Comment. Math. Helv. 72 (1997) 605–617
0010-2571/97/040605-13 \$ 1.50+0.20/0

**Commentarii Mathematici Helvetici**

# Cohen–Macaulay coordinate rings of blowup schemes

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**Abstract.** Suppose that $Y$ is a projective $k$-scheme with Cohen–Macaulay coordinate ring $S$. Let $I \subset S$ be a homogeneous ideal of $S$. I can be blown up to produce a projective $k$-scheme $X$ which birationally dominates $Y$. Let $I_c$ be the degree $c$ part of $I$. Then $k[I_c]$ is a coordinate ring of a projective embedding of $X$ for all $c$ sufficiently large. This paper considers the question of when there exists a constant $f$ such that $k[(I^e)_c]$ is Cohen–Macaulay for $c \geq ef$. A very general result is proved, giving a simple criterion for a linear bound of this type. As a consequence, local complete intersections have this property, as well as many other ideals.

**Mathematics Subject Classification (1991).** 14M05, 13H10.

**Keywords.** Cohen–Macaulay, coordinate ring.

## Introduction

Suppose that $Y$ is a projective $k$-scheme with Cohen–Macaulay coordinate ring $S$. Let $I \subset S$ be a homogeneous ideal of $S$. Then I can be blown up to produce a projective $k$-scheme $X$ which birationally dominates $Y$. Let $I_c$ be the degree $c$ part of $I$. Then $k[I_c]$ is a coordinate ring of a projective embedding of $X$ for all $c$ sufficiently large. In general, $k[I_c]$ is not Cohen–Macaulay even when $X$ is Cohen–Macaulay (a simple example is given in Section 1). Recently, [3], [5], [6], [13] have given criteria for $k[I_c]$ to be Cohen–Macaulay in many important situations.

Powers $I^e$ of $I$ blow up to the same scheme $X$, and the rings $k[(I^e)_c]$ for $c \gg e > 0$ are coordinate rings of projective embeddings of $X$.

In Theorem 4.6 [3] an explicit necessary and sufficient linear bound in $c$ and $e$ is given for $k[(I^e)_c]$ to be Cohen–Macaulay, when $S$ is a polynomial ring of dimension $n$ and $I$ is a complete intersection in $S$. Suppose that the complete intersection ideal $I$ is minimally generated by forms of degree $d_1, \ldots, d_r$. Assume that $c \geq ed + 1$, $d = \max\{d_j | j = 1, \ldots, r\}$. Then $k[(I^e)_c]$ is a Cohen–Macaulay ring if and only if $c > \sum_{j=1}^{r} d_j + (e-1)d - n$.

This leads to the question of when there exists a constant $f$ such that $k[(I^e)_c]$

* First author partially supported by NSF.

is Cohen–Macaulay for $c \geq ef$. In other words, when is there a linear bound on $c$ and $e$ ensuring that $k[(I^e)_c]$ is Cohen–Macaulay?

For instance, it is natural to expect that ideals $I$ that are local complete intersections (that is, $IS_p$ is a complete intersection if $p$ is not the irrelevant ideal of $S$) will have this property. The Kodaira Vanishing Theorem suggests that there should be a linear bound ensuring that $k[(I^e)_c]$ is Cohen–Macaulay, at least when a regular ideal is blown up in a nonsingular projective variety of characteristic zero.

In this paper, we prove a very general result (Theorem 4.1) giving a simple criterion for a linear bound of this type. As a consequence, we show (Corollary 4.2) that local complete intersections have this property, as well as many other ideals (Corollaries 4.3 and 4.4).

## 1. Coordinate rings of a blowup

Throughout this paper we will have the following assumptions. Let $k$ be a field, $S$ a noetherian graded $k$-algebra which is generated in degree 1 with graded maximal ideal $M$. Then $S$ has a presentation $S = k[x_0, x_1, \ldots, x_n]/K$, where each $x_i$ is homogeneous of degree 1. Let $\beta = \dim(S)$, $\overline{n} = \beta - 1$, $I \subset S$ be a homogeneous ideal, and let $\tilde{I}$ be the sheaf associated to $I$ in $Y = \mathrm{Proj}\,(S)$. Let $X = \mathrm{Proj}\,(\bigoplus \tilde{I}^n)$ be the blowup of $\tilde{I}$, with natural map $\pi\colon X \to Y$. $I_c$ will denote the $c$-graded part of $I$.

