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## A theory of cobordism for non-spherical links

Vincent Blanlœil and Françoise Michel

**Abstract.** We define an equivalence relation, called algebraic cobordism, on the set of bilinear forms over the integers. When  $n \geq 3$ , we prove that two  $2n - 1$  dimensional, simple fibered links are cobordant if and only if they have algebraically cobordant Seifert forms. As an algebraic link is a simple fibered link, our criterion for cobordism allows us to study isolated singularities of complex hypersurfaces up to cobordism.

**Mathematics Subject Classification (1991).** 57R, 57R80, 57R90, 57M25, 57Q45, 32S, 32S55, 14B05.

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### 0. Introduction

In this work we present a cobordism theory for links which is motivated by the study of the topology of isolated singularities of complex hypersurfaces. Let us be more precise:

(0.1) Let  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ , be a holomorphic germ with an isolated singular point at the origin. We denote by  $D_\delta^{2k}$  the compact ball of radius  $\delta$  centred at 0 in  $\mathbb{C}^k$ , and by  $S_\delta^{2k-1}$  its boundary. The orientation-preserving homeomorphism class of the pair  $(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$  does not depend on the choice of a sufficiently small  $\varepsilon$ , by definition it is *the topological type* of  $f$ . The orientation preserving diffeomorphism class of the pair  $(S_\varepsilon^{2n+1}, K(f))$ , where  $K(f) = (f^{-1}(0)) \cap S_\varepsilon^{2n+1}$  is the link of  $f$ . The Milnor's conic structure theorem (see [M3, 68]) shows that the link  $K(f)$  determines the topological type of  $f$ . Moreover, J. Milnor has also proved that:

1.  $f/|f| : S_\varepsilon^{2n+1} \setminus K(f) \rightarrow S^1$  is a differentiable fibration which is trivial on  $U \setminus K(f)$ , when  $U$  is a sufficiently "small" open tubular neighbourhood of  $K(f)$ .
2. The manifold  $K(f)$  is  $(n - 2)$ -connected.
3. The adherence  $F$  of a fiber of  $f/|f|$  is a compact, oriented,  $(n - 1)$ -connected

smooth submanifold of  $S_{\varepsilon}^{2n+1}$  having  $K(f)$  as boundary. By definition  $F$  is the *Milnor fiber* of  $K(f)$ .

(0.2) More generally, we will say that a *link* is a  $(n-2)$ -connected, oriented, smooth, closed,  $(2n-1)$  dimensional submanifold of  $S^{2n+1}$ . A *knot* is a spherical link (i.e. a link abstractly homeomorphic to  $S^{2n-1}$ ). It is well-known that, for any link  $K$ , there exists a smooth, compact, oriented  $2n$ -submanifold  $F$  of  $S^{2n+1}$ , having  $K$  as boundary ; such a manifold  $F$  is called a *Seifert surface* for  $K$ .

(0.3) Following M. Kervaire [K1, 65], we say that two links  $K_0$  and  $K_1$ , abstractly diffeomorphic to the same manifold  $\mathcal{K}$ , are *cobordant* if there exists an embedding  $\Phi, \Phi : \mathcal{K} \times [0, 1] \rightarrow S^{2n+1} \times [0, 1]$ , such that:

$$\Phi(\mathcal{K} \times \{0\}) = K_0 \text{ and } \Phi(\mathcal{K} \times \{1\}) = -K_1,$$

where  $-K_1$  is the link  $K_1$  with the orientation reversed.

(0.4) Let  $F$  be a  $2n$  dimensional oriented smooth manifold of  $S^{2n+1}$ , and let  $G$  be the quotient of  $H_n(F, \mathbb{Z})$  by its  $\mathbb{Z}$ -torsion.

The *Seifert form* associated to  $F$  is the bilinear form  $A : G \times G \rightarrow \mathbb{Z}$  defined as follows (see also [K2, 70] p.88 or [L2, 70], p.185): let  $(x, y)$  be in  $G \times G$ , then  $A(x, y)$  is the linking number in  $S^{2n+1}$  of  $x$  and  $i_+(y)$ , where  $i_+(y)$  is the cycle  $y$  "pushed" in  $(S^{2n+1} \setminus F)$  by the positively oriented vector field normal to  $F$  in  $S^{2n+1}$ .

By definition a *Seifert form for a link  $K$*  is the Seifert form associated to a Seifert surface for  $K$ .

When  $n \geq 2$ , J. Levine ([L1, 69]) and M. Kervaire ([K2, 70]) gave a complete characterization of cobordism classes of knots in terms of Witt-equivalence classes of Seifert forms.

(0.5) A *simple link* is a link which has a  $(n-1)$ -connected Seifert surface. A link  $K$  is a *simple fibered link* if there exists a differentiable fibration  $\varphi : S^{2n+1} \setminus K \rightarrow S^1$ ,  $\varphi$  being trivial on  $U \setminus K$ , where  $U$  is a "small" open tubular neighbourhood of  $K$ , and having  $(n-1)$ -connected fibers, the adherence of which are Seifert surfaces for  $K$ . In this paper we define in §1 (see (1.2)) an equivalence relation on integral bilinear forms which is much more sophisticated than "Witt-equivalence" and the theorems 2 and 3, stated in §1, imply:

**Theorem A.** *If  $n \geq 3$ , two simple fibered links are cobordant if and only if they have algebraically cobordant Seifert forms.*

(0.6) By definition an *algebraic link* is a link  $K(f)$  associated, as described above, to a holomorphic germ  $f$  with an isolated singularity. Furthermore, Milnor's theory of singular complex hypersurfaces implies that algebraic links are simple fibered links. So theorem 2' and 3 stated in §1 imply:

**Theorem B.** *If  $n \geq 3$ , two algebraic links are cobordant if and only if the Seifert forms associated to their Milnor's fibers are algebraically cobordant.*

In [Lê, 72], D.T. Lê showed that two cobordant algebraic links of plane curves (i.e. when  $n = 1$ ) are isotopic. In [DB-M, 93], P. du Bois and F. Michel found (using the classical cobordism theory for knots of M. Kervaire and J. Levine), for all  $n \geq 3$ , examples of non isotopic but cobordant algebraic knots. But in general algebraic links are not spherical links. So theorem B gives a cobordism theory for algebraic links.

Furthermore, having algebraically cobordant Seifert forms is also a necessary condition of cobordism for simple fibered links when  $n$  is 1 or 2. So we obtain in §5, without any restriction of dimension, a "Fox-Milnor" relation (see [F-M, 66]) for the Alexander polynomials of cobordant simple fibered links which implies:

(0.7) **Corollary.** *Let  $K_0$  and  $K_1$  be two algebraic links having respectively  $\Delta_0$  and  $\Delta_1$  as characteristic polynomials of monodromy. If  $K_0$  and  $K_1$  are cobordant then the product  $\Delta_0 \cdot \Delta_1$  is a square in  $\mathbb{Z}[X]$ .*

(0.8) **Comments.** In [V1, 77] and [V2, 78] R. Vogt gave, when  $n \geq 3$ , a sufficient, but not necessary, condition of cobordism for simple links having torsion free homology groups. As shown in [DB-M, 93] the sufficient condition of cobordism for algebraic links given in [Sz, 89] by S. Szczepanski, cannot be true. So the problem of finding a criterion for cobordism of simple fibered links was largely open. Our definition of algebraic cobordism for Seifert forms solves the problem.

(0.9) In this paper we use the following **notations**: If  $X$  is a differentiable manifold we denote by  $\partial X$  its boundary, by  $\overset{\circ}{X}$  its interior and by  $H_k(X)$  the  $k^{th}$ -homology group of  $X$  with coefficients in  $\mathbb{Z}$ . If  $a$  is a  $k$ -cycle of  $X$  we denote by  $[a]$  its homology class in  $H_k(X)$ . If  $G$  is an abelian group let  $\text{rk}(G)$  be the rank of  $G$ , and  $\text{Tors}(G)$  be the torsion subgroup of  $G$ .

## 1. Definitions and statement of results

Let  $\mathcal{A}$  be the set of bilinear forms defined on free  $\mathbb{Z}$ -modules  $G$  of finite rank.

Let  $\varepsilon$  be  $+1$  or  $-1$ .

(1.1) If  $A$  is in  $\mathcal{A}$ , let us denote by  $A^T$  the transpose of  $A$ , by  $S$  the  $\varepsilon$ -symmetric form  $A + \varepsilon A^T$  associated to  $A$ , by  $S^* : G \rightarrow G^*$  the adjoint of  $S$  ( $G^*$  being the dual  $\text{Hom}_{\mathbb{Z}}(G; \mathbb{Z})$  of  $G$ ), by  $\overline{S} : \overline{G} \times \overline{G} \rightarrow \mathbb{Z}$  the  $\varepsilon$ -symmetric non degenerated form induced by  $S$  on  $\overline{G} = G/\text{Ker } S^*$ . A submodule  $M$  of  $G$  is pure if  $G/M$  is torsion free. If  $M$  is any submodule of  $G$  let us denote by  $M^\wedge$  the smallest pure submodule of  $G$  which contains  $M$ . In fact  $M^\wedge$  is equal to  $(M \otimes \mathbb{Q}) \cap G$ . For a submodule  $M$  of  $G$  we denote by  $\overline{M}$  the image of  $M$  in  $\overline{G}$ .

**Definition.** *Let  $A : G \times G \rightarrow \mathbb{Z}$  be a bilinear form in  $\mathcal{A}$ . The form  $A$  is Witt associated to 0 if the rank  $m$  of  $G$  is even and if there exists a pure submodule  $M$  of rank  $\frac{m}{2}$  in  $G$  such that  $A$  vanishes on  $M$ ; such a module  $M$  is called a*

metabolizer for  $A$ .

(1.2) **Definition.** Let  $A_i : G_i \times G_i \rightarrow \mathbb{Z}$ ,  $i=0,1$ , be two bilinear forms in  $\mathcal{A}$ . Let  $G$  be  $G_0 \oplus G_1$  and  $A$  be  $(A_0 \oplus -A_1)$ . The form  $A_0$  is algebraically cobordant to  $A_1$  if there exists a metabolizer  $M$  for  $A$  such that  $\bar{M}$  is pure in  $\bar{G}$ , an isomorphism  $\varphi$  from  $\text{Ker } S_0^*$  to  $\text{Ker } S_1^*$  and an isomorphism  $\theta$  from  $\text{Tors}(\text{Coker } S_0^*)$  to  $\text{Tors}(\text{Coker } S_1^*)$  which satisfy the two following conditions:

- c.1:  $M \cap \text{Ker } S^* = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$ ,
- c.2:  $d(S^*(M)^\wedge) = \{(x, \theta(x)); x \in \text{Tors}(\text{Coker } S_0^*)\}$ , where  $d$  is the quotient map from  $G^*$  to  $\text{Coker } S^*$ .

