v$  is constant on  $G$ -conjugacy classes (it is the restriction of a class function on  $G$ ), it now follows that  $v\langle \phi \rangle = 0 \in \mathbb{R}(N_\phi)$ .

Let  $\phi_1, \dots, \phi_k \in T^*$  be  $N/T$ -orbit representatives for the support of  $v$ , and write  $N_i = N_{\phi_i}$  (the subgroup of elements which fix  $\phi_i$ ). Then  $v = \sum_{i=1}^k \text{Ind}_{N_i}^N(v\langle \phi_i \rangle)$ . We have just seen that  $v\langle \phi_i \rangle = 0 \in \mathbb{R}(N_i)$  for each  $i$ , and hence  $v = 0$ .

*Real case:* Write  $N_\pm = N_G(T, \pm C)$ , for short. Fix a class function  $f : G \rightarrow \mathbb{C}$  such that  $f|_{N_\pm}$  is a real character. Then  $f$  is a character by the above, and  $f(G) \subseteq \mathbb{R}$  since any element of  $G$  is conjugate to an element of  $N \subseteq N_\pm$  (Proposition 1.1). By Lemma 1.3(b), we can assume (after replacing  $f$  by its sum with an appropriate real character) that  $f = \chi_V$ , where  $V = \sum_{i=1}^k V_i$ , the  $V_i$  are distinct irreducible  $G$ -representations of quaternion type, and  $V|_{N_\pm}$  is a representation of real type. We claim that  $V = 0$  (i.e., that  $k = 0$ ).

Assume otherwise: that  $k > 0$ . Choose a  $W_G$ -orbit  $\Psi$  of maximal weights in one of the  $V_i$  — say  $V_1$  — which does not occur in any of the others except possibly as maximal weights. Set  $\Phi = \Psi \cap C^*$  and  $\Phi_\pm = \Psi \cap (\pm C^*)$ . By Proposition 1.4 (and the original assumption on  $\Psi$ ),  $V_1\langle \Phi \rangle$  is irreducible as an  $N$ -representation, and does not occur as a summand of  $V_i|_N$  for any  $i \neq 1$ . So the  $N_\pm$ -representation  $V'_1 \stackrel{\text{def}}{=} V_1\langle \Phi_\pm \rangle$  is irreducible — since

$$V'_1|_N \cong V_1\langle \Phi \rangle \oplus V_1\langle \Phi_\pm \setminus \Phi \rangle$$

— and  $V'_1$  does not occur as a summand of  $V_i|_{N_\pm}$  for any  $i \neq 1$ . Also, since  $V_1$  is self-conjugate, the elements of  $\Psi$ , and hence of  $\Phi_\pm$ , occur in pairs  $\pm\phi$ . This shows that  $V'_1 = V_1\langle \Phi_\pm \rangle$  is invariant under the conjugate linear automorphism  $j : V_1 \rightarrow V_1$ , and hence that it also has quaternion type. Thus,  $V|_{N_\pm}$  contains with multiplicity one the irreducible summand  $V'_1$  of quaternion type, and this contradicts the assumption that  $V|_{N_\pm}$  is a representation of real type.  $\square$

It remains to extend this criterium to a result which detects characters by restriction to finite subgroups of  $G$ . As usual, a finite group is called elementary if it is the product of a  $p$ -group (for some prime  $p$ ) and a cyclic group. A finite group  $G$  is called  $\mathbb{R}$ -elementary if it is elementary, or if it contains a normal cyclic subgroup  $C \triangleleft G$  of 2-power index with the property that for any  $g \in G$ , conjugation by  $g$  acts on  $C$  via the identity or via  $(x \mapsto x^{-1})$ .

**Proposition 1.5.** *For any class function  $f : G \rightarrow \mathbb{C}$ ,  $f$  is a character of  $G$  if and only if its restriction to any finite elementary subgroup of  $G$  is a character; and  $f$  is a real character of  $G$  if and only if its restriction to each finite  $\mathbb{R}$ -elementary subgroup of  $G$  is a real character.*

*Proof.* When  $G$  is finite, the proposition holds by the classical Brauer theorems for detecting characters of finite groups (cf. [Ser, Theorem 21 and Proposition 36]). So it will suffice to show that  $f$  is a (real) character of  $G$  if and only if its restrictions to all finite subgroups of  $G$  are (real) characters. By Proposition 1.2, it suffices to prove this when the connected component  $G_0$  of  $G$  is a torus.

Assume now that  $G_0 = T$  is a torus. We can choose a sequence  $H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$  of subgroups of  $G$  such that each  $H_i$  intersects all connected components of  $G$ , and such that the union of the  $H_i$  is dense in  $G$ . The simplest way to see this is to set  $n = |G/T|$ , let  ${}_nT \subseteq T$  denote the  $n$ -torsion subgroup, and note that the homomorphism  $H^2(G/T; {}_nT) \rightarrow H^2(G/T; T)$  is surjective since  $n \cdot H^2(G/T; T) = 0$ . Hence there is a subgroup  $H_0 \subseteq G$  such that  $H_0 \cap T = {}_nT$  and  $\langle H_0, T \rangle = G$ ; and we can define  $H_k = \langle H_0, {}_{n \cdot 2^k}T \rangle$  for each  $k > 0$ .

Let  $f \in \text{Cl}(G)$  be any class function whose restriction to each  $H_i$  is a character. For each character  $\chi$  of  $G$ ,

$$\langle f, \chi \rangle_G \stackrel{\text{def}}{=} \int_G f \cdot \bar{\chi} = \lim_{i \rightarrow \infty} \left( \frac{1}{|H_i|} \sum_{g \in H_i} f(g) \cdot \overline{\chi(g)} \right) = \lim_{i \rightarrow \infty} \langle f, \chi \rangle_{H_i}$$

(by definition of the Riemann integral); and  $\langle f, \chi \rangle_{H_i} \in \mathbb{Z}$  for each  $i$  since  $f|_{H_i}$  is a character of  $H_i$ . Thus,  $\langle f, \chi \rangle_G \in \mathbb{Z}$  for each  $\chi$ , and so  $f$  is a character of  $G$  by Lemma 1.3(a). And if  $f|_{H_i}$  is a real character for each  $i$ , then  $f$  is real valued (the union of the  $H_i$  being dense in  $G$ ),  $\langle f, \chi \rangle_G = \lim_{i \rightarrow \infty} \langle f, \chi \rangle_{H_i} \in 2\mathbb{Z}$  for each character  $\chi$  of quaternion type by Lemma 1.3(b), and so  $f$  is a real character by Lemma 1.3(b). □

## Section 2. Induction for representations of compact Lie groups

Again, throughout the section,  $G$  denotes a fixed compact Lie group. We construct an induction homomorphism  $R(H) \rightarrow R(G)$ , for an arbitrary closed subgroup  $H \subseteq G$ , by first defining it between the groups of class functions, and then using the results of Section 1 to show that it sends characters to characters.

The following lemma is useful for constructing continuous functions on  $G$ , and on certain closed subsets of  $G$ .

**Lemma 2.1.** *Let  $\mathcal{F}$  be any set of closed subgroups of  $G$ , closed under conjugation and closed in the space of all subgroups (with the Hausdorff topology). Set  $G_{\mathcal{F}} = \cup_{H \in \mathcal{F}} H$ : the union of the subgroups in  $\mathcal{F}$ . Then for any function  $f : G_{\mathcal{F}} \rightarrow$*

$\mathbb{C}$  invariant under conjugation,  $f$  is continuous on  $G_{\mathcal{F}}$  if  $f|H$  is continuous for all  $H \in \mathcal{F}$ .

*Proof.* Fix any conjugation invariant function  $f : G_{\mathcal{F}} \rightarrow \mathbb{C}$  such that  $f|H$  is continuous for all  $H \in \mathcal{F}$ . It will suffice to show, for any sequence  $g_i \rightarrow g$  in  $G_{\mathcal{F}}$ , that some subsequence of the  $f(g_i)$  converges to  $f(g)$ . Since if  $f$  is not continuous at  $g$ , then there is  $\epsilon > 0$  and a sequence  $\{g_i\}$  in  $G_{\mathcal{F}}$  converging to  $g$  such that  $|f(g_i) - f(g)| > \epsilon$  for all  $i$ .

Fix such  $g_i$  and  $g$ ; and for each  $i$  choose  $H_i \in \mathcal{F}$  such that  $g_i \in H_i$ . Since  $\mathcal{F}$  is closed in the space of closed subgroups of  $G$ , and since this space is compact (cf. [tD, Proposition IV.3.2(i)]), we can replace the  $g_i$  by a subsequence and assume that the  $H_i$  converge to some subgroup  $H \in \mathcal{F}$ . By [tD, Theorem I.5.9], there exist elements  $a_i \rightarrow e$  such that  $a_i H_i a_i^{-1} \subseteq H$  for  $i$  sufficiently large. And hence

$$\lim_{i \rightarrow \infty} f(g_i) = \lim_{i \rightarrow \infty} f(a_i g_i a_i^{-1}) = f(g)$$

since  $f|H$  is continuous. □

The next lemma is also rather technical, and will be used later to show that the induction homomorphism we define for class functions is well defined.