**Lemma 1.1.** *Suppose that $I$ is generated in degree $\leq d$. Then*
(1) *$(I_c) \cdot \mathcal{O}_X = \tilde{I}(c) \cdot \mathcal{O}_X$ for $c \geq d$.*
(2) *$I_c$ is a very ample subspace of $\Gamma(X, \tilde{I}(c) \cdot \mathcal{O}_X)$ for $c \geq d+1$.*

*Proof.* Let $I$ be generated by $G_1, \ldots, G_m$ where the $G_i$ are homogeneous of degree $d_i \leq d$. Let

$$R_{ij} = \left(k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] / K_i\right)\left[\frac{G_1 x_i^{d_j - d_1}}{G_j}, \ldots, \frac{G_m x_i^{d_j - d_m}}{G_j}\right],$$

$U_{ij} = \mathrm{Spec}\,(R_{ij})$. $\{U_{ij} : 0 \leq i \leq n, 1 \leq j \leq m\}$ is an affine cover of $X$. $\Gamma(U_{ij}, \tilde{I}(c) \cdot \mathcal{O}_X) = G_j x_i^{c-d_j} R_{ij}$.

Since $I_c \subset \Gamma(X, \tilde{I}(c) \cdot \mathcal{O}_X)$ and $G_j x_i^{c-d_j} \in I_c$ for $1 \leq j \leq m$ whenever $c \geq d$, we have (1).

To establish (2) we will use the criteria of Proposition II.7.2 [7]. Suppose that $c \geq d+1$. By (1), $I_c$ gives a morphism of $X$. $I_c$ is generated over $k$ by $\{G_j x_0^{l_0} x_1^{l_1} \cdots x_n^{l_n} : d_j + l_0 + l_1 + \cdots + l_n = c\}$. Suppose that $s = G_j x_0^{l_0} x_1^{l_1} \cdots x_n^{l_n}$ is one of these generators. Then some $l_i > 0$, and $X_s \subset U_{ij}$.

$$s = G_j x_0^{l_0} x_1^{l_1} \cdots x_n^{l_n} = G_j x_i^{c-d_j} \left(\frac{x_0}{x_i}\right)^{l_0} \cdots \left(\frac{x_n}{x_i}\right)^{l_n},$$

so that $X_s = \mathrm{Spec}\,(A)$ where

$$A = R_{ij}\left[\left(\frac{x_i}{x_0}\right)^{l_0}\cdots\left(\frac{x_i}{x_n}\right)^{l_n}\right].$$

We have

$$\frac{G_j x_i^{c-d_j}}{G_j x_0^{l_0}\cdots x_n^{l_n}} = \left(\frac{x_i}{x_0}\right)^{l_0}\cdots\left(\frac{x_i}{x_n}\right)^{l_n}$$

$$\frac{G_j x_0^{l_0}\cdots x_t^{l_t+1}\cdots x_i^{l_i-1}\cdots x_n^{l_n}}{G_j x_0^{l_0}\cdots x_n^{l_n}} = \frac{x_t}{x_i}$$

$$\frac{G_t x_i^{c-d_t}}{G_j x_0^{l_0}\cdots x_n^{l_n}}\cdot\left(\frac{G_j x_0^{l_0+1}\cdots x_i^{l_i-1}\cdots x_n^{l_n}}{G_j x_0^{l_0}\cdots x_n^{l_n}}\right)^{l_0}\cdots\left(\frac{G_j x_0^{l_0}\cdots x_i^{l_i-1}\cdots x_n^{l_n+1}}{G_j x_0^{l_0}\cdots x_n^{l_n}}\right)^{l_n} = \frac{G_t x_i^{d_j-d_t}}{G_j}$$

generate $A$ as a $k$-algebra. □

Let $\mathcal{L} = \tilde{I}\cdot\mathcal{O}_X$, $\mathcal{M} = \pi^*\mathcal{O}_Y(1)$, so that $(I^e)_c\cdot\mathcal{O}_X = \mathcal{L}^e\otimes\mathcal{M}^c$ for $e>0$ and $c\geq de$, and $X\cong\mathrm{Proj}\,(k[(I^e)_c])$ for $c\geq de+1$. In Lemma 1.2 we state the usual exact sequences relating local cohomology and global cohomology (cf. A4.1 [4]).

**Lemma 1.2.** *Suppose that $I$ is generated in degree $\leq d$, $e>0$ and $c\geq de+1$. Let $A = k[(I^e)_c]$, with graded maximal ideal $m$. There are exact sequences*

$$0\to H_m^0(A)\to A\to\bigoplus_{s\in\mathbf{Z}}\Gamma(X,\mathcal{L}^{se}\otimes\mathcal{M}^{sc})\to H_m^1(A)\to 0$$

*and isomorphisms*

$$H_m^{i+1}(A)\cong\bigoplus_{s\in\mathbf{Z}}H^i(X,\mathcal{L}^{se}\otimes\mathcal{M}^{sc})$$

*for $i\geq 1$.*

Let $I^\star$ denote the intersection of the primary components which are not $M$-primary of an irredundant primary decomposition of $I$.