In §2 (see (2.3)) we prove:

**Theorem 1.** Algebraic cobordism is an equivalence relation on the set  $\mathcal{A}$ .

(1.3) From now on,  $A_0$  and  $A_1$  will always be two Seifert forms associated to some  $(n-1)$ -connected Seifert surfaces  $F_0$  and  $F_1$ , of two simple links  $K_0$  and  $K_1$ . Let us justify the definition of algebraic cobordism. As a generalization of the Kervaire-Levine theory of knot cobordism we obtain in §3 (see (3.10)):

**Proposition.** If  $K_0$  and  $K_1$  are cobordant simple links, then  $A = A_0 \oplus -A_1$  has a metabolizer.

**Remark.** Let  $\varepsilon$  be  $(-1)^n$ , then for  $i=0,1$ ,  $S_i = A_i + \varepsilon A_i^T$  is the intersection form on  $H_n(F_i)$ ,  $\text{Ker } S_i^*$  is the image of  $H_n(K_i)$  in  $H_n(F_i)$  and  $\text{Coker } S_i^*$  is isomorphic to  $\tilde{H}_{n-1}(K_i)$ . So for spherical links, both  $\text{Ker } S_i^*$  and  $\text{Coker } S_i^*$  are zero, and conditions c.1 and c.2 in definition (1.2) vanish. Then, for spherical links, two Witt associated Seifert forms are algebraically cobordant, and we recover the Kervaire-Levine criterion for cobordism.

In the non-spherical case, the topology of the cobordism implies that the restriction of  $A_0$  on  $\text{Ker } S_0^*$  is isomorphic (on  $\mathbb{Z}$ ) to the restriction of  $A_1$  on  $\text{Ker } S_1^*$  (it is easy to check it directly, and it is also implied by the more general proposition (3.10)). This necessary condition for cobordism is not implied by the fact that  $A_0 \oplus -A_1$  is Witt associated to 0, but by condition c.1 in definition (1.2). The topology of the cobordism also implies that the linking forms on  $\text{Tors}(H_{n-1}(K_i))$  are isomorphic. This necessary condition for cobordism is contained in point c.2 of definition (1.2).

(1.4) The major result of this work is theorem 2 proved in §3 (see (3.10) and (3.13)):

**Theorem 2.** Let  $K_0$  and  $K_1$  be two cobordant simple links. If  $K_0$  and  $K_1$  have  $(n-1)$ -connected Seifert surfaces  $F_0$  and  $F_1$  with unimodular Seifert forms  $A_0$

and  $A_1$ , then  $A_0$  is algebraically cobordant to  $A_1$ .

**Remark.** Let  $i$  be 0 or 1. Let us suppose that  $K_i$  is a simple fibered link and let  $F_i$  be a  $(n-1)$ -connected fiber of a fibration  $\varphi_i : S^{2n+1} \setminus K_i \rightarrow S^1$ ; then, the Seifert form  $A_i$  associated to  $F_i$  is unimodular. Conversely, if  $n \geq 3$  and if  $A_i$  is unimodular then  $K_i$  is a simple fibered link (see [K-W, 77] chap. V, §3, p.118).

So, theorem 2 implies:

**Theorem 2'.** *Let  $K_0$  and  $K_1$  be two simple fibered links having  $F_0$  and  $F_1$  as  $(n-1)$ -connected fibers of differentiable fibrations  $\varphi_0$  and  $\varphi_1$ . If  $K_0$  is cobordant to  $K_1$ , then the Seifert forms  $A_0$  and  $A_1$ , associated respectively to  $F_0$  and  $F_1$ , are algebraically cobordant.*

(1.5) Using classical methods of surgery, we prove in §4 (see (4.4) and (4.5)):

**Theorem 3.** *Let  $n$  be greater or equal to 3 and let  $K_0$  and  $K_1$  be two  $2n-1$  dimensional simple links. If the Seifert forms  $A_0$  and  $A_1$ , associated to some  $(n-1)$ -connected Seifert surfaces  $F_0$  and  $F_1$  of  $K_0$  and  $K_1$ , are algebraically cobordant then  $K_0$  is cobordant to  $K_1$ .*

(1.6) Proposition (3.10), which does not use (as remarked in (3.12)) any hypothesis on the Seifert forms, gives:

**Theorem 4.** *Let  $K_0$  and  $K_1$  be two cobordant simple links. If  $A_0$  (resp.  $A_1$ ) is a Seifert form associated to any  $(n-1)$ -connected Seifert surface for  $K_0$  (resp.  $K_1$ ), then  $A_0 \oplus -A_1$  has a metaboliser  $M$  such that  $M \cap \text{Ker } S^* = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$ , where  $\varphi$  is an isomorphism between  $\text{Ker } S_0^*$  and  $\text{Ker } S_1^*$ .*

## 2. Algebraic cobordism

(2.0) Let  $A_0$  and  $A_1$  be two algebraically cobordant forms, let  $A$  be the form  $A_0 \oplus -A_1$  defined on  $G = G_0 \oplus G_1$  and  $S$  be  $A + \varepsilon A^T$ . In this section we prove proposition (2.1) which shows that the algebraic cobordism between  $A_0$  and  $A_1$  allows us to describe  $S$ ; this characterization of  $S$  is fundamental to prove theorem 3 (see §4). Let  $M$ ,  $\varphi$  and  $\theta$  be as in (1.2), let  $m$  be  $\text{rk}(G)$  and  $r$  be  $\text{rk}(\text{Ker } S_0^*)$ . Then definition (1.2) implies that  $s = \text{rk}(S^*(M)) = \frac{1}{2} \text{rk}(S^*(G))$  and  $\text{rk}(M) = r + s = \frac{m}{2}$ .

We use the following notations: if  $E$  is any subset of  $G$  we denote by  $\langle E \rangle$  the submodule of  $G$ , generated by  $E$ . If  $L$  is any submodule of  $G$  then:

$$L^\perp = \{x \in G \text{ s.t. } S(x, l) = 0 \forall l \in L\}$$

$$\text{Hom}_{\mathbb{Z}}(G|_L, \mathbb{Z}) = \{f \in G^* \text{ s.t. } f(l) = 0 \ \forall l \in L\}$$

Moreover if  $L_1$  and  $L_2$  are two submodules of  $G$ , orthogonal for  $S$ , we denote by  $L_1 \oplus^\perp L_2$  their (orthogonal) direct sum.

**Lemma.** *We have:  $S^*(G) \cap S^*(M)^\wedge = S^*(M^\perp)$ .*

*Proof.* Let  $r$  be the rank of  $\text{Ker } S_0^*$  and  $s$  be the rank of  $S^*(M)$ . As  $M$  is a metabolizer for  $S$  which fulfills condition c.1 in (1.2) we have:

$$\text{rk}(\text{Ker } S^*) = 2 \text{rk}(M \cap \text{Ker } S^*) = 2r, \text{rk}(S^*(G)) = 2s \text{ and } \text{rk}(M^\perp) = s + 2r. \text{ Hence } M^\perp = (M + \text{Ker } S^*)^\wedge \text{ and } S^*(M^\perp) \subset S^*(G) \cap S^*(M)^\wedge.$$

Moreover,  $S^*(M)$  is of finite index in  $\text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$ . As  $\text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$  is a pure submodule of  $G^*$ , we get  $S^*(M)^\wedge = \text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$ . So if  $S^*(x) \in S^*(M)^\wedge$ , then  $S^*(x, l) = 0$  for all  $l$  in  $M^\perp$  and  $x$  is in  $M^\perp$ . □

Since  $S^*(M)$  is of finite index in  $S^*(M)^\wedge$ , one can write  $(S^*(M)^\wedge)/S^*(M) \cong \bigoplus_{i=1}^s \mathbb{Z}/a_i \mathbb{Z}$  where  $a_i \in \mathbb{N} \setminus \{0\}$  and  $a_i$  divides  $a_{i+1}$  (we do not exclude that there exists an integer  $l$  such that  $a_i = 1$  for  $i = 1, \dots, l$ ).

**Proposition.** *The submodule  $\overline{M}$  is pure in  $\overline{G}$  if and only if  $S^*(M^\perp) = S^*(M)$ .*

*Proof.* We suppose that  $\overline{M}$  is pure in  $\overline{G}$ . As  $M \cap \text{Ker } S^* = \Delta(\varphi)$  has rank  $r$ , the rank of  $M + \text{Ker } S^*$  is  $s + 2r$ . So  $M + \text{Ker } S^*$  is of finite index in  $M^\perp$ . Let  $x$  be in  $M^\perp$ ; there exists a positive integer  $k$  such that  $kx = y + m$ , where  $y$  is in  $\text{Ker } S^*$ ,  $m$  is in  $M$ ; so  $\overline{m} = k\overline{x}$ . Since  $\overline{M}$  is pure in  $\overline{G}$  then  $\overline{x}$  is in  $\overline{M}$ , so there exists  $y'$  in  $\text{Ker } S^*$  such that  $x + y'$  is in  $M$ . Finally  $S^*(x) = S^*(x + y') \in S^*(M)$ , and  $S^*(M^\perp) \subset S^*(M)$ . But  $M \subset M^\perp$  so  $S^*(M^\perp) = S^*(M)$ .

We suppose that  $S^*(M) = S^*(M^\perp)$ . First we prove that  $\overline{M^\perp}$  is pure in  $\overline{G}$ . Let  $z$  be in  $M^\perp$  with  $\overline{z} = k\overline{x}$  where  $x$  is in  $G$  and  $k$  is a positive integer. So there exists  $y$  in  $\text{Ker } S^*$  such that  $kx = z + y$ . For all  $m$  in  $M$  we have  $S(kx, m) = S(z + y, m) = 0$ , so  $S(x, m) = 0$  and  $x$  is in  $M^\perp$ . Now we prove that  $S^*(M^\perp) = S^*(M)$  implies  $\overline{M} = \overline{M^\perp}$ . Let  $z$  be in  $M^\perp$ . If  $S^*(z) = f$  there exists  $m$  in  $M$  such that  $S^*(m) = f$ . So  $z - m = y$  is in  $\text{Ker } S^*$ , and  $\overline{z} = \overline{m}$  is in  $\overline{M}$ . Finally, since  $\overline{M^\perp}$  is pure in  $\overline{G}$  and  $\overline{M^\perp} \subset \overline{M}$  we get  $\overline{M^\perp} = \overline{M}$  is pure in  $\overline{G}$ . □

By definition (1.2)  $\overline{M}$  is pure in  $\overline{G}$ , so lemma (2.0) and proposition (2.0), and, conditions c.1 and c.2 in definition (1.2) imply that  $\text{Coker } S^*$  is isomorphic to

$$\mathbb{Z}^{2r} \oplus \left( \bigoplus_{i=1}^s \mathbb{Z}/a_i \mathbb{Z} \right)^2.$$

(2.1) **Proposition.** *There exists a basis  $\mathcal{B} = \{m_i, m_i^*; i=1, \dots, s+r\}$  of  $G$  such that:*

1.  $\{m_i; i=1, \dots, s+r\}$  is a basis of  $M$ ,
2.  $\{m_i, m_i^*; i=s+1, \dots, s+r\}$  is a basis of  $\text{Ker } S^*$  and  $\{m_i^*; i=s+1, \dots, s+r\}$  is a basis of  $\text{Ker } S_0^*$ ,
3. the submodules  $\langle m_i, m_i^* \rangle, i=1, \dots, s+r$ ; are orthogonal for  $S$ , i.e.:  $G = \bigoplus_{1 \leq i \leq s+r}^\perp \langle m_i, m_i^* \rangle$ ,
3. when  $i=1, \dots, s, S(m_i, m_i^*) = a_i$ .