**Lemma 2.2.** *Fix a closed subgroup  $H \subseteq G$  and an element  $g \in G$ , and let  $(G/H)^g$  be the fixed point set of the action of  $g$  on  $G/H$ . A coset  $aH \in G/H$  lies in  $(G/H)^g$  if and only if  $a^{-1}ga \in H$ . And if  $a_1H$  and  $a_2H$  lie in the same connected component of  $(G/H)^g$ , then  $a_2H = xa_1H$  for some  $x \in C_G(g)$ . In particular, in this situation,  $a_1^{-1}ga_1$  is conjugate in  $H$  to  $a_2^{-1}ga_2$ .*

*Proof.* For any  $a \in G$ ,  $aH \in (G/H)^g$  if and only if  $gaH = aH$ , if and only if  $a^{-1}ga \in H$ . Also, if  $a_2H = xa_1H$  for any  $a_1, a_2 \in G$  and any  $x \in C_G(g)$ , then  $a_1^{-1}ga_1$  and  $a_2^{-1}ga_2$  are conjugate by an element of  $H$ .

Now fix an element  $aH \in (G/H)^g$ . Let  $C_G(g)_0$  be the identity connected component of the centralizer of  $g$ . We must show that the connected component of  $aH$  in  $(G/H)^g$  is  $C_G(g)_0 \cdot aH$ . Equivalently, via translation by  $a^{-1}$ , we must show that the connected component of  $eH$  in  $(G/H)^{a^{-1}ga}$  is  $C_G(a^{-1}ga) \cdot eH$ . So upon replacing  $a^{-1}ga$  by  $g$ , we are reduced to the case where  $a = e$  and  $g \in H$ .

Let  $\mathfrak{h} \subseteq \mathfrak{g}$  denote the Lie algebras of  $H \subseteq G$ . For all  $x \in G$ ,  $xH \in (G/H)^g$  if and only if  $xH = gxH = gxg^{-1}H$ . In particular,  $C_G(g) \cdot H \subseteq (G/H)^g$ ; and the tangent plane at  $eH$  to the manifold  $(G/H)^g$  is  $(\mathfrak{g}/\mathfrak{h})^{\text{Ad}(g)}$  (the fixed point set of the adjoint action of  $g$  on  $\mathfrak{g}/\mathfrak{h}$ ). Also, the projection of  $\mathfrak{g}$  onto  $\mathfrak{g}/\mathfrak{h}$  is split, equivariantly with respect to the action of the compact group  $H$ , and so  $\mathfrak{g}^{\text{Ad}(g)}$  surjects onto  $(\mathfrak{g}/\mathfrak{h})^{\text{Ad}(g)}$ . Since  $\mathfrak{g}^{\text{Ad}(g)}$  is the Lie algebra of  $C_G(g)$ , this shows that the two submanifolds  $C_G(g)_0 \cdot H \subseteq (G/H)^g$  have the same dimension, and hence that  $C_G(g)_0 \cdot H$  is the connected component of  $eH$  in  $(G/H)^g$ . □

We can now define the induction homomorphism for class functions, motivated by the formula given by Segal [Seg, p. 119].

**Proposition 2.3.** *Let  $H \subseteq G$  be any closed subgroup. Then there is a homomorphism*

$$\text{Ind}_H^G : \text{Cl}(H) \longrightarrow \text{Cl}(G)$$

*determined (uniquely) by the following formula. Fix any  $g \in G$ , let  $F_1, \dots, F_k$  be the connected components of  $(G/H)^g$ , and choose elements  $a_i H \in F_i$ . Then for any  $f \in \text{Cl}(H)$ ,*

$$\text{Ind}_H^G(f)(g) = \sum_{i=1}^k \chi(F_i) \cdot f(a_i^{-1} g a_i). \tag{1}$$

*Proof.* Fix any  $f \in \text{Cl}(H)$ . By Lemma 2.2, for each  $g \in G$ ,  $\text{Ind}_H^G(f)(g)$  is independent of the choice of representatives  $a_i H$  for the components of  $(G/H)^g$ . Also,  $\text{Ind}_H^G(f)$  is conjugation invariant by definition; and it only remains to check that it is continuous.

Let  $\mathcal{F}$  be the family of abelian subgroups of  $G$ . Clearly,  $\mathcal{F}$  is closed in the Hausdorff topology, and its union is all of  $G$ . By Lemma 2.1, it will suffice to show that  $f|A$  is continuous for each  $A \in \mathcal{F}$ . Let  $X$  be a connected component of some subgroup  $A \in \mathcal{F}$ ; we can assume that  $X$  generates  $\pi_0(A)$ . For any  $g \in X$ ,  $A/\langle g \rangle$  is connected (where  $\langle g \rangle$  is the closure of the subgroup generated by  $g$ ); and hence is a torus (or trivial). If  $(G/H)^g = \coprod_{i=1}^k F_i$ , where the  $F_i$  are connected components, then  $(G/H)^A = \coprod_{i=1}^k (F_i)^{A/\langle g \rangle}$  and  $\chi((F_i)^{A/\langle g \rangle}) = \chi(F_i)$  for each  $i$ . Thus, if we write  $(G/H)^A = \coprod_{j=1}^m E_j$  (where the  $E_j$  are the connected components), and choose elements  $b_j H \in E_j$ , then

$$\text{Ind}_H^G(f)(g) = \sum_{j=1}^m \chi(E_j) \cdot f(b_j^{-1} g b_j).$$

This formula holds for all  $g \in X$ , and shows that  $\text{Ind}_H^G(f)$  is continuous on  $X$ .  $\square$

The following double coset formula for induction and restriction of class functions is analogous to that shown by Feshbach [Fe] for equivariant cohomology theories. It was shown for representations by Snaith [Sn, Theorem 2.4], using Segal's definition. We prove it here for class functions, using directly the definition in Proposition 2.3.

**Lemma 2.4.** *Fix closed subgroups  $H, K \subseteq G$ , and write*

$$K \backslash G / H = \coprod_{i=1}^k U_i$$

where each  $U_i$  is a connected component of one orbit type for the action of  $K$  on  $G/H$ . Fix elements  $a_1, \dots, a_k \in G$  such that  $Ka_iH \in U_i$ . For each  $i$ , let  $\varphi_i : \text{Cl}(H) \rightarrow \text{Cl}(K)$  denote the composite

$$\varphi_i : \text{Cl}(H) \xrightarrow{\text{Res}} \text{Cl}(a_i^{-1}Ka_i \cap H) \xrightarrow{\text{conj}(a_i^{-1})} \text{Cl}(K \cap a_iHa_i^{-1}) \xrightarrow{\text{Ind}} \text{Cl}(K).$$

Then, as functions from  $\text{Cl}(H)$  to  $\text{Cl}(K)$ ,

$$\text{Res}_K^G \circ \text{Ind}_H^G = \sum_{i=1}^k \chi^\sharp(U_i) \cdot \varphi_i; \tag{1}$$

where for each  $i$ ,

$$\chi^\sharp(U_i) = \chi(\overline{U_i}, \overline{U_i} \setminus U_i) = \chi(\overline{U_i}) - \chi(\overline{U_i} \setminus U_i).$$

*Proof.* Fix elements  $f \in \text{Cl}(H)$  and  $g \in K$ . We will compare the two maps in (1), when evaluated on a given class function  $f$  and a given element  $g$ .

Let  $\tilde{U}_i \subseteq G/H$  denote the inverse image of  $U_i$  under the projection to  $K \backslash G/H$ . Let  $F_1, \dots, F_m$  be the connected components of  $(G/H)^g$ . Thus,  $G/H = \coprod_{i=1}^k \tilde{U}_i$  and  $(G/H)^g = \coprod_{j=1}^m F_j$ . For each  $i, j$ , set

$$V_{ij} = (K \cdot a_iH) \cap F_j \subseteq \tilde{U}_i \cap F_j \subseteq G/H$$

(note that  $V_{ij}$  need not be connected). Then the  $V_{ij} \rightarrow \tilde{U}_i \cap F_j \rightarrow U_i$  are fibration sequences, and so

$$\chi(F_j) = \sum_{i=1}^k \chi^\sharp(\tilde{U}_i \cap F_j) = \sum_{i=1}^k \chi(V_{ij}) \cdot \chi^\sharp(U_i).$$

for each  $j$ . Fix elements  $b_{ij} \in K$ , for each  $i, j$ , such that  $b_{ij}a_iH \in V_{ij}$ . Then by definition of the induction map (and Lemma 2.1),

$$\begin{aligned} (\text{Res}_K^G \circ \text{Ind}_H^G)(f)(g) &= \text{Ind}_H^G(f)(g) = \sum_{i=1}^k \sum_{j=1}^m \chi(V_{ij}) \cdot \chi^\sharp(U_i) \cdot f(a_i^{-1}b_{ij}^{-1}gb_{ij}a_i) \\ &= \sum_{i=1}^k \chi^\sharp(U_i) \cdot \sum_{j=1}^m \chi(V_{ij}) \cdot (f \circ \text{conj}(a_i^{-1}))(b_{ij}^{-1}gb_{ij}). \end{aligned}$$

