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# Fox's congruence classes and the quantum-$SU(2)$ invariants of links in 3-manifolds

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## 1. Introduction

From the time Jones first discovered his polynomial, it has been hoped that it provides information about the unknotting properties of knots and links. This hope was founded on the fact that the polynomial of a link can be calculated from that of two closely related links which differ only in the neighbourhood of a single crossing. It is the purpose of this paper to demonstrate that the Jones polynomial does indeed contain unknotting information. However, the methods we employ do not exploit the recurrence relation of the polynomial, but instead take advantage of its relation to the quantum-$SU(2)$ invariants of links in 3-manifolds.

These invariants were discovered by Witten [15] using techniques from theoretical physics. A rigorous mathematical proof of their existence was first given by Reshetikhin and Turaev [11], and then by Kirby and Melvin [4]. These proofs relied heavily upon the representation theory of quantum groups. A very simple and elegant proof of the existence of the invariants has been given by Lickorish [7]. We follow his approach in this paper.

In [2], Fox introduced the notion of congruence classes of knots in $S^3$. He termed two knots congruent if they differ by a sequence of $1/n$ surgeries about certain unknotted curves. Here, we generalise his definition.

DEFINITION 1.1. Let $n$ and $q$ be non-negative integers. Let $K$ and $L$ be tame oriented framed links in a closed connected oriented 3-manifold $M$. Then $K$ and $L$ are said to be **congruent modulo** $(n, q)$, written $K \equiv L \pmod{(n, q)}$, if there are oriented framed links $K_0, K_1, \ldots, K_m$ and trivial knots $J_1, J_2, \ldots, J_m$ in $M$ such that

(1) $K_{i-1}$ and $J_i$ are disjoint,
(2) $K_i$ is obtained from $K_{i-1}$ by $1/n$ surgery along $J_i$,
(3) the linking number $lk(K_{i-1}, J_i) \equiv 0 \pmod q$, and
(4) $K_0 = K$ and $K_m = L$.

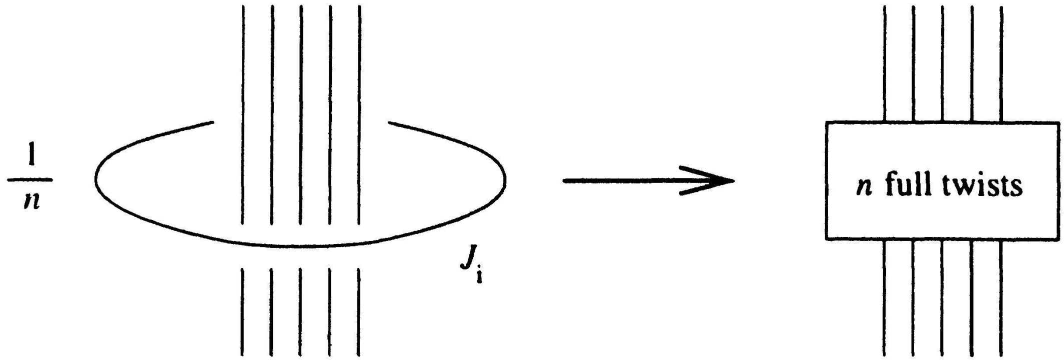


Figure 1

Thus, $K$ and $L$ are congruent if they differ by a sequence of moves as shown in Figure 1, with suitable restrictions on the linking number of the link with $J_i$.

Fox asked whether the set of congruence classes of a knot in $S^3$ determines the knot type. That is, if $K$ and $L$ are knots in $S^3$, and $K \equiv L$ (mod($n$, $q$)) for all $n > 0$ and $q \geq 0$, then are $K$ and $L$ equivalent? He gave evidence supporting this conjecture by showing that the Alexander polynomial of a knot restricts its possible congruence classes. His result has since been corrected and extended by Nakanishi and Suzuki [10]. See also [9]. In this paper, we shall show that the quantum-$SU(2)$ invariants of knots and links in 3-manifolds also provide information about their congruence classes. As a corollary, we show that if two knots fail Fox's conjecture, then they must have the same Jones polynomial. In fact, we prove the following result.