**Definition.** *Such a basis is called a good basis of  $G$  associated to  $M$ .*

The form  $S = A + \varepsilon A^T$  is always an even form. Moreover, when the  $a_i$  are odd we get the following corollary:

**Corollary.** *When the  $a_i$  are odd, the isomorphic class of  $S$  is given by  $m = \text{rk}(G)$  and the isomorphic class of  $\text{Coker } S^*$ .*

*Proof of proposition (2.1).* In (2.0) we have seen that  $S^*(M)^\wedge = \text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$ . Let  $M_0$  be any direct summand complement of  $(M \cap \text{Ker } S^*)$  in  $M$ . There exists a basis  $\{m_i; i=1, \dots, s\}$  of  $M_0$  and a basis  $\{h_i; i=1, \dots, s\}$  of  $\text{Hom}_{\mathbb{Z}}(G|_{M^\perp}; \mathbb{Z})$  such that  $S^*(m_i) = a_i h_i$  where  $a_i \in \mathbb{N} \setminus \{0\}$  and  $a_i$  divides  $a_{i+1}$ . Let  $m_1^*$  be any element in  $G$  such that  $G = \text{Ker } h_1 \oplus \langle m_1^* \rangle$  and  $h_1(m_1^*) = S(m_1, m_1^*).a_1^{-1} = 1$ .

**Claim.** For all  $x$  in  $G$ ,  $a_1$  divides  $S(x, m_1^*)$ .

If  $a_1 = 1$  it is obvious. If  $a_1 > 1$ , condition c.2 in (1.2) implies that  $(S^*(G)^\wedge)/S^*(G)$  is isomorphic to  $(S^*(M)^\wedge)/S^*(M)^2 \cong \left(\bigoplus_{i=1}^s \mathbb{Z}/a_i \mathbb{Z}\right)^2$  and the rank of  $S^*(G)$  is  $2s$ .

So  $a_1$  divides  $S^*(x)$  for all  $x$  in  $G$ .

Now, we will construct an orthogonal complement  $(M_1 \oplus R_1)$  for  $\langle m_1, m_1^* \rangle$  in  $G$  such that:

- i)  $M = \langle m_1 \rangle \oplus M_1$ ,
- ii)  $\text{Ker } h_1 = M \oplus R_1$ .

Let  $M_1$  be the submodule of  $M$  generated by  $m_i' = m_i - a_1^{-1} S(m_i, m_1^*).m_1$ ,  $2 \leq i \leq s$ , and  $M \cap \text{Ker } S^*$ . By construction  $M_1$  is orthogonal to  $\langle m_1, m_1^* \rangle$  and  $M = \langle m_1 \rangle \oplus M_1$ .

By construction  $\text{Ker } h_1$  is orthogonal to  $m_1$  and  $M$  is in  $\text{Ker } h_1$ .

If  $\{x_i, i=2, \dots, s+r\}$  is a basis of any direct summand complement of  $M$  in  $\text{Ker } h_1$ , let  $R_1$  be the submodule of  $\text{Ker } h_1$  generated by  $x_i'$  where:  $x_i' = x_i - a_1^{-1} S(x_i, m_1^*).m_1$ . Then  $\text{Ker } h_1 = \langle m_1 \rangle \oplus M_1 \oplus R_1$  and  $R_1$  is orthogonal to  $m_1^*$ .

Now we have an orthogonal decomposition of  $G$  in  $\langle m_1, m_1^* \rangle \oplus^\perp (M_1 \oplus R_1)$ . By



induction on  $s$  we obtain an orthogonal decomposition:

$$G = (\oplus^\perp \langle m_i, m_i^* \rangle) \oplus^\perp (M_s \oplus R_s) \text{ where } \text{Ker } S^* = M_s \oplus R_s.$$

Let  $\{m_{s+1}, \dots, m_{s+r}\}$  be any basis of  $\text{Ker } S^* \cap M$ . Thanks to condition c.1,  $\text{Ker } S^* \cap M = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$ . So we can choose any basis  $\{m_{s+1}^*, \dots, m_{s+r}^*\}$  of  $\text{Ker } S_0^*$  to build up a basis of  $G$  which fulfills proposition (2.1).  $\square$

(2.2) Now, we use the notations established in §1 and the following convention: if  $f : R \rightarrow S$  is an isomorphism of  $\mathbb{Z}$ -modules,  $\Delta(f)$  is the submodule  $\{(x, f(x)); x \in R\}$  in  $R \oplus S$ . To prove theorem 1, we need the following proposition which gives an equivalent definition of algebraic cobordism.

**Proposition.** *Let  $A_0$  and  $A_1$  be in  $\mathcal{A}$ . Then  $A_0$  is algebraically cobordant to  $A_1$  if and only if there exists a pure submodule  $H$  of  $G = G_0 \oplus G_1$  on which  $A = A_0 \oplus -A_1$  vanishes, an isomorphism  $\varphi$  from  $\text{Ker } S_0^*$  to  $\text{Ker } S_1^*$  and an isomorphism  $\theta$  from  $\text{Tors}(\text{Coker } S_0^*)$  to  $\text{Tors}(\text{Coker } S_1^*)$  such that:*

- c.11:  $\Delta(\varphi) \subset H$ ,
- c.12: *the image  $\overline{H}$  of  $H$  in  $\overline{G} = G/\text{Ker } S^*$  is a metabolizer for  $\overline{S} = \overline{S_0} \oplus -\overline{S_1}$ ,*
- c.2:  $d(S^*(H)^\wedge) = \Delta(\theta)$ .

*Proof.* Let  $M, \varphi, \theta$  be as in definition (1.2). Then  $M$  satisfies c.1 and c.2. The existence of  $\varphi$  shows that  $\text{Ker } S_0^*$  and  $\text{Ker } S_1^*$  have the same rank,  $r$ . So the rank of  $\overline{G}$  is  $(m_0 + m_1 - 2r)$ . By c.1  $M \cap \text{Ker } S^* = \Delta(\varphi)$  and  $\text{rk}(M) = \frac{m_0 + m_1}{2}$  because  $M$  is a metabolizer for  $A$ . So  $\text{rk}(\overline{M}) = \frac{m_0 + m_1}{2} - r$  and  $\overline{S}$  vanishes on  $\overline{M}$ . It implies that  $\overline{M}$  is a metabolizer for  $\overline{S}$ .

Conversely let  $H, \varphi$  and  $\theta$  be as in the statement of proposition (2.1). As  $\Delta(\varphi)$  is pure in  $H$  and in  $\text{Ker } S^*$ , there exists a direct sum decomposition  $H \cap \text{Ker } S^* = \Delta(\varphi) \oplus M_0$ . As  $\text{Ker } S^*$  is pure in  $G$ , there exists also a direct sum decomposition  $H = M_1 \oplus (H \cap \text{Ker } S^*)$ . Let  $M$  be  $M_1 \oplus \Delta(\varphi)$ . By construction  $A$  vanishes on  $M$ ,  $M \cap \text{Ker } S^* = \Delta(\varphi)$  and  $S^*(M) = S^*(H)$ . So  $M, \varphi$  and  $\theta$  satisfy c.1 and c.2 of definition (1.2). Furthermore,  $\overline{H} = \overline{M_1} = \overline{M}$  and by c.12 the rank of  $\overline{H}$  is  $\frac{m_0 + m_1}{2} - r$ . But  $M_1$  being isomorphic to  $\overline{M_1}$ , the rank of  $M$  is  $\frac{m_0 + m_1}{2}$  and  $M$  is a metabolizer for  $A$ .  $\square$

(2.3) *Proof of theorem 1.* The only non trivial property to check is the transitivity of the relation "algebraic cobordism".

(2.4) **Lemma.** *Let  $B_i : G_i \times G_i \rightarrow \mathbb{Z}$  be in  $\mathcal{A}$ ,  $i = 0, 1, 2$ . Let  $m_i$  be the rank of  $G_i$ . If there exists a metabolizer  $H_{01}$  (resp.  $H_{12}$ ) for  $B_0 \oplus -B_1$  (resp.  $B_1 \oplus -B_2$ ) and if the  $B_i$  are non-degenerate, the form  $B_0 \oplus -B_2$  vanishes on  $H_{02} = \pi(L)$  and  $\text{rk } H_{02} = \frac{1}{2} \text{rk}(G_0 \oplus G_2)$ , where:  $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$ ,  $H = H_{01} \oplus H_{12}$ ,*

$\Delta = \{(y, y) \in G_1 \oplus G_1 ; y \in G_1\}$ ,  $L = H \cap (G_0 \oplus \Delta \oplus G_2)$  and  $\pi$  is the projection of  $G$  on  $G_0 \oplus G_2$ .

*Proof.* As  $B_0 \oplus -B_2$  vanishes on  $H_{02}$  by construction, it is sufficient to prove that the rank of  $H_{02}$  is  $\frac{m_0+m_1}{2}$ . The definition of  $H_{02}$  gives the following exact sequence:

$$0 \rightarrow L \cap \Delta \xrightarrow{i} L \xrightarrow{\pi} H_{02} \rightarrow 0.$$

So we get:

$$(*) \quad \text{rk}(L) = \text{rk}(L \cap \Delta) + \text{rk}(H_{02}).$$

If  $v$  is in  $H$ , there exists unique  $x$  in  $G_0$ ,  $y_1$  and  $y_2$  in  $G_1$  and  $z$  in  $G_2$  such that  $v = (x, y_1, y_2, z)$ . Let  $\rho : H \rightarrow G_1 \oplus G_1$  be defined by  $\rho(v) = (y_1 - y_2, 0)$ . Let us denote by  $L_1$  the image  $\rho(H)$ . By construction  $L$  is the kernel of  $\rho$  and we get the exact sequence:  $0 \rightarrow L \xrightarrow{i} H \xrightarrow{\rho} L_1 \rightarrow 0$ . Both this sequence and  $(*)$  show:

$$(**) \quad \frac{m_0 + m_2 + 2m_1}{2} - \text{rk}(L_1) = \text{rk}(L \cap \Delta) + \text{rk}(H_{02}).$$

**Claim.** By  $(B_1 \oplus -B_1)$ ,  $\Delta \cap L$  is orthogonal to  $L_1 \oplus \Delta$ .

Indeed,  $\Delta$  is self-orthogonal ; if  $(y, y)$  is in  $\Delta \cap L$ , then  $(0, y)$  is in  $H_{01}$  and  $(y, 0)$  is in  $H_{12}$ . On the other hand, an element of  $L_1$  is of the form  $(y_1, -y_2)$  where there exists  $(x, y_1)$  in  $H_{01}$  and  $(y_2, z)$  in  $H_{12}$ . So  $B_1(y, y_1) = B_1(y_1, y) = 0$  and  $-B_1(y, y_2) = -B_1(y_2, y) = 0$ .

The rank of  $L_1 \oplus \Delta$  is  $m_1 + \text{rk}(L_1)$ . The claim implies that the rank of the restriction of  $B_1 \oplus -B_1$  to  $(\Delta \cap L) \times (G_1 \oplus G_1)$  is smaller or equal to  $m_1 - \text{rk}(L_1)$ . But  $B_1 \oplus -B_1$  is non-degenerate by hypothesis, so:  $\text{rk}(\Delta \cap L) \leq m_1 - \text{rk}(L_1)$ . By  $(**)$  it implies:  $\frac{m_0+m_2}{2} \leq \text{rk}(H_{02})$ .

As  $B_0$  and  $B_2$  are non-degenerate by hypothesis and as  $B_0 \oplus -B_2$  vanishes on  $H_{02}$ ,  $\text{rk}(H_{02}) \leq \frac{m_0+m_2}{2}$ . It ends the proof of the lemma.  $\square$

Let us go back to the proof of theorem 1. Let  $A_i$  be algebraically cobordant to  $A_{i+1}$ ,  $i = 0, 1$ . Let  $M_{i,i+1}$  be a metabolizer for  $A_i \oplus -A_{i+1}$  with the isomorphisms  $\varphi_i$  and  $\theta_i$  fulfilling conditions c.1 and c.2 in definition (1.2).