And for each  $i$ , if we set  $K_i = K \cap a_iHa_i^{-1}$  (the isotropy subgroup of the action of  $K$  on  $a_iH \in G/H$ ), then  $(K/K_i)^g \cong (K \cdot a_iH)^g = \coprod_{j=1}^m V_{ij} (\subseteq G/H)$ ; and so

$$\sum_{j=1}^m \chi(V_{ij}) \cdot (f \circ \text{conj}(a_i^{-1}))(b_{ij}^{-1}gb_{ij}) = \text{Ind}_{K_i}^K(f \circ \text{conj}(a_i^{-1}))(g) = \varphi_i(f)(g).$$

□

When  $G$  is finite, the formula given in Proposition 2.3 is just the usual formula for the induction of characters (cf. [Ser, Theorem 12]). Hence by the double coset formula in Lemma 2.4, for each character (real character)  $\chi$  of  $H$  and each finite subgroup  $K \subseteq G$ ,  $(\text{Ind}_H^G(\chi))|_K$  is a character (or real character) of  $K$ . The detection result of Proposition 1.5 now applies to show:

**Theorem 2.5.** *The homomorphism  $\text{Ind}_H^G$  of Proposition 2.3 sends characters to characters, and sends real characters to real characters. It thus restricts to homomorphisms*

$$\text{Ind}_H^G : \mathbb{R}(H) \longrightarrow \mathbb{R}(G) \quad \text{and} \quad \text{Ind}_H^G : \text{RO}(H) \longrightarrow \text{RO}(G). \quad \square$$

These induction homomorphisms are in fact functorial; i.e., they compose in the expected way.

**Lemma 2.6.** *For any closed subgroups  $K \subseteq H \subseteq G$ ,*

$$\text{Ind}_K^G = \text{Ind}_H^G \circ \text{Ind}_K^H : \text{Cl}(K) \longrightarrow \text{Cl}(G),$$

and hence

$$\text{Ind}_K^G = \text{Ind}_H^G \circ \text{Ind}_K^H : \mathbb{R}(K) \longrightarrow \mathbb{R}(G).$$

*Proof.* Fix any element  $g \in G$ , and consider the projection  $(G/K)^g \xrightarrow{\text{pr}} (G/H)^g$ . For any  $aH \in (G/H)^g$ ,

$$\text{pr}^{-1}(aH) = \{ahK \mid h \in H, h^{-1}(a^{-1}ga)h \in K\} = a \cdot (H/K)^{a^{-1}ga}.$$

If  $aH$  and  $a'H$  lie in the same connected component of  $(G/H)^g$ , then  $a'H = xaH$  for some  $x \in C_G(g)$  (Lemma 2.2), and so  $\text{pr}^{-1}(a'H) = x \cdot \text{pr}^{-1}(aH)$ . It follows that  $\text{pr}$  is a fibration (fiber bundle) over each connected component of  $(G/H)^g$ . The result now follows from the definition of the induction homomorphisms (Proposition 2.3), together with the multiplicativity of Euler characteristics in a fibration.  $\square$

We leave it as an exercise to check that this induction homomorphism is the same as that defined by Segal in [Seg] (use the formula given in [Seg, p. 119]).

It is not hard to prove Frobenius reciprocity for induction and restriction of representations, using the definition given here. And that in turn implies, for example, that the induction map  $\text{Ind}_{N(T)}^G : \mathbb{R}(N(T)) \rightarrow \mathbb{R}(G)$  is always surjective, and split by the restriction map. See also [Sn, Section 2.3] for the proofs of these results using Segal's definition of induction.

**Section 3. Representations supported by  $p$ -toral subgroups**

Again, throughout this section,  $G$  will be a fixed compact Lie group, and  $G_0$  will denote its identity connected component. Let  $\mathcal{S}_p(G)$  denote the family of  $p$ -toral subgroups of  $G$ , for all primes  $p$ . We now consider the groups

$$R_{\mathcal{P}}(G) = \varinjlim_{P \in \mathcal{S}_{\mathcal{P}}(G)} R(P) \quad \text{and} \quad RO_{\mathcal{P}}(G) = \varinjlim_{P \in \mathcal{S}_{\mathcal{P}}(G)} RO(P),$$

where the limits are taken with respect to inclusion and conjugation; and the natural “restriction” maps

$$rs_G^U : R(G) \longrightarrow R_{\mathcal{P}}(G) \quad \text{and} \quad rs_G^O : RO(G) \longrightarrow RO_{\mathcal{P}}(G).$$

These groups were shown in [JO] to be naturally isomorphic to the Grothendieck groups  $\mathbb{K}(BG)$  and  $\mathbb{K}\mathbb{O}(BG)$ , respectively, of vector bundles over  $BG$  (and  $rs_G^U$  and  $rs_G^O$  are isomorphic to the natural homomorphisms  $R(G) \rightarrow \mathbb{K}(BG)$  and  $RO(G) \rightarrow \mathbb{K}\mathbb{O}(BG)$ ).

The homomorphisms  $rs_G$  are shown here to split as a direct sums of homomorphisms between finitely generated groups, one for each  $G/G_0$ -orbit of irreducible  $G_0$ -representations, and the cokernel of each summand is computed (Theorem 3.9). In particular, this yields necessary and sufficient conditions for  $rs_G^U$  to be onto (Theorem 3.10 and Corollary 3.11). The orthogonal case seems to be much more complicated; but we do at least show that  $rs_G^O$  is onto whenever  $G$  is finite or  $\pi_0(G)$  has prime power order (Propositions 3.2 and 3.4), and then give some examples which show that  $rs_G^O$  can fail to be onto even when  $rs_G^U$  is onto.

It will be useful to define the “character” of an element of  $R_{\mathcal{P}}(G)$ . For any compact Lie group  $G$ , let  $G_{\mathcal{P}}$  denote the union of the connected components in  $G$  of prime power order in  $\pi_0(G)$ . Let  $Cl(G_{\mathcal{P}})$  denote the space of continuous functions  $G_{\mathcal{P}} \rightarrow \mathbb{C}$  invariant on conjugacy classes (i.e., the “class functions” on  $G_{\mathcal{P}}$ ).

**Lemma 3.1.** *There is a (unique) character homomorphism*

$$\chi : R_{\mathcal{P}}(G) \longrightarrow Cl(G_{\mathcal{P}}),$$

*such that for any  $v = (v_P)_{P \in \mathcal{S}_{\mathcal{P}}(G)} \in R_{\mathcal{P}}(G)$ ,  $\chi(v)|P = \chi_{v_P}$  for all  $P$  in  $\mathcal{S}_{\mathcal{P}}(G)$ . Also,  $\chi$  sends  $R_{\mathcal{P}}(G)$  ( $RO_{\mathcal{P}}(G)$ ) isomorphically to the subgroup of those class functions on  $G_{\mathcal{P}}$  whose restriction to each  $p$ -toral subgroup  $P \subseteq G$ , for all primes  $p$ , is a character of  $P$  (a real character of  $P$ ).*

*Proof.* Let  $\mathcal{F}$  be the set of  $p$ -toral subgroups of  $G$  (for all primes  $p$ ), whose identity connected component is a maximal torus of  $G$ . Clearly,  $\mathcal{F}$  is closed in the Hausdorff topology (note that for  $P \in \mathcal{F}$ , the order of  $\pi_0(P)$  is bounded by  $|N_G(T)/T|$ ). And by Proposition 1.1,  $G_{\mathcal{P}}$  is the union of the  $P \in \mathcal{F}$ .

Now, for any  $v = (v_P)_{P \in \mathcal{S}_P(G)} \in R_P(G)$ , define  $\chi(v) : G_P \rightarrow \mathbb{C}$  to be the union of the characters  $\chi_{v_P}$ . This is well defined, and invariant under conjugation, by definition of the inverse limit. Also,  $\chi(v)$  is continuous by Lemma 2.1, applied to the family  $\mathcal{F}$ ; and so  $\chi(v) \in \text{Cl}(G_P)$ .

The character homomorphism  $\chi$  is clearly a monomorphism, and the descriptions of the images of  $R_P(G)$  and  $\text{RO}_P(G)$  are immediate from the construction.  $\square$

We are now ready to study the groups  $R_P(G)$  and  $\text{RO}_P(G)$ , beginning with the following case.

**Proposition 3.2.** *If  $\pi_0(G)$  has prime power order, then*

$$rs_G^U : R(G) \xrightarrow{\cong} R_P(G) \quad \text{and} \quad rs_G^O : \text{RO}(G) \xrightarrow{\cong} \text{RO}_P(G)$$

are isomorphisms. Furthermore, for any  $G$ ,

$$R_P(G) \cong \varprojlim_{P \in \mathcal{F}_P(G)} R(P) \quad \text{and} \quad \text{RO}_P(G) \cong \varprojlim_{P \in \mathcal{F}_P(G)} \text{RO}(P); \quad (1)$$

where  $\mathcal{F}_P(G)$  denotes the family of subgroups  $H \subseteq G$  of finite index such that  $H/G_0$  has prime power order (and the limits are taken with respect to inclusion and conjugation).