COROLLARY 2.4. *If two oriented links $K$ and $L$ in $S^3$ have different Jones polynomials, then, for any framings on $K$ and $L$, $K \equiv L$* (mod($n$, 2)) *for at most finitely many $n$.*

It is worth noting what the effect a move as in Figure 1 has on the framing of a link $K$. Now, the framing of $K$ is determined by a set of annuli, each annulus having a boundary component equal to a component of $K$. The effect of surgery along $J_i$ on the framing of the link is determined by the effect on these annuli. An example is given in Figure 2. In this figure and, indeed, in all the diagrams of this paper, the links are given blackboard framings. Note also that the framing on an oriented link in $S^3$ uniquely determines the writhe of any diagram which represents it.

In this paper, we shall be examining the cases $q = 1$ and 2. Note that when $q = 1$, there is no restriction on $lk(K_{i-1}, J_i)$. Thus, if $K \equiv L$ (mod($n$, $q$)) for some non-negative $q$, then $K \equiv L$ (mod($n$, 1)). Note that if $J_1$ bounds a disc which intersects $K$ in two points, and $n = 1$, then $1/n$ surgery along $J_1$ is the standard notion of a crossing change.

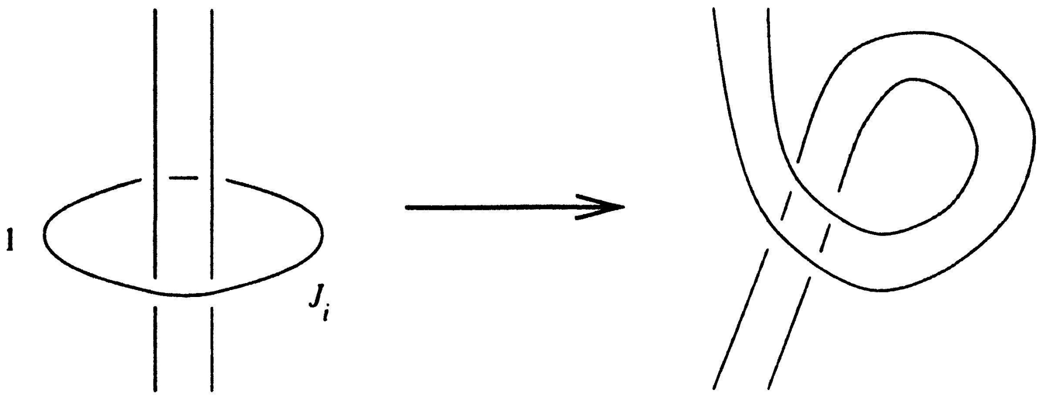


Figure 2

In §2, the main theorem is proved, and a number of corollaries are deduced. In §3, we generalise the notion of congruence to an equivalence relation between closed connected oriented 3-manifolds. We show that the quantum-$SU(2)$ invariants also provide information about these congruence classes.

## 2. The main theorem

For a framed link $K$ in a closed connected oriented 3-manifold $M$, we shall often be considering the quantum-$SU(2)$ invariant at a specified root of unity $A$. We shall adopt the terminology of Lickorish in his paper [7]. Strictly speaking, the invariant for the framed link is a linear form on $\mathcal{S}(S^1 \times I)^{\otimes \# K}$, that is the tensor product of $\# K$ copies of the linear skein of the annulus. However, we shall evaluate this form by inserting $\alpha$, that is a single strand going round the annulus, into each copy of $\mathcal{S}(S^1 \times I)$. This gives a complex number which we shall denote, by a mild abuse of terminology, $\mathcal{J}_A(M, K)$. The following is the main theorem of this paper.

THEOREM 2.1. *Let $n$ be an integer greater than one. Let $A$ be a primitive $4n^{th}$ root of unity. Let $K$ and $L$ be framed links in a closed connected oriented 3-manifold $M$. If $K \equiv L$ (mod($n$, 2)) or $K \equiv L$ (mod($4n$, 1)), then $\mathcal{J}_A(M, K) = \mathcal{J}_A(M, L)$.*

*Proof.* It suffices to check that the invariant is preserved by a single move on $K$ as shown in Figure 1, subject to one of the following conditions:

(i) there is an even number of strands running from top to bottom, or

(ii) $n$ is replaced by $4n$ in Figure 1.