Let us take the following notations:  $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$ ,  $S_{02} = S_0 \oplus -S_2$ ,  $G_{02} = G_0 \oplus G_2$ ,  $S = S_0 \oplus -S_1 \oplus S_1 \oplus -S_2$ ,  $\Delta = \{(x, x) ; x \in G_1\} \subset G_1 \oplus G_1$ ,  $d$  be the quotient map from  $G$  to  $\text{Coker } S^*$  and  $d_{02}$  the quotient map from  $G_{02}^*$  to  $\text{Coker } S_{02}^*$ . Let  $\pi$  (resp.  $\tilde{\pi}$ ) be the obvious projection from  $G$  (resp.  $\text{Coker } S^*$ ) to  $G_0 \oplus G_2$  (resp.  $\text{Coker } S_{02}^*$ ). Since  $\overline{M}_{i,i+1}$  is pure in  $\overline{G}_i \oplus \overline{G}_{i+1}$  we have the following decompositions  $M_{i,i+1}^\perp = \Delta(\varphi_i) \oplus \text{Ker } S_i^* \oplus R_{i,i+1}$  with  $M_{i,i+1} = \Delta(\varphi_i) \oplus R_{i,i+1}$ , and  $\overline{R}_{i,i+1}$  is pure in  $\overline{G}_i \oplus \overline{G}_{i+1}$ . Let  $Q_{i,i+1}$  be any direct summand complement of  $M_{i,i+1}^\perp$  in  $\overline{G}_i \oplus \overline{G}_{i+1}$ . If  $T_{i,i+1} = R_{i,i+1} \oplus Q_{i,i+1}$ , then we have the following decomposition  $G = \text{Ker } S_{01}^* \oplus \text{Ker } S_{12}^* \oplus T_{01} \oplus T_{12}$ . Let us denote by  $T_0$  (resp.  $T_1$ ,  $T_1'$ ,  $T_2$ ) the projection of  $T_{01}$  (resp.  $T_{01}$ ,  $T_{12}$ ,  $T_{12}$ ) to  $G_0$  (resp.  $G_1$ ,  $G_1$ ,  $G_2$ ). We

modify  $R_{12}$  and  $Q_{12}$  by adding to them some elements of  $\Delta(\varphi_1)$  in order to have  $T_1 = T'_1$ . Moreover, we have the following equalities:  $G_i = \text{Ker } S_i^* \oplus T_i$   $i = 0, 1, 2$ .

Let  $T_{02}$  be  $T_{02} = \pi(T_{01} \oplus T_{12}) = T_0 \oplus T_2$ . Let  $R_{02}$  be the smallest pure submodule of  $T_{02}$  which contains the projection of  $(R_{01} \oplus R_{12}) \cap (G_0 \oplus \Delta \oplus G_2)$  on  $T_{02}$ :  $R_{02} = (\pi((R_{01} \oplus R_{12}) \cap (G_0 \oplus \Delta \oplus G_2)))^\wedge$ ; and let  $A$  be  $A_0 \oplus -A_2$ ,  $\varphi$  be  $\varphi_1 \circ \varphi_0$  and  $\theta$  be  $-(\theta_1 \circ \theta_0)$ .

By proposition (2.2), to prove that  $A_0$  is algebraically cobordant to  $A_2$  it is sufficient to prove that  $H = \Delta(\varphi) \oplus R_{02}$  is a metabolizer for  $A_0 \oplus -A_2$ , and,  $H$  fulfill conditions c.11, c.12 and c.2 of (2.2). First we remark that  $H$  fulfills c.11 by definition.

(2.5) **Lemma.** *We have the equality  $d_{02}(S_{02}^*(H)^\wedge) = \Delta(-\theta_1 \circ \theta_0)$ .*

(2.6) **Lemma.** *The submodule  $H$  is a metabolizer for  $A$ , and  $\overline{H}$  is a metabolizer for  $\overline{S_0} \oplus -\overline{S_2}$ .*

*Proof of lemma (2.5).* By construction:  $d(S^*(G)^\wedge) = \text{Tors}(\text{Coker } S^*)$  and  $d_{02}(S_{02}^*(H)^\wedge) = \tilde{\pi}(d(S^*(L)^\wedge))$ . But c.2 implies:

$d(S^*(L)^\wedge) = (\Delta(\theta_0) \oplus \Delta(\theta_1)) \cap d(S^*(G_0 \oplus \Delta \oplus G_2)^\wedge)$ , so:

$d(S^*(L)^\wedge) = \{(x, \theta_0(x), y, \theta_1(y)); x \in \text{Tors}(\text{Coker } S_0^*), y = -\theta_0(x)\}$ .

Finally:  $d_{02}(S_{02}^*(H)^\wedge) = \{(x, -\theta_1 \circ \theta_0(x)); x \in \text{Tors}(\text{Coker } S_0^*)\} = \Delta(-\theta_1 \circ \theta_0)$ .  $\square$

*Proof of lemma (2.6).* The restriction  $S_{i,i+1}|_{T_{i,i+1}}$  on  $T_{i,i+1}$ , of the  $\varepsilon$ -symmetric bilinear form  $S_{i,i+1}$ , is non-degenerate; and the submodule  $R_{i,i+1}$  is a metabolizer for  $S_{i,i+1}|_{T_{i,i+1}}$ ,  $i = 0, 1$ . By construction  $T_0$  (resp.  $T_1, T_2$ ) is the projection of  $T_{01}$  (resp.  $T_{01}, T_{12}$ ) onto  $G_0$  (resp.  $G_1, G_2$ ). So we have  $S_{i,i+1}|_{T_{i,i+1}} = S_i|_{T_i} \oplus -S_{i+1}|_{T_{i+1}}$ . We use lemma (2.4) replacing  $B_i$  by  $S_i|_{T_i}$ , so  $S_{02}|_{T_{02}}$  vanishes on  $R_{02}$  and  $\text{rk } R_{02} = \frac{1}{2} \text{rk } T_{02}$ . Since the pure submodule  $H$  of  $G_{02} = \text{Ker } S_{02}^* \oplus T_{02}$  is defined by the equality  $H = \Delta(\varphi) \oplus R_{02}$  then  $\text{rk } H = \frac{1}{2} \text{rk } G_{02}$ . Moreover for all  $h_1, h_2$  in  $H$  there exist two integers  $a_1$  and  $a_2$  such that for  $i = 1, 2$  we have:  $a_i h_i = \pi(m_i)$  and  $m_i = (x_i, \varphi_0(x_i), \varphi_0(x_i), \varphi(x_i)) + (m_{0,i}, m_{1,i}, m_{1,i}, m_{2,i})$  is in  $M_{01} \oplus M_{12}$ . So  $A(h_1, h_2) = \frac{1}{a_1 a_2} (A_{01} \oplus -A_{12})(m_1, m_2) = 0$ , so  $A$  vanishes on the pure submodule  $H$  of  $G_{02}$ . Finally  $H$  is a metabolizer for  $A$ . By construction  $S_{02}|_{T_{02}}$  is isomorphic to  $\overline{S_{02}}$ , so as  $R_{02}$  is pure in  $T_{02}$  then  $\overline{R_{02}}$  is a metabolizer for  $\overline{S_{02}}$ .  $\square$

The above properties of  $H$ , and, lemmas (2.5) and (2.6) imply conditions c.12 and c.2 of proposition (2.2), and  $A_0$  is algebraically cobordant to  $A_2$ . This ends the proof of theorem 1.  $\square$

### 3. The necessary condition to have a cobordism

Let  $K_0$  and  $K_1$  be two cobordant links. Let us denote by  $\mathcal{S}$  the product  $S^{2n+1} \times [0, 1]$  and by  $\Sigma$  its oriented boundary. The definition of cobordism gives a submanifold  $C = \Phi(\mathcal{K} \times [0, 1])$  of  $\mathcal{S}$  such that  $\Sigma \cap C = K_0 \amalg (-K_1)$ . Let  $N$  be  $F_0 \cup C \cup (-F_1)$  where  $F_i$  is a Seifert surface for  $K_i$ . By construction  $N$  is a closed, compact, oriented,  $2n$ -submanifold of  $\mathcal{S}$ .

(3.1) **Lemma.** *There exists a smooth oriented, compact, submanifold  $W$  of  $\mathcal{S}$  such that  $N$  is the boundary of  $W$ .*

*Proof.* This lemma is a consequence of classical obstruction theory. If  $n \geq 3$  a proof is written in [L2, 70], p. 183. As the existence of  $W$  is fundamental to obtain theorem 2, we write a proof which works in any dimension.

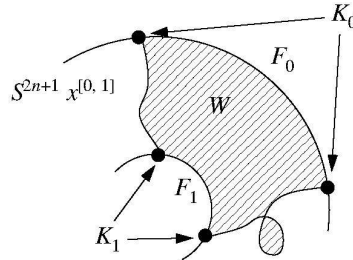
Let  $C_j$  for  $j = 1, \dots, k$  be the  $k$  connected components of  $C$ . As  $C$  has a trivial normal bundle in  $\mathcal{S}$ , it is possible to choose disjoint, closed, tubular neighbourhoods  $U_j$  of  $C_j$  and a diffeomorphism  $\Psi : C \times D^2 \rightarrow U = \coprod_{1 \leq j \leq k} U_j$ .

Now we have meridians  $m_j$  on  $\partial U_j$  defined by:  $m_j = \Psi(P_j \times S^1)$  where  $P_j$  is some point of  $C_j$  and  $m_j$  is oriented such that the linking number of  $m_j$  and  $C_j$  (in  $\mathcal{S}$ ) is  $+1$ . Let  $X$  be  $S \setminus \overset{\circ}{U}$ ,  $v$  be the diffeomorphism induced by the inclusion of  $\partial X$  in  $U$ ,  $e$  be the excision isomorphism and  $\partial^i$  (resp.  $\partial_X^i$ ) be the connectant homomorphism for the pair  $(\mathcal{S}, U)$  (resp.  $(X, \partial X)$ ). Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \xrightarrow{\partial_X^0} & H^1(X, \partial X) & \xrightarrow{\rho} & H^1(X) & \xrightarrow{\sigma} & H^1(\partial X) & \xrightarrow{\partial_X^1} & H^2(X, \partial X) & \rightarrow \\
 & \cong \uparrow e & & \uparrow & & v \uparrow & & \cong \uparrow e & \\
 \xrightarrow{\partial^0} & H^1(\mathcal{S}, U) & \rightarrow & 0 = H^1(\mathcal{S}) & \rightarrow & H^1(U) & \xrightarrow{\cong \partial^1} & H^2(\mathcal{S}, U) & \rightarrow 0
 \end{array}$$

The commutativity of all the squares of the above diagram implies that the homomorphism  $\rho$  is zero so  $\sigma$  is injective and  $\partial_X^i$  is surjective for  $0 \leq i \leq 2n-1$ . We have the following direct sum decomposition:  $H^1(\partial X) = \sigma(H^1(X)) \oplus v(H^1(U))$ . Any element of  $\sigma(H^1(X))$  is represented by a differentiable map from  $\partial X$  to  $S^1$ , which is, up to homotopy, characterized by its degree on each meridian  $m_j$ , and which has a unique extension to  $X$ . Let  $g : X \rightarrow S^1$  be the unique, up to homotopy, differentiable map which has degree  $+1$  on each meridian. Thanks to the Thom-Pontriagin construction there exists a differentiable map  $f : \Sigma \setminus (K_0 \amalg -K_1) \rightarrow S^1$  which has  $\overset{\circ}{F}_0 \amalg (-\overset{\circ}{F}_1)$  as regular fiber and  $f$  has degree  $+1$  on the meridians of the connected components of  $K_0 \amalg (-K_1)$ . So  $f$  and  $g$  have homotopic restrictions on  $X \cap \Sigma$  and we can choose  $g$  such that its restriction on  $X \cap \Sigma$  coincides with  $f$ .