*Proof.* If  $\pi_0(G)$  is a  $p$ -group for any prime  $p$ , then  $G = G_P$ , and so  $rs_G^U$  and  $rs_G^O$  are both monomorphisms by Lemma 3.1. To prove that they are isomorphisms, we must show that a class function  $f \in \text{Cl}(G)$  is a (real) character if its restriction to all  $p$ -toral subgroups of  $G$  is a (real) character.

Fix a maximal torus  $T$  and a Weyl chamber  $C \subseteq \mathfrak{t}$ , and set  $N = N_G(T, C)$ . Then  $N$  is  $p$ -toral by Proposition 1.1; and by Proposition 1.2 a class function  $f \in \text{Cl}(G)$  is a character of  $G$  if  $f|_N$  is a character of  $N$ . This shows that  $rs_G^U$  is an isomorphism. If  $\pi_0(G)$  is a 2-group, then  $N_G(T, \pm C)$  is 2-toral, and the same argument shows that  $rs_G^O$  is an isomorphism. Finally, if  $\pi_0(G)$  is a  $p$ -group for an odd prime  $p$ , then for any  $v \in \text{RO}_P(G)$ ,  $\chi(v)$  is a real valued character of  $G$  whose restriction to  $G_0$  is a real character of  $G_0$  (since  $\pi_0(G_0) = 1$  is a 2-group); and so  $\chi(v)$  is a real character of  $G$  by Lemma 1.3(c).

This finishes the proof of the first statement above. The formulas in (1) now follow immediately (by the transitivity of inverse limits).  $\square$

The importance of the formulas in (1) above is that they show that the groups  $R_P(G)$  and  $\text{RO}_P(G)$ , and also the maps  $rs_G^U$  and  $rs_G^O$ , split as sums of groups and maps indexed by the irreducible representations of the identity component  $G_0$ . This will be made more explicit in Theorem 3.9 below.



The next proposition describes how standard induction techniques apply to study  $R_{\mathcal{P}}(G)$  and  $rs_G^U$ . Recall that a finite group  $\Gamma$  is  $p$ -elementary if it is a product of a  $p$ -group and a cyclic group, and is elementary if it is  $p$ -elementary for some prime  $p$ . Also,  $\Gamma$  is  $2$ - $\mathbb{R}$ -elementary if it contains a normal cyclic subgroup  $C_m$  of  $2$ -power index such that any element of  $G$  either centralizes  $C_m$  or acts on it via  $(a \mapsto a^{-1})$ ; and is  $\mathbb{R}$ -elementary if it is elementary or  $2$ - $\mathbb{R}$ -elementary.

**Proposition 3.3.** (a) For any subgroup  $H \subseteq G$  of finite index, there is an induction homomorphism

$$\text{Ind}_H^G : R_{\mathcal{P}}(H) \longrightarrow R_{\mathcal{P}}(G),$$

with the property that for any  $v \in R_{\mathcal{P}}(H)$  and any  $g \in G_{\mathcal{P}}$ ,

$$\chi(\text{Ind}_H^G(v))(g) = (\text{Ind}_H^G(\chi(V)))(g) = \sum_{aH \in (G/H)^g} \chi(v)(a^{-1}ga). \tag{1}$$

(b) Let  $\mathcal{E}(G)$  and  $\mathcal{E}_{\mathbb{R}}(G)$  denote the sets of subgroups  $E \subseteq G$  of finite index such that  $E/G_0$  is elementary or  $\mathbb{R}$ -elementary, respectively. Then restriction induces isomorphisms

$$\text{Coker}(rs_G^U) \xrightarrow{\cong} \varinjlim_{E \in \mathcal{E}(G)} \text{Coker}(rs_E^U) \quad \text{and} \quad \text{Coker}(rs_G^O) \xrightarrow{\cong} \varinjlim_{E \in \mathcal{E}_{\mathbb{R}}(G)} \text{Coker}(rs_E^O);$$

where the limits are taken with respect to inclusion and conjugation in  $G$ .

*Proof.* We regard  $\text{Ind}_H^G$  as a homomorphism  $\text{Cl}(H_{\mathcal{P}}) \rightarrow \text{Cl}(G_{\mathcal{P}})$ , defined via formula (1). Note that this is just the restriction to  $G_{\mathcal{P}}$  of the formula given in Proposition 2.3 (though only in the case where  $[G:H] < \infty$ ). In particular, the double coset formula of Lemma 2.4 applies in this situation.

(a) Fix any  $v \in R_{\mathcal{P}}(H)$ , and let  $\chi = \chi(v) \in \text{Cl}(H_{\mathcal{P}})$  be its character. We must show that  $\text{Ind}_H^G(\chi)$  is the character of an element of  $R_{\mathcal{P}}(G)$ ; or equivalently (by Lemma 3.1) that  $\text{Ind}_H^G(\chi)|P$  is a character for all  $p$ -toral subgroups  $P \subseteq G$  (for all primes  $p$ ). And for any such  $P$ ,  $gPg^{-1} \cap H$  is  $p$ -toral for each  $g \in G$ , so  $\chi|(gPg^{-1} \cap H)$  is a character, and hence  $\text{Ind}_H^G(\chi)|P$  is a character of  $P$  by the double coset formula.

(b) Let  $\mathcal{F}(G)$  be the class of subgroups of  $G$  of finite index. The functor  $H \mapsto R(H/G_0)$  satisfies the double coset formula and Frobenius reciprocity relations for induction and restriction, and hence is a Green ring over  $\mathcal{F}(G)$  in the sense of Dress [Dr]. Also, the double coset formula of Lemma 2.4 says that  $H \mapsto R_{\mathcal{P}}(H)$  and  $H \mapsto \text{Coker}(rs_H^U)$  are both Mackey functors over  $\mathcal{F}(G)$  (again in the sense of Dress); and both are modules over  $R(-/G_0)$  satisfying Frobenius reciprocity. Since  $R(G/G_0)$  is generated by induction from the  $R(E/G_0)$  for  $E \in \mathcal{E}(G)$  [Ser, §10.5, Theorem 19], the “fundamental theorem” of Mackey functors and Green rings says that  $F(G) \cong \varinjlim_{E \in \mathcal{E}(G)} (F(E))$  for any such module over  $R(-/G_0)$ .

This is shown in [Dr, Propositions 1.1' and 1.2], and a more direct proof is given in [Ol, Theorem 11.1].

Similarly,  $H \mapsto \text{RO}_{\mathcal{P}}(H)$  and  $H \mapsto \text{Coker}(\text{rs}_H^{\text{O}})$  are Mackey functors over  $\mathcal{F}(G)$ , and modules over  $\text{RO}(-/G_0)$  satisfying Frobenius reciprocity. Since  $\text{RO}(G/G_0)$  is generated by induction from the  $\text{RO}(E/G_0)$  for  $E \in \mathcal{E}_{\mathbb{R}}(G)$  [Ser, §12.6, Theorem 27], the same argument applies to show that  $\text{Coker}(\text{rs}_G^{\text{O}}) \cong \varinjlim_{E \in \mathcal{E}_{\mathbb{R}}(G)} \text{Coker}(\text{rs}_E^{\text{O}})$ .  $\square$

In fact, the induction map  $\text{Ind}_H^G : \text{R}_{\mathcal{P}}(H) \rightarrow \text{R}_{\mathcal{P}}(G)$  is defined for any closed subgroup  $H \subseteq G$ , using the formula for induction of characters in Proposition 2.3. To see this, one must check, for any  $f \in \text{Cl}(H)$ , that  $\text{Ind}_H^G(f)|_{G_{\mathcal{P}}} = 0$  if  $f|_{H_{\mathcal{P}}} = 0$ . This would be immediate if we knew that  $H \cap G_{\mathcal{P}} \subseteq H_{\mathcal{P}}$ ; but that is not the case in general. The existence of the induction map is thus slightly more tricky than in the case where  $[G : H] < \infty$ , but is not difficult.

We now turn to the case of finite groups.

**Proposition 3.4.** *If  $G$  is finite, then  $\text{rs}_G^{\text{U}}$  and  $\text{rs}_G^{\text{O}}$  are both surjective.*

*Proof.* By Proposition 3.3(b), it suffices to show that  $\text{rs}_G^{\text{U}}$  is onto when  $G$  is elementary, and that  $\text{rs}_G^{\text{O}}$  is onto when  $G$  is  $\mathbb{R}$ -elementary. We do this in the orthogonal case only; the unitary case is similar (but simpler).