For, $K \equiv L$ (mod($n$, 2)) if and only if $K$ and $L$ differ by a sequence of moves satisfying (i), and $K \equiv L$ (mod($4n$, 1)) if and only if $K$ and $L$ differ by a sequence of moves satisfying (ii).

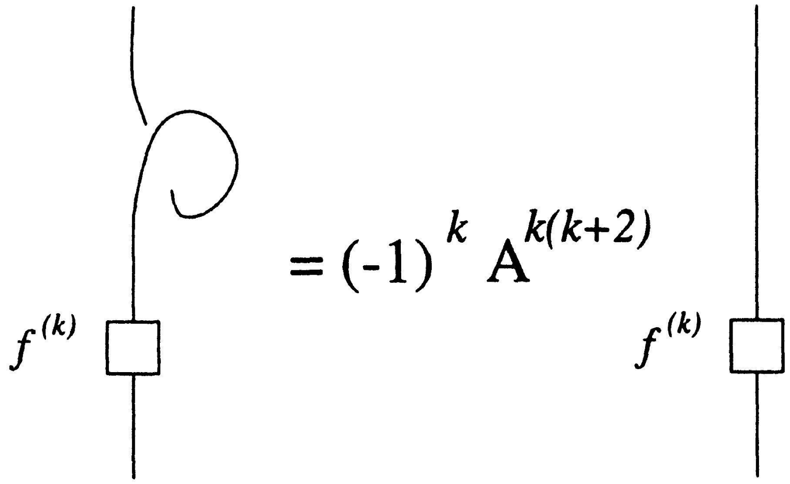


Figure 3

We shall use the equality shown in Figure 3, a proof of which is in [16]. The symbol $f^{(k)}$ refers to the element of the linear skein of the disc with $2k$ marked points in its boundary which is defined in Lemma 1 of [7].

Recall the map [7]

$$\langle , , \ldots , \rangle_D : \mathscr{S}(S^1 \times I) \times \cdots \times \mathscr{S}(S^1 \times I) \to \mathscr{S}(\mathbb{R}^2)$$

associated with a planar link diagram $D$. Recall also that $S_k(\alpha)$ is the element of $\mathscr{S}(S^1 \times I)$ obtained by inserting $f^{(k)}$ into the annulus and then joining up the $2k$ points in the standard way with strings encircling the annulus. Then, $\Delta_k$ denotes $\langle S_k(\alpha) \rangle_U$, where $U$ is a diagram of the unknot with zero framing. Recall also the definitions of the elements $\omega$, $\omega_0$ and $\omega_1$ of $\mathscr{S}(S^1 \times I)$.

$$\omega = \sum_{k=0}^{n-2} \Delta_k S_k(\alpha) \qquad \omega_0 = \sum_{\substack{k=0 \\ k \text{ even}}}^{n-2} \Delta_k S_k(\alpha) \qquad \omega_1 = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n-2} \Delta_k S_k(\alpha).$$

The equality in Figure 3, together with the assumption that $A$ is a $4n^{th}$ root of unity, implies the equality shown in Figure 4. It also implies a similar equality, with $\omega_0$ replaced by $\omega$, and with $n$ kinks replaced by $4n$. It is exactly this freedom to change the framings of surgery curves which is the basis of this paper.

Let $H$ be the standard diagram of the Hopf link with each component having zero framing.

CLAIM. *$\langle \omega, \omega \rangle_H$ and $\langle \omega_0, \omega_0 \rangle_H$ are both non-zero.*

*Proof.* Now, $\langle \omega, \omega \rangle_H = \langle \omega \rangle_U$, by Lemma 6 of [7]. It is proved in 4.1 of [7] that this is non-zero. The claim will be proved if we can show that $\langle \omega_0, \omega_0 \rangle_H$