Then  $g$  has a regular fiber  $\overline{W}$  such that  $\overline{W} \cap \Sigma = (F_0 \amalg -F_1) \cap X$ . The union of  $\overline{W}$  with a small collar in  $U$  is the manifold  $W$  such that  $N = \partial W$ .  $\square$



(3.2) Let us take  $A_0$  (resp.  $A_1$ ) the Seifert form associated to a  $(n - 1)$ -connected Seifert surface  $F_0$  (resp.  $F_1$ ) for  $K_0$  (resp.  $K_1$ ). Let  $\tau : K_0 \rightarrow K_1$  be the diffeomorphism defined by:  $\tau(P) = \Phi(\Phi^{-1}(P) \times \{1\})$  where  $P$  is any point of  $K_0$ . The diffeomorphism  $\tau$  induces isomorphisms  $\theta_j : H_j(K_0) \rightarrow H_j(K_1)$  such that for any  $j$ -cycle  $x$  of  $K_0$ ,  $(x, \theta_j(x))$  is a boundary in  $C = \Phi(K \times [0, 1])$ . Let  $\chi_i : H_n(K_i) \rightarrow H_n(F_i)$  and  $\lambda_i : H_n(F_i) \rightarrow H_n(N)$ ,  $i = 0, 1$ , be the homomorphisms induced by the inclusions  $K_i \subset F_i \subset N$ . The Mayer-Vietoris exact sequence associated to the decomposition of  $N$  in the union of  $F_0 \cup C$  and  $C \cup (-F_1)$  gives:

$$\rightarrow H_n(K_0) \xrightarrow{\chi} H_n(F_0) \oplus H_n(F_1) \xrightarrow{\lambda} H_n(N) \xrightarrow{\delta} H_{n-1}(K_0) \rightarrow$$

where  $\chi = (\chi_0, \chi_1 \circ \theta_n)$  and  $\lambda = (\lambda_0, \lambda_1)$

(3.3) **Remark.** Let  $m_i$  be  $\text{rk}(H_n(F_i))$ ,  $m$  be  $\text{rk}(H_n(N))$  and  $r$  be  $\text{rk}(\chi(H_n(K_0)))$ . By Poincaré duality  $m = m_0 + m_1$ ,  $r = \text{rk}(\delta(H_n(N)))$  and  $r = \text{rk}(\text{Ker } S_i^*)$  where  $S_i^*$  is the adjoint of the intersection form  $S_i$  on  $H_n(F_i)$ .

(3.4) Construction of the isomorphisms  $\varphi : \text{Ker } S_0^* \rightarrow \text{Ker } S_1^*$  and  $\theta : \text{Tors}(\text{Coker } S_0^*) \rightarrow \text{Tors}(\text{Coker } S_1^*)$ .

Let  $S_{i*} : H_n(F_i) \rightarrow H_n(F_i, K_i)$  and  $\partial : H_n(F_i, K_i) \rightarrow H_{n-1}(K_i)$  be the homomorphisms given by the long exact sequence for the pair  $(F_i, K_i)$ . Let  $U : H^n(F_i) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(F_i); \mathbb{Z})$  be the universal coefficient isomorphism ( $F_i$  is  $(n - 1)$ -connected) and let  $P : H_n(F_i, K_i) \rightarrow H^n(F_i)$  be the Poincaré duality isomorphism. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \chi_i(H_n(K_i)) & \rightarrow & H_n(F_i) & \xrightarrow{S_{i*}} & H_n(F_i, K_i) & \xrightarrow{\partial} & \partial(H_n(F_i, K_i)) & \rightarrow & 0 \\ & & \parallel & & \parallel & & \cong \downarrow U \circ P & & \downarrow \Delta_i & & \\ 0 & \rightarrow & \text{Ker } S_i^* & \rightarrow & H_n(F_i) & \xrightarrow{S_i^*} & \text{Hom}_{\mathbb{Z}}(H_n(F_i); \mathbb{Z}) & \xrightarrow{d} & \text{Coker } S_i^* & \rightarrow & 0 \end{array}$$

By definition  $\Delta_i : \partial(\mathbb{H}_n(F_i, K_i)) \rightarrow \text{Coker } S_i^*$  is the quotient of the isomorphism  $U \circ P$ , so  $\Delta_i$  is an isomorphism.

Let us consider again the isomorphism  $\theta_j : \mathbb{H}_j(K_0) \rightarrow \mathbb{H}_j(K_1)$ , which is defined in (3.2) thanks to the existence of the cobordism. Since  $F_i$  is  $(n-1)$ -connected then  $\partial(\mathbb{H}_n(F_i, K_i)) = \tilde{\mathbb{H}}_{n-1}(K_i)$  and  $\theta_n(\text{Ker } \chi_0) = \text{Ker } \chi_1$ , so  $\theta_{n-1} \circ \partial(\mathbb{H}_n(F_0, K_0)) = \partial(\mathbb{H}_n(F_1, K_1))$ .

Let  $\theta$  be the restriction of the isomorphism  $\Delta_1 \circ \theta_{n-1} \circ \Delta_0^{-1}$  on the  $\mathbb{Z}$ -torsion of  $\text{Coker } S_0^*$ .

Let  $\varphi$  be the restriction of  $\theta_n$  on  $\chi_0(\mathbb{H}_n(K_0))$ . As  $\chi_i(\mathbb{H}_n(K_i)) = \text{Ker } S_i^*$ , so  $\varphi$  is defined on  $\text{Ker } S_0^*$ .

We denote by  $\Delta(\varphi)$  the submodule  $\{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$  of  $\mathbb{H}_n(F_0) \oplus \mathbb{H}_n(F_1)$ .

(3.5) **Remark.** By construction  $\varphi$  fulfills:  $\varphi \circ \chi_0 = \chi_1 \circ \theta_n$  and  $\Delta(\varphi) = \chi(\mathbb{H}_n(K_0))$  where  $\chi = (\chi_0, \chi_1 \circ \theta_n)$  as in (3.2).

(3.6) To prove theorem 2, we will construct a metabolizer  $M$  (in  $\mathbb{H}_n(F_0 \amalg -F_1)$ ) for  $A = A_0 \oplus -A_1$ . This metabolizer  $M$  will fulfill conditions c.1 and c.2 in definition (1.2) of the algebraic cobordism, for the isomorphisms  $\varphi$  and  $\theta$  defined in (3.4). To do that, we have to choose an oriented submanifold  $W$  of  $\mathcal{S}$  with  $\partial(W) = N$  (thanks to (3.1) such a  $W$  exists). Let  $j : \mathbb{H}_n(N) \rightarrow \mathbb{H}_n(W)$  be the homomorphism induced by the inclusion of  $N$  in  $W$ .

(3.7) **Lemma.** *The form  $A = A_0 \oplus -A_1$  vanishes on  $\lambda^{-1}(\text{Ker } j^\wedge)$ .*

*Proof.* It is sufficient to prove that  $A$  vanishes on  $\lambda^{-1}(\text{Ker } j)$ . Let  $a = [x]$  and  $b = [y]$  be two homology classes in  $\lambda^{-1}(\text{Ker } j)$ . As  $\lambda$  is induced by the inclusion of  $F_0 \amalg -F_1$  in  $N$  (see (3.2)), there exists two  $(n+1)$ -chains  $\alpha$  and  $\beta$  in  $W$  such that  $\partial\alpha = x$  and  $\partial\beta = y$ . Let  $i_+$  be the positively oriented normal vector field to  $W$  in  $\mathcal{S}$ . The intersection of  $\alpha$  and  $i_+(\beta)$  is zero. Hence the linking number in  $\Sigma$  of  $x$  and  $i_+(y)$  is zero. But this linking number is, by definition, equal to  $A(a, b)$ , so  $A(a, b) = 0$  and the lemma is proved.  $\square$

(3.8) **Lemma.** *Let  $m$  be the rank of  $\mathbb{H}_n(N)$ . The rank of  $\text{Ker } j$  is  $\frac{m}{2}$ .*

*Proof.* The long exact sequence for the pair  $(W, N)$  gives the exactness of:

$$0 \rightarrow \mathbb{H}_{2n+1}(W) \rightarrow \mathbb{H}_{2n+1}(W, N) \rightarrow \mathbb{H}_{2n}(N) \rightarrow \dots \rightarrow \mathbb{H}_{n+1}(W, N) \rightarrow \text{Ker } j \rightarrow 0$$

The alternating sum of the ranks in this exact sequence together with the Poincaré duality give:

$$\text{rk}(\text{Ker } j) = \frac{\text{rk}(\mathbb{H}_n(N))}{2} = \frac{m}{2}.$$

$\square$

(3.9) **Lemma.** *There exists a direct summand decomposition of  $\lambda^{-1}(\text{Ker } j^\wedge)$  in*

$\Delta(\varphi) \oplus R_0 \oplus R$  where  $\Delta(\varphi) = \{(x, \varphi(x)); x \in \text{Ker } S_0^*\}$ ,  $R_0 = \lambda^{-1}(\text{Ker } j^\wedge) \cap \text{Ker } S_0^*$ , and  $R$  is any direct summand complement of  $\lambda^{-1}(\text{Ker } j^\wedge) \cap \text{Ker } S^*$  in  $\lambda^{-1}(\text{Ker } j^\wedge)$ .

*Proof.* As the considered submodules of  $\lambda^{-1}(\text{Ker } j^\wedge)$  are pure, the lemma comes from the following equalities:

$$\begin{aligned} \chi(\text{H}_n(K_0)) &= \text{Ker } \lambda \subset \lambda^{-1}(\text{Ker } j^\wedge) \text{ (see (3.2)),} \\ \Delta(\varphi) &= \chi(\text{H}_n(K_0)) \text{ (see (3.5)),} \\ \text{Ker } S^* &= \chi(\text{H}_n(K_0)) \oplus \text{Ker } S_0^*. \end{aligned} \quad \square$$

(3.10) **Proposition.** *The submodule  $M = \Delta(\varphi) \oplus R$  of  $\lambda^{-1}(\text{Ker } j^\wedge)$  is a metabolizer for  $A = A_0 \oplus -A_1$ , which fulfills:  $M \cap \text{Ker } S^* = \Delta(\varphi)$ .*

*Proof.* By lemma (3.9),  $M \cap \text{Ker } S^* = \Delta(\varphi)$ . By (3.6),  $A$  vanishes on  $M$ . So we only have to show that  $M$  is of rank  $\frac{m}{2}$ . As remarked in (3.3),  $r = \text{rk}(\delta(\text{H}_n(N)))$ , so  $\text{rk}(\delta(\text{Ker } j^\wedge)) \leq r$ . Let us consider the following exact sequence induced by (3.2):  $0 \rightarrow \Delta(\varphi) \xrightarrow{\lambda} \lambda^{-1}(\text{Ker } j^\wedge) \xrightarrow{\lambda} \text{Ker } j^\wedge \xrightarrow{\delta} \delta(\text{Ker } j^\wedge) \rightarrow 0$ . This exact sequence together with the equalities:  $\text{rk}(\text{Ker } j^\wedge) = \frac{m}{2}$  (see (3.8)),  $\text{rk}(\Delta(\varphi)) = r$ ; give  $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) = r + \frac{m}{2} - \text{rk}(\delta(\text{Ker } j^\wedge))$ . So  $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) \geq \frac{m}{2}$ .