Assume that  $G$  is  $\mathbb{R}$ -elementary, and fix an element  $v = (v_P)_{P \in \mathcal{S}_{\mathcal{P}}(G)} \in \text{RO}_{\mathcal{P}}(G)$ . In other words,  $v_P \in \text{RO}(P)$  for each  $p$ -subgroup  $P \subseteq G$  (for each prime  $p| |G|$ ); and by subtracting a constant character we can assume that  $\chi_{v_P}(1) = 0$  for each  $P$ . For each  $p| |G|$ , write  $v_p = v_{\text{Syl}_p(G)} \in \text{RO}(\text{Syl}_p(G))$ . It will suffice to show that each  $v_p$  extends to an element  $v'_p \in \text{R}(G)$  whose character vanishes on all elements of order prime to  $p$  (then  $v = \text{rs}_G^{\text{O}}(\sum v'_p)$ ). This is clear if  $\text{Syl}_p(G)$  has a normal complement, since in that case  $v'_p$  can be taken to be the composite of  $v_p$  with a surjection  $G \twoheadrightarrow \text{Syl}_p(G)$ .

The only case left to consider is that where  $p$  is odd,  $G$  is 2- $\mathbb{R}$ -elementary, and  $\text{Syl}_p(G)$  has no normal complement. Set  $p^k = |\text{Syl}_p(G)|$ ; then there is a surjection  $G \twoheadrightarrow D(2p^k)$ , where  $D(2p^k)$  is dihedral of order  $2p^k$ . One easily checks that any  $v_p \in \text{RO}(C_{p^k})$  such that  $\chi_{v_p}(1) = 0$  extends to an element  $v''_p \in \text{RO}(D(2p^k))$  such that  $\chi_{v''_p}(g) = 0$  for all  $g$  of order prime to  $p$ . And hence if  $v'_p \in \text{RO}(G)$  is the composite of  $v''_p$  with the surjection  $G \twoheadrightarrow D(2p^k)$ , then  $v'_p|_{\text{Syl}_p(G)} = v_p$  and  $\chi_{v'_p}$  vanishes on all elements of order prime to  $p$ .  $\square$

Recall that for any torus  $T$ , we let  $\mathfrak{t}$  denote the Lie algebra of  $T$ , and regard the group  $T^* = \text{Hom}(T, S^1)$  of irreducible characters of  $T$  as a lattice in  $\mathfrak{t}^* = \text{Hom}(\mathfrak{t}, \mathbb{R})$ . The following definitions establish some of the notation which will be used when dealing with irreducible characters and representations of groups with torus identity component.

**Definition 3.5.** If  $G$  is a compact Lie group with identity component  $T$ , then the support of a  $G$ -representation  $V$  is the ( $G/T$ -invariant) subset  $\text{Supp}(V) \subseteq T^*$  of all characters of irreducible summands of  $V|T$ . More generally, for any  $v \in R(G)$ ,  $\text{Supp}(v) \in T^*$  is the union of the supports of the irreducible  $G$ -representations which occur in the decomposition of  $v$ . For any  $G/T$ -invariant subset  $\Phi \subseteq T^*$ ,  $\text{Irr}(G, \Phi)$  denotes the set of irreducible  $G$ -representations with support in  $\Phi$ , and  $R(G, \Phi) \subseteq R(G)$  denotes the subgroup of elements with support in  $\Phi$ . For  $\phi \in T^*$ , we write  $(\phi)$  for the  $G/T$ -orbit of  $\phi$  (and write  $\text{Irr}(G, \phi)$ , etc., if  $\phi$  is  $G/T$ -invariant). Finally, if  $V$  is any  $G$ -representation, then  $V\langle\Phi\rangle$  and  $V\langle\phi\rangle$  denote the largest summands of  $V$  with support in  $\Phi$  or  $\phi$ , respectively.

The descriptions of  $\text{Coker}(\text{rs}_G^U)$  in Lemma 3.8 and Theorem 3.9 below will be given in terms of a certain function  $\delta(G)$ , defined for compact Lie groups whose identity component is a torus and central.

**Definition 3.6.** Assume that  $G$  lies in a central extension  $1 \rightarrow T \rightarrow G \rightarrow \Gamma \rightarrow 1$ , where  $T$  is a torus and  $\Gamma$  is a finite group. For each  $\phi \in T^*$ , define

$$\delta(G, \phi) = \gcd\{\dim(V) \mid V \in \text{Irr}(G, \phi)\};$$

and set

$$\delta(G) = \text{lcm}\{\delta(G, \phi) \mid \phi \in T^*\}.$$

The next lemma gives a partial description of this function, independently of representations; and also lists some of its more technical properties which will be needed in later proofs.

**Lemma 3.7.** Assume that  $G_0 \subseteq Z(G)$ ; i.e., that  $G$  lies in a central extension  $1 \rightarrow T \rightarrow G \rightarrow \Gamma \rightarrow 1$ , where  $T$  is a torus and  $\Gamma$  is finite. Set  $e = \text{expt}(T \cap [G, G])$ . For each prime  $p \mid |\Gamma|$ , let  $G_p$  be a maximal  $p$ -toral subgroup of  $G$ : the extension of  $T$  by a Sylow  $p$ -subgroup of  $\Gamma$ . Then

- (a)  $\delta(G) = 1$  if and only if  $e = 1$ , if and only if  $G \cong T \times \Gamma$
- (b)  $e \mid \delta(G)$  and  $\delta(G)^2 \mid |\Gamma|$
- (c)  $\delta(G) = \prod_{p \mid |\Gamma|} \delta(G_p)$ , and  $\delta(G, \phi) = \prod_{p \mid |\Gamma|} \delta(G_p, \phi)$  for all  $\phi \in T^*$
- (d)  $\delta(G, \phi') = \delta(G, \phi)$  for all  $\phi', \phi \in T^*$  with  $\phi' \equiv \phi \pmod{e}$
- (e)  $\delta(G, n\phi) = \delta(G, \phi)$  for all  $\phi \in T^*$ , and all  $n \in \mathbb{Z}$  with  $(n, e) = 1$ .

*Proof.* Note first that for any  $H \subseteq G$  of finite index, and any  $\phi \in T^*$ ,

$$\delta(H, \phi) \mid \delta(G, \phi) \mid [G : H] \cdot \delta(H, \phi). \quad (1)$$

The first relation holds since each  $G$ -representation with support in  $\phi$  can be regarded as an  $H$ -representation; and the second since  $\text{Ind}_H^G(V)$  has support in  $\phi$  for any  $H$ -representation  $V$  with support in  $\phi$ .

**(b)** Fix any  $\phi \in T^*$ , and choose  $a \in T \cap [G, G]$  such that  $\phi(a)$  generates  $\phi(T \cap [G, G])$ . Then for any  $G$ -representation  $V$  with support in  $\phi$ ,  $a$  acts on  $V$  via multiplication by  $\phi(a)$ ; and since  $a \in [G, G]$ ,  $\phi(a) \cdot \text{Id}_V$  has determinant  $\phi(a)^{\dim(V)} = 1$ . Thus,  $|\phi(a)| \mid \dim(V)$  for all such  $V$ , and so

$$|\phi(a)| = |\phi(T \cap [G, G])| \mid \delta(G, \phi). \tag{2}$$

In particular,  $e = \text{expt}(T \cap [G, G])$  divides  $\delta(G)$ .

Now fix any  $\phi \in T^*$ , and let  $V_\phi$  be the 1-dimensional irreducible  $T$ -representation with character  $\phi$ . Let  $V_1, \dots, V_k$  be the irreducible  $G$ -representations with support in  $\phi$ . For each  $i$ , the multiplicity of  $V_i$  in  $\text{Ind}_T^G(V_\phi)$  is

$$\dim_{\mathbb{C}}(\text{Hom}_G(\text{Ind}_T^G(V_\phi), V_i)) = \dim_{\mathbb{C}}(\text{Hom}_T(V_\phi, V_i)) = \dim_{\mathbb{C}} V_i.$$

Thus,  $|\Gamma| = \dim(\text{Ind}_T^G(V_\phi)) = \sum_{i=1}^k \dim(V_i)^2$ . And so  $\delta(G, \phi)$ , the greatest common divisor of the  $\dim(V_i)$ , is such that  $\delta(G, \phi)^2 \mid |\Gamma|$ .

**(a)** We prove here the slightly more general equivalence that

$$\delta(G, \phi) = 1 \iff \phi(T \cap [G, G]) = 1 \iff G/\text{Ker}(\phi) \cong T/\text{Ker}(\phi) \times \Gamma. \tag{3}$$

The third statement clearly implies the first, and the first implies the second by (2).

By the universal coefficient theorem,  $H^2(\Gamma; T) \cong \text{Hom}(H_2(\Gamma), T)$ ; and  $T \cap [G, G]$  is the image of the homomorphism  $\eta_G : H_2(\Gamma) \rightarrow T$  which corresponds to  $[G]$  as an element of  $H^2(\Gamma; T)$ . So  $G \cong T \times \Gamma$  if  $T \cap [G, G] = 1$ , and  $G/\text{Ker}(\phi) \cong T/\text{Ker}(\phi) \times \Gamma$  if  $\phi(T \cap [G, G]) = 1$ .