**Lemma 1.3.** *There exists a positive integer $f$ such that $(I^a)_b = (I^a)^\star_b$ for all $a,b$ with $b\geq fa$.*

*Proof.* This is immediate from the Main Theorem of Swanson [14] (cf. also Theorem 1.5 [10]), which states that there exists an integer $f$ such that $I^a$ has an irredundant primary decomposition $I^a = q_1\cap\cdots\cap q_s$ with $(\sqrt{q_i})^{af}\subset q_i$ for all $i$. □

**Lemma 1.4.** *Suppose that no associated prime of $S$ contains $I$,* $\operatorname{depth}_M(S) \geq 2$ *and* $\pi_*(\tilde{I}^e \cdot \mathcal{O}_X) = \tilde{I}^e$ *for all* $e \geq 0$. *Then there exists a positive integer $f$ such that, with the notation of Lemma 1.2,* $H^0_m(A) = 0$ *and* $H^1_m(A) = 0$ *whenever* $e > 0$ *and* $c \geq ef$.

*Proof.* Suppose $I$ is generated in degree $\leq d$. By Lemma 1.3 there exists an integer $f'$ such that $(I^t)_s = (I^t)^\star_s$ for $s \geq f't$. Set $f = \max(f', d+1)$.

By consideration of the natural inclusion $A \subset S[It]$, we have $H^0_M(A) = 0$ since $H^0_I(S) = 0$. $H^0(Y, \mathcal{O}_Y(s)) = H^1_M(S)_s = 0$ for all $s < 0$ since $\operatorname{depth}_M(S) \geq 2$. From the inclusions

$$\Gamma(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) \hookrightarrow \Gamma(X, \mathcal{M}^{sc}) = \Gamma(Y, (\pi_*\mathcal{O}_X) \otimes \mathcal{O}_Y(sc)) = \Gamma(Y, \mathcal{O}_Y(sc))$$

where the first equality is by the projection formula (cf. Exercise II.5.1 [7]), we get $\Gamma(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$ for $s < 0$.

Let $\Delta$ be the $(c, e)$ diagonal of $\mathbf{Z}^2$ ([3]). For $c \geq ed$, we have $k[(I^e)_c] = S[It]_\Delta$, as in Lemma 1.2 of [3].

$$\begin{aligned}
\bigoplus_{s \geq 0} H^0(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) &= \bigoplus_{s \geq 0} H^0(Y, \pi_*(\mathcal{L}^{se}) \otimes \mathcal{O}_Y(sc)) \\
&= \bigoplus_{s \in \mathbf{Z}} H^0(Y, \tilde{I}^{se}(sc)) \\
&= (\Gamma(\operatorname{Spec}(S) - M, \tilde{I}))_\Delta \\
&= \left( \bigoplus_{t \geq 0} (I^t)^\star \right)_\Delta = S[It]_\Delta .
\end{aligned}$$

Now Lemma 1.2 implies $H^1_m(A) = 0$. □

The condition of the existence of a Cohen–Macaulay coordinate ring is somewhat delicate, as shown by the following simple example of a Cohen–Macaulay scheme obtained by blowing up an ideal sheaf on a scheme with a Cohen–Macaulay coordinate ring, which does not have a Cohen–Macaulay coordinate ring. Let $T$ be a nonsingular "irregular" projective surface ($H^1(T, \mathcal{O}_T) \neq 0$). Let $\pi\colon T \to U$ be a birational projection onto a hypersurface in $\mathbf{P}^3$. $\pi$ is the blowup of an ideal sheaf on $U$. The coordinate ring of $U$ is Cohen–Macaulay, and certainly $T$ is Cohen–Macaulay, but no coordinate ring of $T$ can be Cohen–Macaulay since $T$ is irregular.

However, we can give a simple proof of the existence of a linear bound ensuring that $k[(I^e)_c]$ is Cohen–Macaulay when $k$ has characteristic zero, $S$ is Cohen–Macaulay, $Y$ is nonsingular, $I$ is equidimensional and $\operatorname{Proj}(S/I)$ is nonsingular. The proof has three ingredients:

(1) The Kodaira Vanishing Theorem.

(2) In this situation (everything nonsingular) $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$ and $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ (cf. Proposition 10.2 [11]).

(3) Lemma 1.4.

If $\mathcal{N}$ is an ample invertible sheaf on a smooth projective variety $Z$ of characteristic zero and dimension $t$, with dualizing sheaf $\omega_Z$, then Kodaira Vanishing states that $H^i(Z,\mathcal{N} \otimes \omega_Z) = 0$ for $i > 0$. The Serre-dual form of Kodaira Vanishing is $H^i(Z,\mathcal{N}^{-1}) = 0$ for $i < t$.