We can remark that if  $A$  is non degenerated (as supposed in theorem 2) then we have  $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) \leq \frac{1}{2}\text{rk}(\text{H}_n(F_0) \oplus \text{H}_n(F_1)) = \frac{m}{2}$ , because  $A$  vanishes on  $\lambda^{-1}(\text{Ker } j^\wedge)$  (see (3.6)). So, if  $A$  is non degenerated,  $\text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) = \frac{m}{2}$ ,  $\text{rk}(\delta(\text{Ker } j^\wedge)) = r$ ,  $\text{rk}(R_0) = 0$  and  $M = \lambda^{-1}(\text{Ker } j^\wedge)$  is a metabolizer for  $A$ .

Come back to the general case. Let  $r_0$  be the rank of  $R_0$ . By construction:  $\text{rk}(M) = \text{rk}(\lambda^{-1}(\text{Ker } j^\wedge)) - r_0 = r + \frac{m}{2} - \text{rk}(\delta(\text{Ker } j^\wedge)) - r_0$ .

(3.11) **Lemma.** *The rank  $l$  of  $\delta(\text{H}_n(N))/\delta(\text{Ker } j^\wedge)$  is greater or equal to  $r_0$ .*

*Proof.* Let  $\{e_j\}$ ,  $j = 1, \dots, r_0$  be a basis of  $R_0$ . Let  $\{e_j^*\}$  be in  $\text{H}_n(N) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $S_N(\lambda(e_j), e_j^*) = \delta_{ij}$  where  $S_N$  is the intersection form on  $\text{H}_n(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ . The  $e_j^*$  exists because  $S_N$  is unimodular. Let  $R^*$  be the submodule of  $\text{H}_n(N) \otimes_{\mathbb{Z}} \mathbb{Q}$  generated by  $\{e_j^*\}$ . Since  $R_0 \cap \text{Ker } \lambda = \{0\}$ , then  $\text{rk}(\lambda(R_0)) = r_0$ . As  $S$  vanishes on  $R_0$ , then  $S_N$  vanishes on  $\lambda(R_0)$ . It implies that  $\text{rk}(R^*) = \text{rk}(R_0) = r_0$ , and  $\text{Ker } j \cap R^* = \{0\}$ . Since  $R_0 \subset \text{Ker } S_0^*$ , we have  $S(x, y) = 0$  for all  $x$  in  $R_0$  and all  $y$  in  $\text{H}_n(F_0 \amalg -F_1)$ . So  $R^* \cap \lambda(\text{H}_n(F_0 \amalg -F_1)) = \{0\}$  and  $\text{rk}(\delta(\text{H}_n(N))/\delta(\text{Ker } j^\wedge)) = l \geq \text{rk}(\delta(R^*)) = \text{rk}(R^*) = r_0$ . □

In order to end the proof of (3.10), we only have to show that  $\text{rk}(R) = \frac{m}{2} - r$ . But  $\text{rk}(\delta(\text{Ker } j^\wedge)) = r - l$ ; so we already have shown that  $\text{rk}(R) = \text{rk}(M) - r = \frac{m}{2} - (r - l) - r_0$ .

By lemma (3.11), we have  $l - r_0 \geq 0$ , so  $\text{rk}(R) \geq \frac{m}{2} - r$ . But  $R \cap \text{Ker } S^* = \{0\}$  by construction, and the form  $\bar{S}$  induced by  $S$  on  $\text{H}_n(F_0 \amalg -F_1)/\text{Ker } S^*$  is non-

degenerate of rank  $m - 2r$ . So  $\text{rk}(R) \leq \frac{m}{2} - r$  because  $\overline{S}$  vanishes on  $\overline{R} = R/(R \cap \text{Ker } S^*)$ .  $\square$

(3.12) **Remark.** We have found a metabolizer  $M = \Delta(\varphi) \oplus R$  for  $A$  which fulfills condition c.1 of the algebraic cobordism without any condition on  $A$ . We already have got theorem 4 (see (1.6)). To prove condition c.2 and  $\overline{M}$  is pure in  $\overline{G}$ , we will have to choose  $(n-1)$ -connected Seifert surfaces  $F_i$  for  $K_i$  on which the Seifert forms  $A_i$  are unimodular. So the following proposition (3.13) together with proposition (3.10) imply theorem 2 stated in (1.4).

Let  $\theta_{n-1}$  be the isomorphism between  $H_{n-1}(K_0)$  and  $H_{n-1}(K_1)$  defined in (3.2), and let  $\theta$  the isomorphism between  $\text{Tors}(\text{Coker } S_0^*)$  and  $\text{Tors}(\text{Coker } S_1^*)$  defined in (3.4). Using the notation of (2.2), let  $\Delta(\theta_{n-1})$  (resp.  $\Delta(\theta)$ ) be the group  $\{(x, \theta_{n-1}(x)) ; x \in \text{Tors}(H_{n-1}(K_0))\}$  (resp.  $\{(x, \theta(x)) ; x \in \text{Tors}(\text{Coker } S_0^*)\}$ ).

(3.13) **Proposition.** *If  $A_0$  and  $A_1$  are unimodular the metabolizer  $M = \Delta(\varphi) \oplus R$  of  $A = A_0 \oplus -A_1$ , fulfills  $d(S^*(M)^\wedge) = \Delta(\theta)$ ; and  $\overline{M}$  is pure in  $H_n(F)/\text{Ker } S^*$ .*

*Proof.* Let us denote  $F_0 \amalg -F_1$  by  $F$ ,  $K_0 \amalg -K_1$  by  $K$ , and  $S_0^* \oplus -S_1^*$  by  $S^*$ . We consider for  $F$  the following commutative diagram already constructed for  $F_i$  in (3.4):

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Ker } S_* & \hookrightarrow & H_n(F) & \xrightarrow{S_*} & H_n(F, K) & \xrightarrow{\partial} & \partial(H_n(F, K)) & \rightarrow & 0 \\ & & \parallel & & \parallel & & \cong \downarrow U \circ P & & \cong \downarrow \Delta_0 \oplus \Delta_1 & & \\ 0 & \rightarrow & \text{Ker } S^* & \hookrightarrow & H_n(F) & \xrightarrow{S^*} & \text{Hom}_{\mathbb{Z}}(H_n(F); \mathbb{Z}) & \xrightarrow{d} & \text{Coker } S^* & \rightarrow & 0 \end{array}$$

(3.14) **Lemma.** *The equality  $d(S^*(M)^\wedge) = \Delta(\theta)$  is equivalent to the equality  $\partial(S_*(M)^\wedge) = \Delta(\theta_{n-1})$ .*

*Proof.* The lemma is a consequence of the two following statements:

The restriction of  $\Delta_0 \oplus \Delta_1$  on  $\Delta(\theta_{n-1})$  is an isomorphism to  $\Delta(\theta)$  because  $\theta \circ \Delta_0 = \Delta_1 \circ \theta_{n-1}$  by construction (see (3.4)).

The restriction of  $\Delta_0 \oplus \Delta_1$  on  $\partial(S_*(M)^\wedge)$  is an isomorphism to  $d(S^*(M)^\wedge)$  because the commutativity of the above diagram gives  $U \circ P(S_*(M)^\wedge) = S^*(M)^\wedge$ .  $\square$

Let  $\kappa : H_n(N) \rightarrow H_n(N, C)$  be the homomorphism which is defined in the long exact sequence for the pair  $(N, C)$  and  $\rho : H_n(N, C) \rightarrow N_n(F, K)$  be the inverse of the excision isomorphism induced by the inclusion of the pair  $(F, K) \subset (N, C)$ . Let  $\xi = \rho \circ \kappa : H_n(N) \rightarrow H_n(F, K)$  and  $\overline{\theta} = (\text{Id}, \theta_{n-1}) : H_{n-1}(K_0) \rightarrow H_{n-1}(K)$ .



With the notations used in (3.2) we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 \rightarrow & H_n(K_0) & \xrightarrow{\chi} & H_n(F) & \xrightarrow{\lambda} & H_n(N) & \xrightarrow{\delta} & H_{n-1}(K_0) & \rightarrow \\
 (\star) & & & \parallel & \text{(I)} & \downarrow \xi & \text{(II)} & \downarrow \bar{\theta} & \\
 \rightarrow & H_n(K) & \xrightarrow{\chi_0 \oplus \chi_1} & H_n(F) & \xrightarrow{S_*} & H_n(F, K) & \xrightarrow{\partial} & H_{n-1}(K) & \rightarrow
 \end{array}$$

The square (I) is commutative by functoriality, and (II) is commutative by definition of  $\xi$  and  $\bar{\theta}$ .

(3.15) **Lemma.** *If  $A_0$  and  $A_1$  are unimodular, then we have  $\delta(\text{Ker } j^\wedge) = \tilde{H}_{n-1}(K_0)$ .*

We first show that lemma (3.15) implies proposition (3.13).

We show that  $\bar{M}$  is pure in  $H_n(F)/\text{Ker } S^*$ , which is equivalent to prove that the quotient  $H_n(F)/(\text{Ker } S^* + M)$  is torsion free. Since  $A = A_0 \oplus -A_1$  is non-degenerate  $M = \lambda^{-1}(\text{Ker } j^\wedge)$ . Furthermore by diagram  $(\star)$  we get  $\lambda(\text{Ker } S^*) = \text{Ker } \xi$ . Let  $\text{pr}$  be the projection of  $H_n(N)$  on  $H_n(N)/(\text{Ker } j^\wedge + \text{Ker } \xi)$ , so  $\text{Ker } (\text{pr} \circ \lambda) = M + \text{Ker } S^*$ . The quotient of  $\text{pr} \circ \lambda$  induces an injective map from  $H_n(F)/(\text{Ker } S^* + M)$  into  $H_n(N)/(\text{Ker } j^\wedge + \text{Ker } \xi)$ .

**Claim.** The module  $H_n(N)/(\text{Ker } j^\wedge + \text{Ker } \xi)$  is torsion free.

*Proof of the claim.* There exists  $x_i, i = 1, \dots, r$ , in  $\text{Ker } j^\wedge$  such that  $\tilde{H}_{n-1}(K_0) = \bigoplus_{i=1}^r \langle \delta(x_i) \rangle \oplus \text{Tors}(\tilde{H}_{n-1}(K_0))$ . Let  $(y_i)_{i=1, \dots, r}$  a basis of  $\text{Ker } \xi$  such that  $S_N(x_i, y_j) = \delta_{ij}$ . By induction on  $r$ , we can construct these bases such that  $H_n(N) = T \oplus^\perp T^\perp$  where  $T = \bigoplus_{i=1}^r \langle x_i, y_i \rangle$ . If we denote by  $D$  the module  $D = T^\perp \cap \text{Ker } j^\wedge$  and by  $D^*$  any direct summand complement of  $D$  in  $T^\perp$ , then we get:  $H_n(N)/(\text{Ker } \xi + \text{Ker } j^\wedge) \cong D^*$  which is torsion free. □

Finally  $H_n(F)/(\text{Ker } S^* + M)$  is torsion free and  $\bar{M}$  is pure in  $H_n(F)/(\text{Ker } S^*)$ .

So if  $n = 1$ , the links  $K_0$  and  $K_1$  have torsion free homology groups ( $\mathcal{K}$  is a one dimensional compact manifold), so  $\text{Tors}(\text{Coker } S^*) = \{0\}$  and we have already proved proposition (3.13).

Now let us take  $n \geq 2$ .