**(c)** This formula follows immediately from (1), and the fact that  $\delta(G_p, \phi) \mid |G_p/T|$  is a power of  $p$  for each  $p$ .

**(d)** If  $\phi \equiv 0 \pmod{e}$ , then  $\phi(T \cap [G, G]) = 1$ , and so  $\delta(G, \phi) = 1$  by (3). If  $\phi' \equiv \phi \not\equiv 0 \pmod{e}$ , then the two composites

$$H_2(\Gamma) \xrightarrow{\eta_G} T \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\phi'} \end{array} S^1$$

are equal. Hence  $(G/\text{Ker}(\phi), \phi) \cong (G/\text{Ker}(\phi'), \phi')$  as pairs, and  $\delta(G, \phi) = \delta(G, \phi')$ .

**(e)** For any  $n \in \mathbb{Z}$  and any  $G$ -representation  $V$  with support  $\phi$ ,  $\psi^n(V)$  is a virtual representation with support  $n\phi$ : since  $\chi_{\psi^n V}(gt) = \chi_V(g^n t^n) = \chi_{\psi^n V}(g) \cdot \phi(t)^n$  for any  $g \in G$  and  $t \in T$ . Cf. [Ad, Lemma 3.61] for details. Also,  $V$  and  $\psi^n(V)$  have the same (virtual) dimension, and hence  $\delta(G, n\phi) \mid \delta(G, \phi)$ . So by (d),  $\delta(G, n\phi) = \delta(G, \phi)$  if  $n$  is invertible mod  $e$ .  $\square$

Ian Leary has pointed out to me that  $\delta(G)$  is the greatest common divisor of the indices  $[G : H]$  of those subgroups  $H \subseteq G$  of finite index such that  $H$  splits as a product  $H \cong T \times (H/T)$ .

Whenever  $G_0 = T$  is a torus,  $R(G)$  splits as the direct sum, taken over all  $G/T$ -orbits  $(\phi) \subseteq T^*$ , of the subgroups  $R(G, (\phi))$  of finite rank. In a similar fashion,  $rs_G^U$  splits as the direct sum over all  $(\phi) \subseteq T^*$  of homomorphisms

$$rs_{G,(\phi)} : R(G, (\phi)) \longrightarrow R_{\mathcal{P}}(G, (\phi)).$$

We are now ready to describe the cokernel of each of these summands for such  $G$ . The key case to consider is that when  $T = G_0$  is central and  $\phi$  is faithful.

**Lemma 3.8.** *Assume that  $G$  lies in a central extension  $1 \rightarrow T \rightarrow G \xrightarrow{\sigma} \Gamma \rightarrow 1$ , where  $T \cong S^1$ , and where  $\Gamma$  is finite. Fix a faithful (injective) character  $\phi \in T^*$ . Let  $S$  be the set of all conjugacy classes of elements  $g \in \Gamma$  such that no two elements in  $\sigma^{-1}g$  are conjugate; and let  $S_{\mathcal{P}} \subseteq S$  be the set of conjugacy classes of elements of prime power order. For each  $g \in S_{\mathcal{P}}$ , let  $\eta(g)$  be the largest divisor of  $\delta(C_G(g), \phi)$  which is prime to the order of  $g$ . Then*

$$R(G, \phi) \cong \mathbb{Z}^{|S|}, \quad R_{\mathcal{P}}(G, \phi) \cong \mathbb{Z}^{|S_{\mathcal{P}}|}, \quad \text{and} \quad \text{Coker}(rs_{G,\phi}) \cong \bigoplus_{1 \neq g \in S_{\mathcal{P}}} \mathbb{Z}/\eta(g).$$

*Proof.* A character  $\chi$  of  $G$  has support in  $\phi$  if and only if it satisfies the relation  $\chi(gt) = \chi(g)\phi(t)$  for all  $g \in G$  and  $t \in T$ . In particular, since  $\phi$  is injective,  $\chi(g) = 0$  for any  $g$  which is conjugate to  $gt$  for some  $1 \neq t \in T$ . Thus,  $Cl(G, \phi)$  is a complex vector space of dimension  $|S|$ ; and by the Peter-Weyl theorem (and the independence of irreducible characters)  $R(G, \phi)$  is a free abelian group of rank  $|S|$ . Also,  $R_{\mathcal{P}}(G, \phi)$  is torsion free (it is detected by characters defined on  $G_{\mathcal{P}}$ ), and  $\text{Ker}(rs_{G,\phi})$  is the set of elements of  $R(G, \phi)$  whose characters vanish on  $G_{\mathcal{P}}$ . So the image of  $rs_{G,\phi}$  is free of rank  $|S_{\mathcal{P}}|$ ; and once we have shown that  $rs_{G,\phi}$  has finite cokernel it will follow that  $R_{\mathcal{P}}(G, \phi)$  is a free abelian group of the same rank.

The computation of the cokernel of  $rs_{G,\phi}$  will be carried out in two steps.

**Step 1.** Assume first that  $\Gamma$  is  $p$ -elementary for some prime  $p$ . Then we can write  $G = C_n \times P$ , where  $C_n$  is cyclic of order  $n$  prime to  $p$ , and where  $P$  is  $p$ -toral. In particular,  $R(G) \cong R(C_n) \otimes R(P)$  and  $R(G, \phi) \cong R(C_n) \otimes R(P, \phi)$ . Let  $\text{IR}(-)$  denote the augmentation ideal of  $R(-)$ , and similarly for  $\text{IR}_{\mathcal{P}}(-)$ . Consider the following commutative diagram with split short exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{IR}(C_n) \otimes R(P, \phi) & \longrightarrow & R(G, \phi) & \longrightarrow & R(P, \phi) \longrightarrow 0 \\ & & \text{rs}_{C_n} \otimes \text{augm.} \downarrow & & \text{rs}_{G,\phi} \downarrow & & = \downarrow \\ 0 & \longrightarrow & \text{IR}_{\mathcal{P}}(C_n) \otimes \mathbb{Z} & \longrightarrow & R_{\mathcal{P}}(G, \phi) & \longrightarrow & R(P, \phi) \longrightarrow 0. \end{array}$$

Here,  $\text{IR}_{\mathcal{P}}(C_n)$  is the product of the  $\text{IR}(\text{Syl}_q(C_n))$  for  $q|n$ , and any  $v \in \text{IR}(\text{Syl}_q(C_n))$  lifts to an element of  $\text{IR}(C_n)$  whose character vanishes on other Sylow subgroups. Hence  $\text{IR}(C_n)$  surjects onto  $\text{IR}_{\mathcal{P}}(C_n)$ , and so

$$\text{Coker}(rs_{G,\phi}) \cong \text{Coker}(rs_{C_n} \otimes \text{augm.}) \cong \text{IR}_{\mathcal{P}}(C_n) \otimes \text{Coker}[R(P, \phi) \xrightarrow{\text{augm.}} \mathbb{Z}].$$

The cokernel of this augmentation map is by definition  $\mathbb{Z}/\delta(P, \phi)$ , and so

$$\text{Coker}(\text{rs}_{G,\phi}) \cong \text{IR}_{\mathcal{P}}(C_n) \otimes (\mathbb{Z}/\delta(P, \phi)). \tag{1}$$

**Step 2.** Now assume that  $G$  is arbitrary. Let  $\mathcal{E}(G)$  be the set of subgroups of  $G$  of finite index such that  $E/T$  is elementary, and (for each prime  $p \mid |\Gamma|$ ) let  $\mathcal{E}_p(G)$  be the set of those  $E \in \mathcal{E}(G)$  such that  $E/T$  is  $p$ -elementary. By Proposition 3.3,  $\text{Coker}(\text{rs}_{G,\phi})$  is the inverse limit of the groups  $\text{Coker}(\text{rs}_{E,\phi})$ , taken over all  $E \in \mathcal{E}(G)$ . By (1),  $\text{Coker}(\text{rs}_{E,\phi})$  is a finite  $p$ -group for all  $E \in \mathcal{E}_p(G)$ . Hence  $\text{Coker}(\text{rs}_{G,\phi})$  is finite; and (for each  $p$ )  $\text{Coker}(\text{rs}_{G,\phi})_{(p)}$  is the inverse limit of the  $\text{Coker}(\text{rs}_{E,\phi})$  for  $E \in \mathcal{E}_p(G)$ .

Fix a prime  $p \mid |\Gamma|$ ; we want to determine the  $p$ -power torsion in  $\text{Coker}(\text{rs}_{G,\phi})$ . If  $K' \subseteq K$  are finite cyclic subgroups of order prime to  $p$ , then the composite

$$\text{IR}(K')_{(p)} \xrightarrow{\text{Ind}} \text{IR}(K)_{(p)} \xrightarrow{\text{Res}} \text{IR}(K')_{(p)} \tag{2}$$

is multiplication by  $[K:K']$ , and hence an isomorphism. Thus, if  $K$  is cyclic of order prime to  $p$ , we can split

$$\text{IR}_{\mathcal{P}}(K)_{(p)} = \bigoplus_{q \mid |K|} \text{IR}(\text{Syl}_q(K)) \cong \bigoplus_{1 \neq K' \subseteq K_p} \widetilde{\text{IR}}(K')_{(p)}$$

(i.e., taking the second sum over subgroups of prime power order). Here,  $\widetilde{\text{IR}}(K') \subseteq \text{IR}(K')$  is the kernel of the map given by restriction to the subgroup of prime index, and is free with rank equal to the number of generators of  $K'$ .