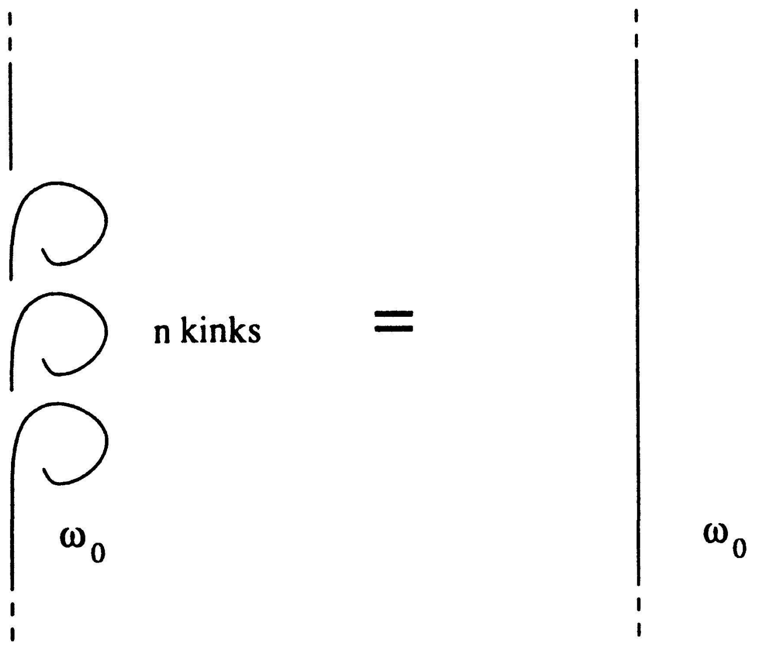


Figure 4

is a non-zero multiple of $\langle\omega, \omega\rangle_H$. Note first that Lemma 1 (iv) of [7] implies that

$$\Delta_k = (-1)^n \Delta_{n-2-k}.$$

This also follows from Proposition 9 (the Symmetry Principle) of [6], which was first introduced by Kirby and Melvin in [4]. The Symmetry Principle also gives that

$$\langle S_k(\alpha), \alpha^j\rangle_H = (-1)^{j+n}\langle S_{n-2-k}(\alpha), \alpha^j\rangle_H$$

and hence that

$$\langle \Delta_k S_k(\alpha), \alpha^j\rangle_H = (-1)^j\langle \Delta_{n-2-k} S_{n-2-k}(\alpha), \alpha^j\rangle_H.$$

Therefore, the following equalities hold.

$$\langle \Delta_k S_k(\alpha), \omega_0\rangle_H = \langle \Delta_{n-2-k} S_{n-2-k}(\alpha), \omega_0\rangle_H,$$

$$\langle \Delta_k S_k(\alpha), \omega_1\rangle_H = -\langle \Delta_{n-2-k} S_{n-2-k}(\alpha), \omega_1\rangle_H.$$

We shall now consider the cases of $n$ odd and $n$ even separately.

Case A: $n$ odd. Then we have the following equalities.

$$\langle\omega_1,\omega_1\rangle_H = -\langle\omega_0,\omega_1\rangle_H,$$

$$\langle\omega_1,\omega_0\rangle_H = \langle\omega_0,\omega_0\rangle_H,$$

$$\langle\omega,\omega\rangle_H \quad = 2\langle\omega_0,\omega_0\rangle_H.$$

This proves the claim in this case.

Case B: $n$ even. In this case,

$$\langle\omega_1,\omega_1\rangle_H = 0,$$

$$\langle\omega_0,\omega_1\rangle_H = 0,$$

$$\langle\omega,\omega\rangle_H \quad = \langle\omega_0,\omega_0\rangle_H,$$

which establishes the claim.

The sequence of equalities in Figure 5 establishes that, if $K \equiv L$ (mod($n$, 2)), then $\mathcal{I}_A(M, K) = \mathcal{I}_A(M, L)$. However, a number of the equalities require further explanation. The first and sixth equalities are trivial, although we are implicitly assuming that $\langle\omega_0,\omega_0\rangle_H$ is non-zero, which was proved in the Claim. The third and fifth equalities are an application of that in Figure 4. The second and fourth follow by repeated use of the fact an element of $\mathcal{S}(\mathbb{R}^2)$ remains unchanged when an even number of strands are slid, via Kirby moves, over a component decorated with $\omega_0$. Hence, in the second figure, we slide the vertical curves over one of the components decorated with $\omega_0$. In the fourth figure, we slide the vertical curves over the component containing $n$ kinks (call this curve $C_1$, say). This operation adds $n$ full twists to the vertical curves. However, $C_1$ becomes entangled with these curves. But the other component decorated with $\omega_0$ ($C_2$, say) now bounds a disc which intersects $C_1$ in a single point and which is disjoint from all other curves. Hence, by the argument of Lemma 4.5 in Chapter I of [3], we may pull $C_1$ and $C_2$ clear of the vertical curves. Note that, in this process, we slide an even number of curves over $C_2$. This establishes the fourth equality.