**Proposition 1.5.** *Suppose that $k$ has characteristic zero, $S$ is Cohen–Macaulay, $Y$ is nonsingular, $I$ is equidimensional and* Proj $(S/I)$ *is nonsingular. Then there exists a positive integer $f$ such that*

$$H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$$

*for all $s \in \mathbf{Z}$ if $0 < i < \overline{n}$, $c \geq ef$, $e > 0$.*

*Proof.* Suppose that $I$ has height $r$, and is generated in degree $< d$. By Lemma 1.1, $\mathcal{L}^a \otimes \mathcal{M}^b$ is very ample if $b > ad$. We immediately get $H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$ if $c > ed$, $s < 0$ and $i < \overline{n}$, since $\mathcal{L}^e \otimes \mathcal{M}^c$ is then ample.

$H^i(X, \mathcal{O}_X) = 0$ for $0 < i$, by the Leray spectral sequence, since $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$, $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $H^j(Y, \mathcal{O}_Y) = 0$ for $0 < j$ (since $S$ is Cohen–Macaulay).

Let $\omega_Y$ be a dualizing sheaf on $Y$. $\omega_Y^{-1}(g)$ is ample on $Y$ for some $g > 0$. $\omega_X = \mathcal{L}^{1-r} \otimes \omega_Y$ is a dualizing sheaf on $X$.

$$\mathcal{L}^e \otimes \mathcal{M}^c \cong \mathcal{L}^{e+r-1} \otimes \mathcal{M}^c \otimes \omega_Y^{-1} \otimes \omega_X.$$

$\mathcal{L}^{e+r-1} \otimes \mathcal{M}^c \otimes \omega_Y^{-1}$ is ample if $c > g + (e+r-1)d$, and then $H^i(X, \mathcal{L}^{se} \otimes \mathcal{M}^{sc}) = 0$ for $s > 0$ and $i > 0$. □

**Theorem 1.6.** *Suppose that $k$ has characteristic zero, $S$ is Cohen–Macaulay, $Y$ is nonsingular, $I$ is equidimensional and* Proj $(S/I)$ *is nonsingular. Then there exists a positive integer $f$ such that $k[(I^e)_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

*Proof.* The assumptions of Lemma 1.4 are satisfied by Proposition 10.2 [11] (or Example 2.3) and Proposition III.8.5 [7]. By Lemmas 1.2, 1.4 and Proposition 1.5 there exists a positive integer $f$ such that $k[(I^e)_c]$ has depth $\overline{n} + 1$ at $m$ whenever $e > 0$ and $c \geq ef$. □

Unfortunately, Kodaira Vanishing fails in positive characteristic or if anything is not (almost) nonsingular. However, we obtain a very general result which is sufficient for this global part of the argument in Section 3. The second ingredient is local. We give a very simple argument to generalize this part in Section 2.

## 2. Local conditions

**Lemma 2.1.** *Suppose that $R$ is a local ring, essentially of finite type over a field $k$ and $J \subset R$ is an ideal. Let $W = \mathrm{Spec}\,(R)$, $V = \mathrm{Proj}\,(\bigoplus_{n\geq 0} J^n)$, $E = \mathrm{Proj}\,(\bigoplus_{n\geq 0} J^n/J^{n+1})$, $\mathcal{L} = \tilde{J}\cdot\mathcal{O}_V$. Suppose that*

$$\Gamma(E, \mathcal{O}_E(m)) = J^m/J^{m+1} \quad \text{for } m \geq 0$$

*and*

$$H^i(E, \mathcal{O}_E(m)) = 0 \quad \text{for } i > 0 \text{ and } m \geq 0.$$

*Then $\Gamma(V, \mathcal{L}^m) = J^m$ if $m \geq 0$ and $H^q(V, \mathcal{L}^m) = 0$ for $q > 0$ and $m \geq 0$.*

*Proof.* Note that $\mathcal{L} = \mathcal{O}_V(1)$ on $V$.

We have exact sequences:

$$0 \to \mathcal{O}_V(m+1) \to \mathcal{O}_V(m) \to \mathcal{O}_E(m) \to 0$$

for all integers $m$. Thus we have surjections

$$H^i(V, \mathcal{O}_V(m+1)) \to H^i(V, \mathcal{O}_V(m))$$

for $i > 0$ and $m \geq 0$. Since $\mathcal{O}_V(m)$ is ample, we have $H^i(V, \mathcal{O}_V(m)) = 0$ for all $m \gg 0$, and $i > 0$, so we have all of the desired vanishing. We also have exact sequences:

$$0 \to \Gamma(V, \mathcal{O}_V(m+1)) \to \Gamma(V, \mathcal{O}_V(m)) \to J^m/J^{m+1} \to 0 \tag{1}$$

for $m \geq 0$. Since $R$ is a localization of a finitely generated $k$-algebra, $\Gamma(V, \mathcal{O}_V(m)) = J^m$ for $m \gg 0$ (cf. Exercise II.5.9 of [7]). Thus it follows from (1) that $\Gamma(V, \mathcal{O}_V(m)) = J^m$ for all $m \geq 0$. □