For each  $n \mid |\Gamma|$  prime to  $p$ , let  $\text{Cyc}_n$  be the set of all cyclic subgroups  $K \subseteq \Gamma$  of order  $n$  if  $n$  is a prime power, and set  $\text{Cyc}_n = \emptyset$  otherwise. By Lemma 3.7(c), for any maximal  $p$ -toral subgroup  $P \subseteq H$ ,  $\delta(P, \phi)$  is the largest power of  $p$  dividing  $\delta(H, \phi)$ . So with the help of (1) we now get

$$\begin{aligned} \text{Coker}(\text{rs}_{G,\phi})_{(p)} &\cong \varprojlim_{E \in \mathcal{E}_p(G)} \text{Coker}(\text{rs}_{E,\phi}) \\ &\cong \bigoplus_{p \nmid n \mid |\Gamma|} \left( \varprojlim_{K \in \text{Cyc}_n} (\widetilde{\text{IR}}(K) \otimes \mathbb{Z}/\delta(\sigma^{-1}(C_{\Gamma}(K)), \phi))_{(p)} \right). \end{aligned} \tag{3}$$

For each  $n = q^k$  (where  $q \neq p$  is prime), set

$$\text{Cyc}'_n = \{K = \langle g \rangle \in \text{Cyc}_n \mid \text{no two elts. in } \sigma^{-1}g \text{ conjugate in } G\}.$$

Fix some  $K \in \text{Cyc}_{q^k} \setminus \text{Cyc}'_{q^k}$ , and let  $K' \subseteq K$  be the subgroup of index  $q$ . Then there exists  $x \in N_G(\sigma^{-1}K)$  such that for each  $g \in \sigma^{-1}(K \setminus K')$ ,  $xgx^{-1} = gt$  for some  $1 \neq t \in T$ . The character of any element  $v \in \widetilde{\text{IR}}(K) \cong \widetilde{\text{IR}}(\sigma^{-1}K, \phi)$  vanishes

on  $\sigma^{-1}K'$ ; and hence (since  $\chi_v(gt) = \chi_v(g) \cdot \phi(t)$ )  $v$  is fixed by the action of  $x$  only if  $v = 0$ . Thus,  $x$  acts on  $\widetilde{\text{IR}}(K)$  with trivial fixed point set; and in particular such terms contribute nothing to the limit in (3).

Formula (3) thus reduces to a sum, over conjugacy class representatives for all  $K \in \text{Cyc}'_n$ , of the groups

$$H^0(N_G(\sigma^{-1}K); \widetilde{\text{IR}}_{\mathcal{P}}(K)) \otimes (\mathbb{Z}/\delta(C_G(\sigma^{-1}K), \phi))_{(p)}.$$

The first factor here is free of rank equal to the number of  $\Gamma$ -conjugacy classes of generators of  $K$ . The formula for  $\text{Coker}(rs_{G,\phi})$  now follows upon taking the product over all primes  $p \mid |\Gamma|$ .  $\square$

As an example, consider the group  $G = C_n \times (S^1 \times_{C_2} Q(8))$ , where  $n$  is odd,  $Q(8)$  is a quaternion group of order 8, and the second product is taken while identifying the central elements of order 2 in  $S^1$  and  $Q(8)$ . By Lemma 3.8, if  $\phi \in T^*$  is a generator, then  $rs_{G,k\phi}$  is onto for  $k$  even, while  $\text{Coker}(rs_{G,k\phi}) \cong \mathbb{Z}/2 \otimes \text{IR}_{\mathcal{P}}(C_n) \neq 0$  if  $k$  is odd.

The groups dealt with in Lemma 3.8 seem quite specialized, but we are now ready to show that the general case — for an arbitrary compact Lie group  $G$  — can always be reduced to the cases handled there.

**Theorem 3.9.** *Let  $G$  be any compact Lie group. Fix a maximal torus  $T \subseteq G$  and a Weyl chamber  $C \subseteq \mathfrak{t}$ , and set  $N = N(T, C) \subseteq G$ . Then  $rs_G^{\cup}$  splits as a direct sum of homomorphisms*

$$rs_{G,(V_0)} : R(G, (V_0)) \longrightarrow R_{\mathcal{P}}(G, (V_0)),$$

taken over all  $G/G_0$ -orbits  $(V_0) \subseteq \text{Irr}(G_0)$ .

For any  $V_0 \in \text{Irr}(G_0)$ , let  $\phi$  be the maximal weight of  $V_0$  in  $C^*$ , let  $N_{\phi} \subseteq N$  be the subgroup of elements which fix  $\phi$ , and set  $K_{\phi} = \text{Ker}(\phi) \subseteq T$ . Then the assignment  $([V] \mapsto [V\langle\phi\rangle])$  induces isomorphisms

$$R(G, (V_0)) \xrightarrow{\cong} R(N_{\phi}/K_{\phi}, \phi), \quad R_{\mathcal{P}}(G, (V_0)) \xrightarrow{\cong} R_{\mathcal{P}}(N_{\phi}/K_{\phi}, \phi),$$

and

$$\text{Coker}(rs_{G,(V_0)}) \xrightarrow{\cong} \text{Coker}(rs_{N_{\phi}/K_{\phi},\phi}).$$

*Proof.* By Lemma 3.2,  $R_{\mathcal{P}}(G)$  is the inverse limit of the representation rings  $R(H)$ , taken over all  $H \subseteq G$  of finite index such that  $H/G_0$  has prime power order. Since each  $R(H)$  splits as a sum of finitely generated groups indexed by the  $G/G_0$ -orbits  $(V_0) \in \text{Irr}(G_0)$ , we now see that  $R_{\mathcal{P}}(G)$  also splits as such a sum. And hence  $rs_G^{\cup}$  also splits as a direct sum of homomorphisms  $rs_{G,(V_0)}$ .

Now fix  $V_0 \in \text{Irr}(G_0)$  and let  $\phi$  be its maximal weight in  $C^*$ . Write  $\Phi = (\phi)$  for short: the  $N/T$ -orbit of  $\phi \in C_T^*$ . By Proposition 1.4, the assignment  $[V] \mapsto [V\langle\Phi\rangle]$  defines a bijection from  $\text{Irr}(G, (V_0))$  to  $\text{Irr}(N, \Phi)$ , and hence an isomorphism  $\text{R}(G, (V_0)) \xrightarrow{\cong} \text{R}(N, \Phi)$ . Similarly, it induces isomorphisms  $\text{R}(H, (V_0)) \xrightarrow{\cong} \text{R}(H \cap N, \Phi)$  for each  $H \subseteq G$  of finite index, and upon taking the inverse limit over all such  $H$  for which  $H/G_0$  has prime power order we get an isomorphism  $\text{R}_{\mathcal{P}}(G, (V_0)) \xrightarrow{\cong} \text{R}_{\mathcal{P}}(N, \Phi)$ . And this in turn induces an isomorphism between the cokernels of  $\text{rs}_{G, (V_0)}$  and  $\text{rs}_{N, \Phi}$ .

The homomorphism  $\text{R}(N, \Phi) \rightarrow \text{R}(N_\phi, \phi) \cong \text{R}(N_\phi/K_\phi, \phi)$ , defined by sending  $[V]$  to  $[V\langle\phi\rangle]$ , is an isomorphism: its inverse is the induction map  $[V] \mapsto [\text{Ind}_{N_\phi}^N(V)]$ . This same assignment also defines an isomorphism  $\text{R}_{\mathcal{P}}(N, \Phi) \xrightarrow{\cong} \text{R}_{\mathcal{P}}(N_\phi/K_\phi, \phi)$  (whose inverse is again the induction map); and hence defines an isomorphism between the cokernels of  $\text{rs}_{N, \Phi}$  and  $\text{rs}_{N_\phi/K_\phi, \phi}$ .  $\square$

The above general description of  $\text{Coker}(\text{rs}_G^U)$  is rather complicated. In contrast, the conditions for the map  $\text{rs}_G^U$  to be onto can be formulated more simply.