A similar sequence of equalities, with $n$ and $\omega_0$ replaced throughout by $4n$ and $\omega$ respectively, and where we allow any number of strings to run from top to bottom, establishes the theorem when $K \equiv L$(mod($4n$, 1)). □

We now use Theorem 2.1 to relate the congruence classes of a link to its Jones polynomial.

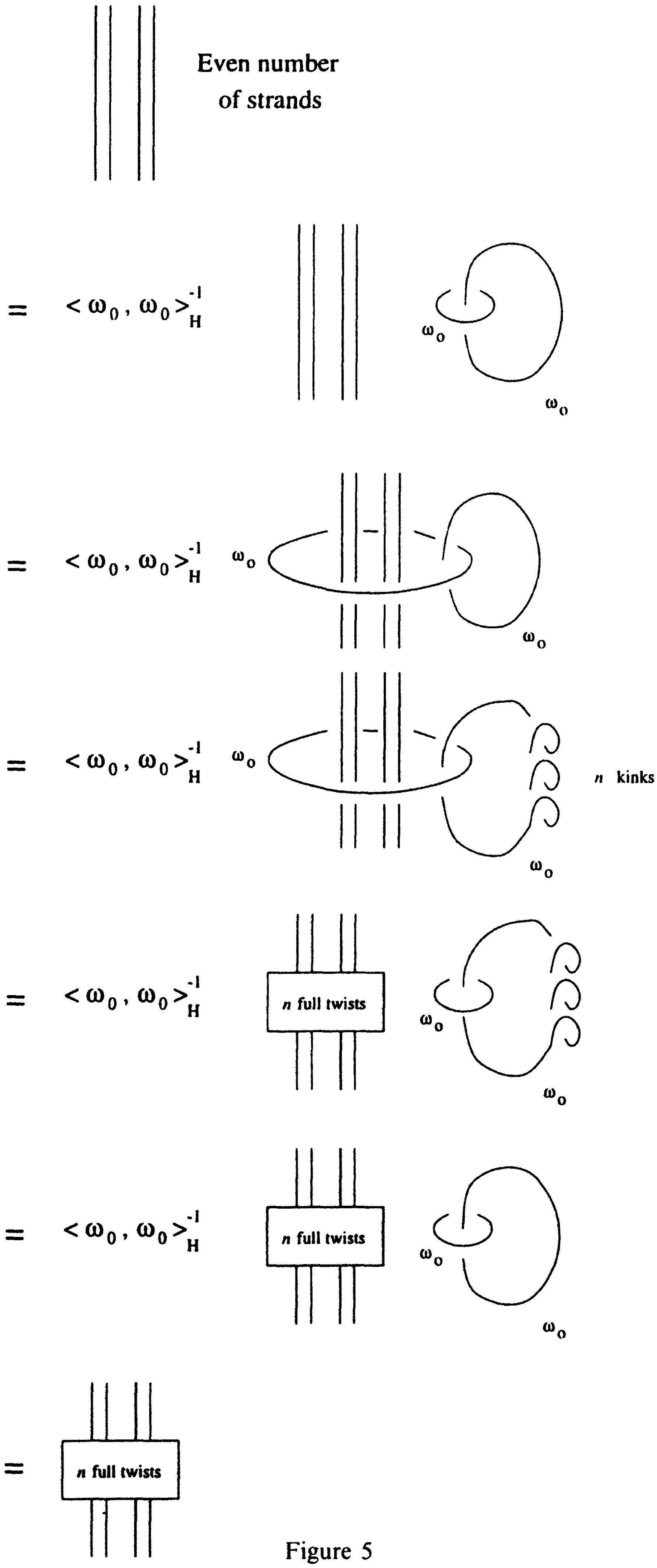


Figure 5

THEOREM 2.2. *Let n be an integer greater than two. Let A be a primitive* $4n^{th}$ *root of unity. Let K and L be oriented framed links in* $S^3$. *If* $K \equiv L \ (mod(n, 2))$ *or* $K \equiv L \ (mod(4n, 1))$, *then* $V_K(A^{-4}) = V_L(A^{-4})$.