**Lemma 2.2.** *Let notation be as in Lemma 2.1. Suppose that $V$ is Cohen–Macaulay. Let $\omega_V$ be a dualizing sheaf on $V$ and $\omega_E$ be a dualizing sheaf on $E$. Suppose that $H^i(E, \omega_E(m)) = 0$ for $i > 0$ and $m \geq 2$. Then $H^q(V, \omega_V \otimes \mathcal{L}^m) = 0$ for $q > 0$ and $m \geq 1$.*

*Proof.* The ideal sheaf of $E$ is $I\cdot\mathcal{O}_V \cong \mathcal{O}_V(1)$. By "adjunction" (cf. Proposition 2.4 [1] or Theorem III 7.11 [7]) we have

$$\omega_E \cong \omega_V \otimes \mathcal{O}_E(-1).$$

Since $\omega_V$ is Cohen–Macaulay, we deduce from the exact sequence

$$0 \to \mathcal{O}_V(1) \to \mathcal{O}_V \to \mathcal{O}_E \to 0$$

exact sequences:

$$0 \to \omega_V(m+1) \to \omega_V(m) \to \omega_E(m+1) \to 0$$

for all integers $m$.

Thus we have surjections

$$H^i(V, \omega_V(m+1)) \to H^i(V, \omega_V(m))$$

for $i > 0$ and $m \geq 1$. Since $\mathcal{O}_V(m)$ is ample, we have $H^i(V, \omega_V(m)) = 0$ for all $m \gg 0$ and $i > 0$, so we have all of the desired vanishing. □

**Example 2.3.** Suppose that $R$ is a Cohen–Macaulay local ring, essentially of finite type over a field $k$ and $J \subset R$ is an ideal generated by a regular sequence. Then the conclusions of Lemmas 2.1 and 2.2 hold.

*Proof.* Let $f_1, \ldots, f_r$ be a minimal set of generators of $I$. $V$ is Cohen–Macaulay, $E = \operatorname{Proj}(\bigoplus_{n \geq 0} J^n/J^{n+1}) \cong \mathbf{P}^{r-1}_{R/J}$, $\omega_E \cong (\omega_W/J\omega_W) \otimes_k \mathcal{O}_E(-r)$. Now the assumptions of Lemmas 2.1 and 2.2 follow from the cohomology of projective space and the isomorphisms

$$H^i(X, \mathcal{O}_E(m)) \cong R/J \otimes_k H^i(\mathbf{P}^{r-1}_k, \mathcal{O}(m)),$$
$$H^i(E, (\omega_W/J\omega_W) \otimes_k \mathcal{O}_E(m)) \cong (\omega_W/J\omega_W) \otimes_k H^i(\mathbf{P}^{r-1}_k, \mathcal{O}(m))$$

by the Künneth formula (cf. p. 77 of [12]). □

Let

$$T = \bigoplus_{n \geq 0} J^n/J^{n+1}$$

be the associated graded ring of $J$, and let $N$ be the ideal of positive degree elements of $T$. Suppose that $T$ is Cohen–Macaulay. Then there is a canonical module $W_T$ of $T$ such that the sheaf associated to $W_T$ is a dualizing sheaf $\omega_E$ on $E$. The vanishing hypotheses of Lemmas 2.1 and 2.2 hold whenever

$$H^i_N(T)_c = 0$$

for $i \geq 0$ and $c \geq 0$ and

$$H^i_N(W_T)_c = 0$$

for $i \geq 2$ and $c \geq 2$.

An ideal $J$ in a ring $R$ is called strongly Cohen–Macaulay if the Koszul homology modules of $I$ with respect to a generating set are Cohen–Macaulay. Let $\mu(J)$ denote the minimal number of generators of an ideal $J$.

**Example 2.4.** Suppose that $R$ is a Gorenstein local ring, essentially of finite type over a field $k$ and $J \subset R$ is a strongly Cohen–Macaulay ideal, with $\mu(J_P) \leq$

height$(P)$ for all primes $P$ containing $J$. Then the conclusions of Lemma 2.1 and 2.2 hold.

*Proof.* Suppose $J$ is of height $g$ generated by $n$ elements. Let $S = R[X_1, \dots, X_n]$ be a polynomial ring over $R$. Let $H(J)$ denote the Koszul homology $H(f_1, \dots, f_n, R)$ where $f_1, \dots, f_n$ are generators of $J$. $W_{R/J} = \mathrm{Ext}^g(R/J, R) \cong H_{n-g}(J)$ is the last non-vanishing $H_i(J)$. The approximation complex $\mathcal{M}$ is

$$0 \to H_{n-g}(J) \otimes S(-n+g) \to \cdots \to H_1(J) \otimes S(-1) \to H_0(J) \otimes S \to 0.$$

In [8], it is shown that $H^0(\mathcal{M}) = \oplus J^n/J^{n+1}$. By Theorems 2.5 and 2.6 [8] $\mathcal{M}$ is acyclic and $\oplus J^n/J^{n+1}$, $\oplus J^n$ are Cohen–Macaulay.