Thanks to lemma (3.14), the equality:  $\Delta(\theta_{n-1}) = \partial(S_*(M)^\wedge)$  gives proposition (3.13). The above diagram  $(\star)$  and lemma (3.15) imply:  $\bar{\theta}(H_{n-1}(K_0)) =$

$\Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge)$ . To show that the inclusion:  $\Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge)$  is an equality, it is sufficient to take any  $x$  in  $(\partial(S_*(M)^\wedge) \cap \partial(H_n(F_0, K_0)))$ , and to show that such a  $x$  is zero.

Let us denote by  $L$  (resp.  $L_i$ ) the linking form on  $\text{Tors}(H_{n-1}(K))$  (resp.  $\text{Tors}(H_{n-1}(K_i))$ ). By definition (see remark (3.16)) such a form  $L = L_0 \oplus -L_1$  is non degenerated and vanishes on  $\partial(S_*(M)^\wedge)$  because  $S_0 \oplus -S_1$  vanishes on  $M$ . Let  $(y, \theta_{n-1}(y))$  be in  $\Delta(\theta_{n-1})$ . Then  $L(x, (y, \theta_{n-1}(y))) = L_0(x, y) = 0$  for all  $y \in \text{Tors}(H_{n-1}(K_0))$ . The non degeneracy of  $L_0$  implies  $x = 0$ . This ends the proof of proposition (3.13).  $\square$

(3.16) **Remark.** The linking form  $L$  is defined as follows (see [L-L, 75] prop. 2.1): Let  $x, y$  be in  $\text{Tors}(H_{n-1}(K))$  such that  $p$  and  $q$  are the smallest positive integers with  $p.x = q.y = 0$ . Let  $\bar{x}$  and  $\bar{y}$  be in  $H_n(F)$  such that  $\partial(S_*(\bar{x}) \otimes \frac{1}{p}) = x$  and  $\partial(S_*(\bar{y}) \otimes \frac{1}{q}) = y$ . Then:  $L(x, y) \equiv \frac{1}{p.q} S(\bar{x}, \bar{y}) \pmod{\mathbb{Z}}$ .

*Proof of lemma (3.15).* As shown in (3.10), if  $A_0 \oplus -A_1$  is non degenerated,  $M = \lambda^{-1}(\text{Ker } j^\wedge)$  has rank  $\frac{m}{2}$  and is the chosen metabolizer. So  $\lambda$  induces a monomorphism  $\bar{\lambda}$  on  $H_n(F)/M$  to  $H_n(N)/\text{Ker } j^\wedge$  and we get the following exact sequence:

$$0 \rightarrow H_n(F)/M \xrightarrow{\bar{\lambda}} H_n(N)/\text{Ker } j^\wedge \xrightarrow{\bar{\delta}} \tilde{H}_{n-1}(K_0)/\delta(\text{Ker } j^\wedge) \rightarrow 0.$$

As  $\bar{\lambda}$  is injective and  $M$  is pure in  $H_n(F)$  there exists two  $\mathbb{Z}$ -bases  $\{\bar{e}_j; j=1, \dots, \frac{m}{2}\}$  of  $H_n(F)/M$  and  $\{\bar{k}_j; j=1, \dots, \frac{m}{2}\}$  of  $H_n(N)/\text{Ker } j^\wedge$  such that  $\bar{\lambda}(\bar{e}_j) = p_j \cdot \bar{k}_j$  with  $p_j \in \mathbb{Z} \setminus \{0\}$ . Let  $E$  (resp.  $H$ ) be a direct summand complement of  $M$  (resp.  $\text{Ker } j^\wedge$ ) in  $H_n(F)$  (resp.  $H_n(N)$ ). Let also  $\{e_j; j=1, \dots, \frac{m}{2}\}$  (resp.  $\{k_j; j=1, \dots, \frac{m}{2}\}$ ) be a  $\mathbb{Z}$ -basis of  $E$  (resp.  $H$ ) such that  $e_j \equiv \bar{e}_j \pmod{M}$  (resp.  $k_j \equiv \bar{k}_j \pmod{\text{Ker } j^\wedge}$ ). By construction  $\lambda(e_j) - p_j \cdot k_j = x \in \text{Ker } j^\wedge$ . So there exists a  $(n+1)$ -chain  $\gamma$  in  $W$  and a positive integer  $a$  such that:  $\partial\gamma = a\lambda(e_j) - a p_j \cdot k_j$ . Let  $\rho$  be a  $(n+1)$ -chain of  $S^{2n+1} \times [0, 1]$  with  $\partial\rho = k_j$ . So  $a e_j$  is the boundary of  $\gamma + a p_j \cdot \rho$  in  $S^{2n+1} \times [0, 1]$ .

Statement: for all  $m$  in  $M$ ,  $p_j$  divides  $A(e_j, m)$ .

Let  $m$  be in  $M = \lambda^{-1}(\text{Ker } j^\wedge)$  and  $\Delta$  be a  $(n+1)$ -chain in  $S^{2n+1} \times [0, 1]$  such that  $\partial\Delta = i_+(m)$ . By definition  $A(a e_j, m)$  is the intersection in  $S^{2n+1} \times [0, 1]$  of  $\gamma + a p_j \cdot \rho$  and  $\Delta$ . But  $\lambda(a m) \in \text{Ker } j$  so there exists a  $(n+1)$ -chain  $\mu$  in  $W$  such that  $\partial\mu = a m$ . We have  $\partial(i_+(\mu)) = a i_+(m)$ . Since  $\partial(a \Delta) = a i_+(m)$ , we get  $\gamma \cap (a \Delta) = \gamma \cap (i_+(\mu)) = 0$ . But  $a > 0$ , so  $a(\gamma \cap \Delta) = 0$  implies  $\gamma \cap \Delta = 0$ . Finally  $A(a e_j, m) = a p_j \cdot (\rho \cap \Delta)$  and  $p_j$  divides  $A(e_j, m)$ .

If  $A$  is unimodular the statement implies that  $p_j = \pm 1$  for all  $j = 1, \dots, \frac{m}{2}$ . So  $\bar{\lambda}$  is an isomorphism and his cokernel is zero. As asked we have got:  $\delta(\text{Ker } j^\wedge) = \tilde{H}_{n-1}(K_0)$ . This ends the proof of lemma (3.15).  $\square$

(3.17) **Remark.** As above we can also prove that: for all  $m$  in  $M$   $p_j$  divides  $A(m, e_j)$ .

**4. The sufficient condition to have a cobordism**

(4.1) Let  $K_0$  and  $K_1$  be two  $2n - 1$  dimensional simple links, with  $n \geq 3$ . We suppose that there exists  $(n - 1)$ -connected Seifert surfaces  $F_0$  and  $F_1$ , for  $K_0$  and  $K_1$ , such that the associated Seifert forms  $A_0$  and  $A_1$  are algebraically cobordant. We consider  $K_0$  (resp.  $-K_1$ ) as embedded in the sphere  $S^{2n+1} \times \{0\}$  (resp.  $S^{2n+1} \times \{1\}$ ) which are oriented as the boundary of  $S^{2n+1} \times [0, 1]$ .

Let  $x$  be in  $S^{2n+1} \times \{0\}$  such that  $(x \times [0, 1]) \cap (F_0 \amalg -F_1)$  is empty, and let  $U$  be a "small" open ball around  $x$  in  $S^{2n+1} \times \{0\}$ . The boundary  $S$  of the disk  $D = (S^{2n+1} \times [0, 1]) \setminus (U \times [0, 1])$  contains  $F_0 \amalg -F_1$ . Let  $G$  be the closure of the connected sum, in  $S$ , of the interiors  $\overset{\circ}{F}_0$  and  $-\overset{\circ}{F}_1$ . By construction  $A = A_0 \oplus -A_1$  is the Seifert form of  $K_0 \amalg -K_1$ , associated to  $G$ .

(4.2) *Proof of theorem 3.* In order to prove theorem 3 we will do in  $D$ , an embed surgery on  $G$ , the result of which being a manifold  $\tilde{G}$  diffeomorphic to  $\mathcal{K} \times [0, 1]$ .

By proposition (2.1) we can choose a good basis  $\mathcal{B} = \{(m_i, m_i^*); i=1, \dots, s+r\}$  of  $H_n(G)$ . Thanks to J. Milnor ([M1, 61] lemma 6 p. 50), any cycle of  $G$  can be represented by the image of an embedding of  $S^n$ . Furthermore:

(4.3) **Proposition.** *There exists  $s+r$  disjoint embeddings  $\psi_i : D^{n+1} \times D^n \rightarrow D$  such that for any  $i \in \{1, \dots, s+r\}$  we have*

- 1-  $[\psi_i(S^n \times \{0\})] = m_i$ ,
- 2-  $(\psi_i(D^{n+1} \times D^n)) \cap G = \psi_i(D^{n+1} \times D^n) \cap S = \psi_i(S^n \times D^n)$ .

*Proof.* Let  $\overline{\psi}_i : S^n \rightarrow G$  be an embedding of  $S^n$  which represents  $m_i$ . Let  $i, j$  with  $i \neq j$ , be in  $\{1, \dots, s+r\}$ , then  $m_i$  and  $m_j$  are in the metabolizer  $M$  and we have:  $S(m_i, m_j) = A(m_i, m_j) + (-1)^n A(m_j, m_i) = 0$ . Since  $n \geq 3$ , thanks to Whitney's procedure [Wh, 44] we can choose the  $\overline{\psi}_i$  such that  $\overline{\psi}_i(S^n) \cap \overline{\psi}_j(S^n) = \emptyset$ . Since  $n \geq 2$ , the Whitney obstruction to extend  $\overline{\psi}_i$  to disjoint embeddings  $\psi_i$  of  $D^{n+1}$  in the  $(2n+2)$ -disk  $D$ , is the matrix  $A(m_i, m_j)$  which is zero. Furthermore,  $A(m_i, m_i) = 0$  is the classical obstruction to extend  $\psi_i$  to  $\psi_i : D^{n+1} \times D^n \rightarrow D$ . (see [Br, 72] and for details see [Bl, 94] proposition 5.1.2, p.58). We choose this extension  $\psi_i$  such that the restriction to  $S^n \times D^n$  is a tubular neighbourhood of  $\psi_i(S^n)$  in  $G$ . □

So thanks to proposition (4.3) we obtain a submanifold  $\tilde{G}$  of  $D$  as follows:

$$\tilde{G} = (G \setminus (\prod_{i=1}^{s+r} \psi_i(S^n \times D^n))) \cup (\prod_{i=1}^{s+r} \psi_i(D^{n+1} \times S^{n-1})).$$

(4.4) **Proposition.** *The inclusion  $k_0$  (resp.  $k_1$ ) of  $K_0$  (resp.  $K_1$ ) in  $\tilde{G}$ , induces isomorphisms  $k_{0,j}$  (resp.  $k_{1,j}$ ) from  $H_j(K_0)$  (resp.  $H_j(K_1)$ ) to  $H_j(\tilde{G})$  for all  $j$ .*

(4.5) **Corollary.** *We have  $H_*(\tilde{G}, K_0) = H_*(\tilde{G}, K_1) = 0$ .*

This corollary (4.5) and the h-cobordism theorem imply that  $\tilde{G}$  is diffeomorphic to  $K_0 \times [0, 1]$ . More precisely  $\dim \tilde{G} = 2n \geq 6$  and:

**h-cobordism Theorem** [M2, 65]. *Let  $\mathcal{M}$  be a  $k$ -dimensional differentiable compact manifold with  $\partial\mathcal{M} = \mathcal{M}_0 \amalg \mathcal{M}_1$  such that  $\mathcal{M}$ ,  $\mathcal{M}_0$  and  $\mathcal{M}_1$  are simply connected. If  $H_*(\mathcal{M}, \mathcal{M}_0) = 0$  and  $k \geq 6$  then  $\mathcal{M}$  is diffeomorphic to  $\mathcal{M}_0 \times [0, 1]$ .*

So to end the proof of theorem 3 we only have to prove proposition (4.4).