**Theorem 3.10.** *Let  $G$  be any compact Lie group. Fix a maximal torus  $T \subseteq G$  and a Weyl chamber  $C \subseteq \mathfrak{t}$ , and set  $N = N(T, C) \subseteq G$ . Let  $\mathcal{E}'(N)$  denote the set of subgroups  $E \subseteq N$  of finite index such that  $E/T$  is elementary but not of prime power order. Then*

$$\text{expt}(\text{Coker}(\text{rs}_G^U)) = \text{lcm}\{\delta(E/[E, T]) \mid E \in \mathcal{E}'(N)\}. \tag{1}$$

*In particular,  $\text{rs}_G^U$  is surjective if and only if  $\text{rs}_N^U$  is surjective, if and only if  $T \cap [E, E] = [E, T]$  for all  $E \in \mathcal{E}'(N)$ .*

*Proof.* It is clear from part (c) that the exponent of  $\text{Coker}(\text{rs}_G^U)$  divides the number given in (1). To show that these are equal, fix any prime  $p$ , and choose  $E \subseteq N$  of finite index such that  $E/T$  is  $p$ -elementary but not a  $p$ -group. We must show that  $\delta(E/[E, T]) \mid \text{expt}(\text{Coker}(\text{rs}_G^U))$ . Choose any  $\phi' \in (T/[E, T])^* \subseteq T^*$  such that  $\delta(E/[E, T], \phi') = \delta(E/[E, T])$ . Since  $N/T$  acts linearly on  $\mathfrak{t}^*$  and leaves  $C^*$  invariant, the fixed set  $(C^*)^E$  is a cone shaped subspace of  $(\mathfrak{t}^*)^E$  with nonempty interior. Hence, we can choose  $\phi \in C^* \cap (T/[E, T])^* = (C_T^*)^E$  such that  $\phi \equiv \phi'$  modulo the exponent of  $\frac{T \cap [E, E]}{[E, T]}$ . If  $q \neq p$  is any other prime dividing  $|E/T|$ , then

$$\delta(E/[E, T], q\phi) = \delta(E/[E, T], \phi) = \delta(E/[E, T], \phi') = \delta(E/[E, T])$$

by Lemma 3.7(d,e). And finally, if  $gT \in E/T$  is the element of order  $q$ , then  $gT \in S$  in the notation of Lemma 3.8: no two elements in  $gT/\text{Ker}(q\phi)$  are conjugate. Thus,

$$\delta(E/[E, T]) = \delta(E/[E, T], q\phi) \mid \text{expt}(\text{Coker}(\text{rs}_{E, q\phi})) \mid \text{expt}(\text{Coker}(\text{rs}_G^U))$$



by Lemma 3.8; and this finishes the proof of formula (1). The necessary and sufficient conditions for  $rs_G^U$  to be surjective now follow from Lemma 3.7(a).  $\square$

Since the general condition for  $rs_G^U$  to be surjective is still somewhat complicated, we now list some special cases which are simpler to formulate.

**Corollary 3.11.** *For any compact Lie group  $G$ ,  $\text{Coker}(rs_G^U)$  has finite exponent, and*

$$\text{expt}(\text{Coker}(rs_G^U))^2 \mid |\pi_0(G)|. \tag{1}$$

Furthermore,  $rs_G^U$  is surjective if  $G$  satisfies any of the following conditions:

- (a)  $G$  is finite or connected.
- (b) All elements of  $\pi_0(G)$  have prime power order.
- (c)  $\pi_0(G)$  is a periodic group: all of its Sylow subgroups are cyclic or quaternion.
- (d)  $Z(G_0) = 1$ .
- (e)  $G$  is a semidirect product of the form  $G = G_0 \rtimes \Gamma$ , where  $\Gamma \subseteq G$  normalizes some maximal torus  $T$  and leaves invariant some Weyl chamber in  $T$ .

*Proof.* Fix a maximal torus  $T \subseteq G_0$ , and a Weyl chamber  $C$ . Set  $N = N(T, C)$ . As in Theorem 3.10, let  $\mathcal{E}'(N)$  be the set of subgroups  $H \subseteq N$  of finite index such that  $H/T$  is elementary but not of prime power order.

By Lemma 3.7(b),  $\delta(H/[T, H])^2 \mid |H/T| \mid |\pi_0(G)|$  for each  $H \subseteq N$  of finite index. So (1) follows from Theorem 3.10.

- (a)  $rs_G^U$  is onto by Lemma 3.4 if  $G$  is finite, and by (1) if  $G$  is connected.
- (b) If all elements of  $\pi_0(G) = \pi_0(N)$  have prime power order, then  $\mathcal{E}'(N) = \emptyset$ , and so  $rs_G^U$  is onto by Theorem 3.10.
- (c) Note that  $H_2(\Gamma) = 0$  for any finite periodic group  $\Gamma$ . Hence, if  $\pi_0(G)$  is periodic, then for any  $H \in \mathcal{E}'(N)$ ,  $H/[H, T] \cong T/[H, T] \times H/T$ . So  $rs_N^U$  and  $rs_G^U$  are onto by Theorem 3.10.
- (e) The conditions on  $\Gamma$  imply that  $N$  is a semidirect product of  $T$  with  $\Gamma$ , and hence that  $rs_G^U$  is onto by Theorem 3.10.
- (d) By [Bo, §4.10, Corollaire], the surjection  $\text{Aut}(G_0) \twoheadrightarrow \text{Out}(G_0)$  is split by outer automorphisms which fix  $T$  and  $C$ . Let  $\Gamma \subseteq G$  be the subgroup of elements whose conjugation action lies in the image of any given splitting map. Then  $G = G_0 \rtimes \Gamma$  (since  $G_0 \cap \Gamma = Z(G_0) = 1$ ); and so  $rs_G^U$  is onto by (e).  $\square$

We remark here that  $G$  being a semidirect product  $G_0 \rtimes \Gamma$  does not in itself imply that  $rs_G^U$  is onto. As an example, set

$$G = C_3 \times (\text{SU}(2) \times_{C_2} Q(8)),$$

where  $C_3$  is cyclic of order 3,  $Q(8)$  is a quaternion group of order 8, and the product is taken by identifying the central subgroups of order 2 in  $\text{SU}(2)$  and

$Q(8)$ . Then Theorem 3.10 applies to show that  $\text{Coker}(\text{rs}_G^{\text{U}})$  has exponent 2. But  $\text{SU}(2) \times_{C_2} Q(8)$  is also a semidirect product of  $\text{SU}(2)$  with  $C_2 \times C_2$ : the splitting comes from the diagonal subgroup

$$\langle (i, i) \rangle \times \langle (j, j) \rangle \subseteq Q(8) \times_{C_2} Q(8) \subseteq \text{SU}(2) \times_{C_2} Q(8).$$

So far, we have dealt mostly with the case of unitary representations. The general conditions for  $\text{rs}_G^{\text{O}}$  to be surjective seem to be much more complicated. For example, with a little more work, one can show that if  $G$  is a central extension of a torus by a finite group, then  $\text{rs}_G^{\text{O}}$  is onto if and only if  $\text{rs}_G^{\text{U}}$  is onto. In contrast, the following example provides a simple way of constructing groups  $G$  for which  $\text{rs}_G^{\text{O}}$  is not onto but  $\text{rs}_G^{\text{U}}$  is onto.

**Example 3.12.** Fix any pair  $(G', V')$ , where  $G'$  is a compact connected Lie group, and  $V'$  an irreducible  $G'$ -representation of real type having the additional property that some central element  $z \in Z(G')$  of order 2 acts on  $V'$  by  $(-\text{Id})$ . Choose any odd prime power  $n > 1$ , and set  $G = G' \times_{C_2} Q(4n)$ : the central product of  $G'$  with the quaternion group of order  $4n$ , where  $z$  is identified with the central element of  $Q(4n)$ . Then  $\text{rs}_G^{\text{O}}$  is not onto.

*Proof.* Let  $W$  be any effective irreducible representation of  $Q(4n)$ , and set  $V = V' \otimes_{\mathbb{C}} W$ . Then  $V$  is an irreducible  $G$ -representation of quaternion type, but its restriction to any  $p$ -toral subgroup of  $G$  (for any prime  $p$ ) has real type. In particular,  $[V]$  represents an element of  $\text{RO}_{\mathcal{P}}(G)$ ; but since  $\text{rs}_G^{\text{O}}$  and  $\text{rs}_G^{\text{U}}$  are injective (all elements of  $\pi_0(G) \cong D(2n)$  have prime power order), it does not lie in the image of  $\text{rs}_G^{\text{O}}$ .  $\square$

For example, we can take  $G' = \text{SO}(2m)$  for any  $m \geq 2$ , and let  $V'$  be the standard  $G'$ -representation on  $\mathbb{C}^{2m}$ . Set  $G = G' \times_{C_2} Q(4n)$ , for some odd prime power  $n \geq 3$ . Then  $\text{rs}_G^{\text{O}} : \text{RO}(G) \rightarrow \text{RO}_{\mathcal{P}}(G)$  fails to be onto, while  $\text{rs}_G^{\text{U}}$  is onto (in fact, an isomorphism) by Corollary 3.11(b) (all elements of  $\pi_0(G)$  have prime power order).

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Bob Oliver  
URM 7539 du CNRS  
Département de Mathématiques  
Université Paris XIII  
Av. J.-B. Clément  
F-93430 Villetaneuse  
e-mail: bob@math.univ-paris13.fr

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