Let $\overline{S} = R/J[X_1, \dots, X_n]$, with canonical module

$$W_{\overline{S}} = W_{R/J} \otimes S(-n) = H_{n-g}(J) \otimes S(-n).$$

$H^i_N(T)$ is dual to $\mathrm{Ext}^{n-i}_{\overline{S}}(T, W_{\overline{S}})$ (cf. Theorem 3.6.19 [2]). We have $\mathrm{Ext}^i_{\overline{S}}(T, W_{\overline{S}}) = 0$ for $i \neq n-g$ since $T$ is a Cohen–Macaulay module of dimension $\dim(R)$, and $\dim(\overline{S}) - \dim(T) = n - g$. By (c) of Theorem 2.6 [8] we can realize $W_T$ as the cokernel of

$$H_1(J) \otimes S(-g-1) \to H_0(J) \otimes S(-g)$$

so that $W_T \cong T(-g)$.

From this we see that $W_T$ is supported in degree $g$, so that $H^g_N(T)_j = 0$ for $j > -g$ and $H^g_N(W_T)_j = 0$ for $j > 0$.

By a Theorem of Huneke (Theorem 1.14 [9]), all ideals in the linkage class of a complete intersection in a Gorenstein local ring are strongly Cohen–Macaulay. For instance, codimension 2 perfect and Gorenstein codimension 3 ideals are strongly Cohen–Macaulay.

**Remark 2.5.** The conclusions of Example 2.4 are true when $R$ is not Gorenstein but only Cohen–Macaulay.

In this case $\mathcal{M}^*$ is acyclic with zeroth homology $W_T$, and we can use $\mathcal{M}$ and $\mathcal{M}^*$ to compute the desired vanishing.

## 3. Global conditions

We return to the notation and hypotheses of Section 1.

**Proposition 3.1.** *Suppose that*

$$R^j\pi_*(\tilde{I}^a \cdot \mathcal{O}_X) = 0 \quad \text{for } 1 \leq a \leq \overline{n}+1, \quad j > 0.$$

*Then there exists a positive integer $f$ such that*

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = 0 \quad \text{for } j > 0,\ a > 0 \text{ and } b \geq fa.$$

*Proof.* After possibly tensoring with an extension field of $k$, we may suppose that $k$ is an infinite field. Suppose that $I$ is generated in degree $\leq d$. Set $d' = d + 1$. Let $D_1, \ldots, D_{\overline{n}}$ be general members of $\Gamma(X, \mathcal{L} \otimes \mathcal{M}^{d'})$. Let

$$L_i = D_1 \cap \cdots \cap D_i$$

be the (scheme theoretic) intersection. $L_i$ has dimension $\overline{n} - i$. Set $L_0 = X$.

We have short exact sequences

$$\begin{aligned} 0 \to (\mathcal{L}^a \otimes \mathcal{M}^b) \otimes \mathcal{O}_{L_i} &\to (\mathcal{L}^{a+1} \otimes \mathcal{M}^{b+d'}) \otimes \mathcal{O}_{L_i} \\ &\to (\mathcal{L}^{a+1} \otimes \mathcal{M}^{b+d'}) \otimes \mathcal{O}_{L_{i+1}} \to 0 \end{aligned} \tag{2}$$

for all integers $a$ and $b$ and $0 \leq i \leq \overline{n} - 1$.

$$R^j\pi_*(\mathcal{L}^a \otimes \mathcal{M}^b) = R^j\pi_*(\tilde{I}^a \cdot \mathcal{O}_X) \otimes \mathcal{O}_Y(b) = 0$$

for $j > 0$ and $1 \leq a \leq \overline{n} + 1$. From the Leray spectral sequence

$$H^i(Y, R^j\pi_*(\mathcal{L}^a \otimes \mathcal{M}^b)) \Rightarrow H^{i+j}(X, \mathcal{L}^a \otimes \mathcal{M}^b)$$

we have

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = H^j(Y, \pi_*(\tilde{I}^a \cdot \mathcal{O}_X) \otimes \mathcal{O}_Y(b)).$$

There thus exists an integer $f \geq d'$ such that

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = 0 \quad \text{for } j > 0,\ 1 \leq a \leq \overline{n} + 1 \text{ and } b \geq f \tag{3}$$

since $\pi_*(\tilde{I}^a \cdot \mathcal{O}_X)$ is coherent and $\mathcal{O}_Y(1)$ is ample on $Y$.

By (3) and induction applied to the long exact cohomology sequences associated to (2) we have

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b \otimes \mathcal{O}_{L_i}) = 0 \text{ for } j > 0,\ i + 1 \leq a \leq \overline{n} + 1 \text{ and } b \geq f + id'. \tag{4}$$

The following inductive statement (5) $\Longrightarrow$ (6) can be established by induction using the exact sequences (2), (4) and the equality

$$(a, b) = (a - i - 1)(1, d') + (i + 1, b - (a - i - 1)d').$$

Note that if $a \geq i + 1$ and $b \geq f + (a - 1)d'$ then $b - (a - i - 1)d' \geq f + id'$.