*Proof of proposition (4.4).* According to proposition (2.1), the intersection form on  $H_n(F)$  splits in an orthogonal sum on the submodules  $\langle m_i, m_i^* \rangle$ ,  $i = 1, \dots, s+r$ . So the proof of (4.4) when  $s+r=1$  implies the general case.

Let us suppose that  $\text{rk}(M) = 1$  and let  $m$  be a generator of  $M$ , then  $H_n(G) = \langle m, m^* \rangle$ . We denote by  $\psi : D^{n+1} \times D^n \rightarrow D$  an embedding choosen as in proposition (4.3), by  $\eta : S^n \rightarrow G$  an embedding such that  $[\eta(S^n)] = m^*$ , and by  $G_T$  the manifold  $G_T = G \setminus \psi(S^n \times D^n)$ .

(4.6) The Mayer-Vietoris sequence associated to the following decomposition of the manifold:  $G = G_T \cup \psi(S^n \times D^n)$  gives:

$$\begin{aligned} 0 \rightarrow H_n(\psi(S^n \times S^{n-1})) \rightarrow H_n(G_T) \oplus H_n(\psi(S^n \times D^n)) \rightarrow H_n(G) \\ \xrightarrow{\delta} H_{n-1}(\psi(S^n \times S^{n-1})) \rightarrow H_{n-1}(G_T) \rightarrow 0. \end{aligned}$$

where  $\delta$  is given by the intersection of cycles with  $m$ .

(4.7) The Mayer-Vietoris sequence associated to the following decomposition of the manifold:  $\tilde{G} = G_T \cup \psi(D^{n+1} \times S^{n-1})$  gives:

$$\begin{aligned} 0 \rightarrow H_n(\psi(S^n \times S^{n-1})) \xrightarrow{\alpha} H_n(G_T) \rightarrow H_n(\tilde{G}) \xrightarrow{\gamma} H_{n-1}(\psi(S^n \times S^{n-1})) \\ \xrightarrow{\beta} H_{n-1}(\psi(D^{n+1} \times S^{n-1})) \oplus H_{n-1}(G_T) \rightarrow H_{n-1}(\tilde{G}) \rightarrow 0. \end{aligned}$$

Remark that the homomorphism  $\beta$  is injective into  $H_{n-1}(\psi(D^{n+1} \times S^{n-1}))$ , hence  $\gamma = 0$  and the sequence (4.7) splits up into:

$$(4.8) \quad 0 \rightarrow H_n(\psi(S^n \times S^{n-1})) \xrightarrow{\alpha} H_n(G_T) \rightarrow H_n(\tilde{G}) \rightarrow 0,$$

$$(4.9) \quad 0 \rightarrow H_{n-1}(\psi(S^n \times S^{n-1})) \xrightarrow{\beta} H_{n-1}(\psi(D^{n+1} \times S^{n-1})) \oplus H_{n-1}(G_T) \rightarrow H_{n-1}(\tilde{G}) \rightarrow 0.$$

Since  $\text{rk}(M) = 1 = s+r$  we have to consider the two following cases:  $s=0, r=1$  and  $s=1, r=0$ .

★ 1<sup>st</sup> case:  $s=0$  and  $r=1$ , then  $\text{Ker } S^* = \langle m, m^* \rangle$ .

In sequence (4.6) we have  $\text{Ker } \delta = \langle m, m^* \rangle$ , then  $H_n(G_T) = \langle [\psi(S^n \times \{1\})], [\eta(S^n)] \rangle$  and  $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ . In sequence (4.8) we have  $\text{Im } \alpha =$

$\langle [\psi(S^n \times \{1\})] \rangle$ , so  $H_n(\tilde{G}) = \langle [\eta(S^n)] \rangle$ . By construction of the good basis (2.1),  $[\eta(S^n)]$  is a generator of  $\text{Im}(H_n(K_0) \rightarrow H_n(G))$ . So the inclusion of  $K_0$  in  $\tilde{G}$  induces the isomorphism:  $k_{0,n} : H_n(K_0) \xrightarrow{\cong} H_n(\tilde{G})$ .

Since  $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$  in sequence (4.9), we have  $H_{n-1}(\tilde{G}) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ . Condition c.1 of the algebraic cobordism gives that there exists  $a$  in  $\text{Ker } S_0^*$  such that  $m = (a, \varphi(a))$ . If we denote by  $\gamma_0 : H_n(K_0) \rightarrow H_n(G)$  the homomorphism induced by the inclusion, then we can choose  $b$  in  $H_{n-1}(K_0)$  such that  $H_{n-1}(K_0) = \langle b \rangle$  and  $b$  is the dual of  $\gamma_0^{-1}(a)$  for the intersection form of  $K_0$ . There exists  $B$  in  $H_n(G, K_0)$  such that  $\partial B = b$  and the intersection between  $B$  and  $m$  is  $+1$ . The boundary of the  $n$ -chain  $(B - (B \cap \psi(S^n \times \overset{\circ}{D}^n)))$  is homologous to the  $(n-1)$ -cycle  $b - (\psi(\{1\} \times S^{n-1}))$ , hence  $b$  and  $[\psi(\{1\} \times S^{n-1})]$  are homologous in  $H_{n-1}(\tilde{G}) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ . Thus the inclusion of  $K_0$  in  $\tilde{G}$  induces the isomorphism:  $k_{0,n-1} : H_{n-1}(K_0) \xrightarrow{\cong} H_{n-1}(\tilde{G})$ .

★  $2^{nd}$  case:  $s = 1$  and  $r = 0$ , then  $\text{Ker } S^* = \{0\}$  and  $H_n(K_0) = 0$ . In sequence (4.6) we have  $\text{Ker } \delta = \langle m \rangle$ , then  $H_n(G_T) = \langle [\psi(S^n \times \{1\})] \rangle$  and  $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ . In sequence (4.8) we have  $\text{Im } \alpha = \langle [\psi(S^n \times \{1\})] \rangle$ . Since  $H_n(G_T) = \langle [\psi(S^n \times \{1\})] \rangle$  we have  $H_n(\tilde{G}) = 0 = H_n(K_0)$ .

- if  $S_*(m)$  is indivisible (i.e.  $H_{n-1}(K_0) = 0$ ), then  $\delta$  in (4.6) is surjective. Thus  $H_{n-1}(\tilde{G}) = 0 = H_{n-1}(K_0)$ .
- If  $a \neq 1$  is the greatest divisor of  $S_*(m)$  (i.e.  $H_{n-1}(K_0) \cong \mathbb{Z}/a\mathbb{Z}$ ) then condition c.2 of algebraic cobordism together with lemma (3.14) give that there exists  $c$  in  $H_{n-1}(K_0)$  such that  $\partial(\frac{1}{a} S_*(m)) = (c, \theta_{n-1}(c))$ . Let  $b$  in  $H_{n-1}(K_0)$  be the dual of  $c$  for the linking form of  $K_0$ . There exists  $B$  in  $H_n(G, K_0)$  such that  $\partial B = b$  and the intersection between  $B$  and  $m$  is  $+1$ . As before the boundary of the  $n$ -chain  $B - (B \cap \psi(S^n \times \overset{\circ}{D}^n))$  is the  $n$ -cycle  $b - \psi(\{1\} \times S^{n-1})$ , hence  $b$  and  $[\psi(\{1\} \times S^{n-1})]$  are homologous in  $H_{n-1}(G)$ . Since  $H_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$  in sequence (4.9) we have  $H_{n-1}(\tilde{G}) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$ . Thus  $b$  and  $[\psi(\{1\} \times S^{n-1})]$  are homologous in  $H_{n-1}(\tilde{G})$  and the inclusion of  $K_0$  in  $\tilde{G}$  induces the isomorphism:  $k_{0,n-1} : H_{n-1}(K_0) \xrightarrow{\cong} H_{n-1}(\tilde{G})$ .

Since  $\tilde{G}$  is obtained by surgery on  $n$ -cycles, this surgery only modifies homology groups of dimensions  $n$  and  $n - 1$ . Hence for  $k \neq n, n - 1$  we have  $H_k(G) \cong H_k(K_0) \xrightarrow{k_{0,k}} H_k(\tilde{G})$ . By symmetry we also have the same results with  $K_1$ . Finally  $k_{0,j}$  and  $k_{1,j}$  are some isomorphisms for all  $j$ . This ends the proof of proposition (4.4), and the proof of theorem 3.  $\square$

### 5. Appendix – Alexander polynomials of cobordant links.

Let  $K$  be a  $2n - 1$  dimensional simple link, and  $\varepsilon = (-1)^n$ . One can associate a polynomial  $\Delta \in \mathbb{Z}[X]$  to any Seifert surface  $F$  for the link  $K$ , defined by:  $\Delta(X) =$

$\det(XA + \varepsilon A^T)$ , where  $A$  is the Seifert form associated to  $F$ . Such a polynomial  $\Delta$  is called a Alexander polynomial for the link  $K$ . Changing the Seifert surface to another multiplies  $\Delta$  by  $\pm X^m$  with  $m$  in  $\mathbb{Z}$ .

For a polynomial  $\gamma$  in  $\mathbb{Z}[X]$  we define the polynomial  $\gamma^*$  by:  $\gamma^*(X) = X^{\deg \gamma} \gamma(X^{-1})$ .

(5.1) **Proposition.** *Let  $K_0$  and  $K_1$  be two cobordant simple  $2n-1$  dimensional links. If  $\Delta_0$  and  $\Delta_1$  are Alexander polynomials for  $K_0$  and  $K_1$ , then there exists  $\gamma$  in  $\mathbb{Z}[X]$  such that:  $\gamma\gamma^* = \pm\Delta_0\Delta_1$ .*

**Remark.** If  $F$  is the Milnor fiber of an algebraic link  $K$ , then the associated Alexander polynomial is the characteristic polynomial of the monodromy. Hence the above proposition and the monodromy theorem imply corollary (0.7).

*Proof of proposition (5.1).* We denote by  $F_0$  and  $F_1$  two  $(n-1)$ -connected Seifert surfaces for  $K_0$  and  $K_1$ , and by  $A_0$  and  $A_1$  the associated Seifert forms. The links  $K_0$  and  $K_1$  are cobordant so proposition (3.10) implies that the form  $A = A_0 \oplus -A_1$  has a metabolizer  $M$ . Therefore, there exists a basis for  $H_n(F_0) \oplus H_n(F_1)$  such that in this basis the matrix for  $A$  is  $\begin{pmatrix} 0 & B_1 \\ B_2 & B_3 \end{pmatrix}$  where  $B_i$ ,  $i=1,2,3$  are square matrices. We have  $\Delta_0(X).\Delta_1(X) = \det(XA + \varepsilon A^T)$ , hence  $\Delta_0(X).\Delta_1(X) = \varepsilon.\det(XB_1 + \varepsilon B_2^T).\det(XB_2 + \varepsilon B_1^T)$ . Let  $\gamma(X)$  be  $\det(XB_1 + \varepsilon B_2^T)$ , then  $\gamma^*(X) = \det(XB_2 + \varepsilon B_1^T)$ . Finally we get  $\gamma.\gamma^* = \pm\Delta_0.\Delta_1$ .  $\square$

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