Suppose that

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b \otimes \mathcal{O}_{L_{i+1}}) = 0 \text{ for } j > 0,\ i+2 \leq a \text{ and } b \geq f + (a-1)d'. \quad (5)$$

Then

$$H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b \otimes \mathcal{O}_{L_i}) = 0 \text{ for } j > 0,\ i+1 \leq a \text{ and } b \geq f + (a-1)d'. \quad (6)$$

$L_{\overline{n}}$ has dimension 0, so that (6) is immediate for $i = \overline{n}$. Thus the proposition follows from descending induction on $i$ using the above statement (5) $\Longrightarrow$ (6). $\square$

**Proposition 3.2.** *Suppose that $X$ is a Cohen–Macaulay scheme and*

$$R^j\pi_*(\omega_X \otimes \mathcal{L}^t) = 0 \quad \textit{for} \ \ 1 \leq t \leq \overline{n}+1,\ j > 0.$$

*Then there exists a positive integer $f$ such that*

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) = 0 \quad \textit{for} \ \ j < \overline{n},\ a > 0 \textit{ and } b \geq fa.$$

*Proof.* After possibly tensoring with an extension field of $k$, we may suppose that $k$ is an infinite field. Suppose that $I$ is generated in degree $\leq d$. Set $d' = d+1$. Let $D_1, \dots, D_{\overline{n}}$ be general members of $\Gamma(X, \mathcal{L} \otimes \mathcal{M}^{d'})$. Let

$$L_i = D_1 \cap \cdots \cap D_i$$

be the (scheme theoretic) intersection. $L_i$ has dimension $\overline{n} - i$. Set $L_0 = X$.

We have short exact sequences

$$\begin{aligned} 0 \to (\mathcal{L}^{-a-1} \otimes \mathcal{M}^{-b-d'}) \otimes \mathcal{O}_{L_i} &\to (\mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) \otimes \mathcal{O}_{L_i} \\ &\to (\mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) \otimes \mathcal{O}_{L_{i+1}} \to 0 \end{aligned} \quad (7)$$

for all integers $a$ and $b$ and $0 \leq i \leq \overline{n} - 1$.

By Serre-duality,

$$\begin{aligned} H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) &= H^{\overline{n}-j}(X, \omega_X \otimes \mathcal{L}^a \otimes \mathcal{M}^b). \\ R^j\pi_*(\omega_X \otimes \mathcal{L}^a \otimes \mathcal{M}^b) &= R^j\pi_*(\omega_X \otimes \mathcal{L}^a) \otimes \mathcal{O}_Y(b) = 0 \end{aligned}$$

for $j > 0$ and $1 \leq a \leq \overline{n}+1$. From the Leray spectral sequence, we hav

$$H^{\overline{n}-j}(X, \omega_X \otimes \mathcal{L}^a \otimes \mathcal{M}^b) = H^{\overline{n}-j}(Y, \pi_*(\omega_X \otimes \mathcal{L}^a) \otimes \mathcal{O}_y(b)).$$

Hence there exists an integer $f \geq d'$ such that

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) = 0 \quad \text{for} \ \ j < \overline{n},\ 1 \leq a \leq \overline{n}+1 \text{ and } b \geq f \quad (8)$$

since $\pi_*(\omega_X \otimes \mathcal{L}^a)$ is coherent and $\mathcal{O}_Y(1)$ is ample on $Y$.

By (8) and induction applied to the long exact cohomology sequences associated to (7) we have

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b} \otimes \mathcal{O}_{L_i}) = 0 \quad \text{for } j < \overline{n} - i,\ 1 \leq a \leq \overline{n} + 1 - i \text{ and } b \geq f. \quad (9)$$

The following inductive statement (10) $\Longrightarrow$ (11) can be established by induction using the exact sequences (7), (9) and the equality

$$(a, b) = (a-1)(1, d') + (1, b - (a-1)d').$$

Note that if $a \geq 1$ and $b \geq f + (a-1)d'$ then $b - (a-1)d' \geq f$.

Suppose that

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b} \otimes \mathcal{O}_{L_{i+1}}) = 0 \quad \text{for } j < \overline{n} - i - 1,\ 1 \leq a \text{ and } b \geq f + (a-1)d'. \quad (10)$$

Then

$$H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b} \otimes \mathcal{O}_{L_i}) = 0 \quad \text{for } j < \overline{n} - i,\ 1 \leq a \text{ and } b \geq f + (a-1)d'. \quad (11)$$

(11) is immediate for $i = \overline{n}$. Thus the proposition follows from descending induction on $i$ using the above statement (10) $\Longrightarrow$ (11). □

## 4. Linear bounds for Cohen–Macaulay coordinate rings

Let $k$ be a field, $S$ a noetherian graded $k$-algebra which is generated in degree 1, with graded maximal ideal $M$. Let $I \subset S$ be a homogeneous ideal, and let $\tilde{I}$ be the sheaf associated to $I$ in $Y = \operatorname{Proj}(S)$. Let $X = \operatorname{Proj}(\bigoplus \tilde{I}^n)$ be the blowup of $\tilde{I}$, with natural map $\pi\colon X \to Y$, and $\mathcal{O}_X(1) = \tilde{I} \cdot \mathcal{O}_X$. Let $\beta$ be the dimension of $S$, $\overline{n} = \beta - 1$ be the dimension of $Y$.

**Theorem 4.1.** *Suppose that $I$ is an ideal of height $> 0$, $S$ is Cohen–Macaulay and $X$ is a Cohen–Macaulay scheme. Let*

$$E = \operatorname{Proj}\Big(\bigoplus_{n \geq 0} \tilde{I}^n / \tilde{I}^{n+1}\Big)$$

*with dualizing sheaf $\omega_E$. Suppose that*

$$\begin{aligned}
\pi_*\mathcal{O}_E(m) &= \tilde{I}^m / \tilde{I}^{m+1} \quad \textit{for } m \geq 0,\\
R^i\pi_*\mathcal{O}_E(m) &= 0 \quad \textit{for } i > 0 \textit{ and } m \geq 0 \textit{ and}\\
R^i\pi_*\omega_E(m) &= 0 \quad \textit{for } i > 0 \textit{ and } m \geq 2.
\end{aligned}$$

*Then there exists a positive integer $f$ such that $k[(I^e)_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

*Proof.* $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$ and $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ by Lemma 2.1 (and Proposition III.8.5 [7]). $S$ is Cohen–Macaulay so that $H^i_M(S) = 0$ for $i < \beta$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < \overline{n}$. Now by the Leray spectral sequence, $H^i(Y, R^j\pi_*\mathcal{O}_X) \Rightarrow H^{i+j}(X, \mathcal{O}_X)$, we get $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \overline{n}$.

By Lemma 2.1 and Proposition 3.1 we have an $f$ such that $H^j(X, \mathcal{L}^a \otimes \mathcal{M}^b) = 0$ for $j > 0$, $a > 0$, $b \geq fa$. By Lemma 2.2 and Proposition 3.2 there exists $f$ such that $H^j(X, \mathcal{L}^{-a} \otimes \mathcal{M}^{-b}) = 0$ for $j < \overline{n}$, $a > 0$, $b \geq fa$.

Now the Theorem follows from Lemmas 1.4 and 1.2. □

**Corollary 4.2.** *Suppose that $S$ is Cohen–Macaulay, $I$ is an ideal of height $> 0$ and $\tilde{I}$ is locally a complete intersection in $Y = \mathrm{Proj}(S)$. Then there exists a positive integer $f$ such that $k[(I^e)_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

*Proof.* This is immediate from Example 2.3. □

The following Corollary is now immediate from the comments following Example 2.3. By a canonical module $W_T$ we mean a canonical module whose associated sheaf is a dualizing sheaf of $\mathrm{Proj}(T)$. $I^n_{(P)}$ denoted the degree 0 elements of the localization $I^n_P$.

**Corollary 4.3.** *Suppose that*

(1) *$S$ is Cohen–Macaulay.*
(2) *$I$ is an ideal of height $> 0$.*
(3) *$\bigoplus_{n\geq 0} I^n_{(P)}$ and $T(P) = \bigoplus_{n\geq 0}(I^n/I^{n+1})_{(P)}$ are Cohen–Macaulay for all $P \in \mathrm{Proj}(S)$.*
(4) *$H^i_{\overline{P}}(T(P))_c = 0$ for $i \geq 0$ and $c \geq 0$ and $H^i_{\overline{P}}(W_{T(P)})_c = 0$ for $i \geq 2$ and $c \geq 2$ for all $P \in \mathrm{Proj}(S)$, where $W_{T(P)}$ is the canonical module of $T(P)$, $\overline{P}$ is the maximal ideal of $S_{(P)}$.*

*Then there exists a positive integer $f$ such that $k[(I^e)_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

**Corollary 4.4.** *Suppose that $S$ is Cohen–Macaulay, $I$ is an ideal of height $> 0$ and $I_{(P)}$ is strongly Cohen–Macaulay with $\mu(I_{(P)}) \leq$ height$(P)$ for all primes $P \in Y$ containing $I$. Then there exists a positive integer $f$ such that $k[(I^e)_c]$ is Cohen–Macaulay whenever $e > 0$ and $c \geq ef$.*

*Proof.* The assumptions of Theorem 4.1 are satisfied by Example 2.4 and Remark 2.5. □

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(Received: January 5, 1997)