*Proof.* Theorem 2.1 implies that

$$\mathcal{I}_A(S^3, K) = \mathcal{I}_A(S^3, L).$$

Now,

$$\langle K \rangle = \frac{\mathcal{I}_A(S^3, K)}{(-A^{-2} - A^2)},$$

where $\langle \ \rangle$ denotes the Kauffman bracket of a framed link evaluated at the complex number $A$. Note that $(-A^{-2} - A^2)$ is non-zero. The theorem is almost proved, since $\langle K \rangle$ and $V_K(A^{-4})$ differ only by a factor of $(-A)^{-3w(K)}$, where $w(K)$ is the writhe of $K$. Now, if two framed links $K_{i-1}$ and $K_i$ differ by a move as shown in Figure 1, then their writhes differ by $n[lk(K_{i-1}, J_i)]^2$. (See [13] for instance). Thus, $w(K)$ and $w(L)$ differ by a multiple of $4n$. This implies that $V_K(A^{-4}) = V_L(A^{-4})$. □

Thus, the Jones polynomial of a link greatly restricts the possible congruence classes to which it belongs.

*Example* 2.3. Let $K$ be the (right-handed) trefoil knot. Then $V_K(t) = -t^4 + t^3 + t$. Then

$$V_K(t) - 1 = (t-1)(-t^3 + 1).$$

Thus, $K$ is not congruent modulo $(n, 2)$ to the unknot for any $n$ greater than 3.

COROLLARY 2.4. *If two oriented links K and L in* $S^3$ *have different Jones polynomials, then, for any framings on K and L,* $K \equiv L \ (mod(n, 2))$ *for at most finitely many n.*

*Proof.* The equation $V_K(t) - V_L(t) = 0$ has only a finite number of roots. □

Further information about the congruence classes of a link can be found by consideration of its parallels.

DEFINITION 2.5. Let $K$ be an oriented framed link in a 3-manifold $M$. For any positive integer $j$, define the $j^{th}$ parallel of $K$, written $K^j$, to be the oriented framed link having $j$ parallel components for each component of $K$, the choice of parallel being determined by the framing on $K$. The framing and orientation of each component of $K^j$ come from the framing and orientation of the relevant component of $K$.

For example, when $K$ is a zero-framed knot in $S^3$, then two components of $K^j$ have linking number zero, and each component has framing zero. The following lemma is immediate.

LEMMA 2.6. *Let $K$ and $L$ be oriented framed links in a 3-manifold $M$. Let $n$ and $q$ be non-negative integers, and let $j$ be a positive integer.*

(1) *If $K \equiv L$ (mod$(n, q)$), then $K^j \equiv L^j$ (mod$(n, qj)$).*

(2) *If $K \equiv L$ (mod$(n, q)$), then $K \equiv L$ (mod$(n', q')$), for any non-negative integers $n'$ and $q'$ satisfying $n'|n$ and $q'|q$.*

COROLLARY 2.7. *Let $n$ be an integer greater than one. Let $A$ be a primitive $4n^{th}$ root of unity. Let $K$ and $L$ be framed links in a closed connected oriented 3-manifold $M$.*

(1) *If $K \equiv L$ (mod$(n, 2)$), then $\mathcal{I}_A(M, K^j) = \mathcal{I}_A(M, L^j)$ for all natural numbers $j$.*

(2) *If $K \equiv L$ (mod$(n, 1)$), then $\mathcal{I}_A(M, K^j) = \mathcal{I}_A(M, L^j)$ for all even natural numbers $j$.*

(3) *If $K \equiv L$ (mod$(4n, 1)$), then $\mathcal{I}_A(M, K^j) = \mathcal{I}_A(M, L^j)$ for all natural numbers $j$.*

*Proof.* If $K \equiv L$ (mod$(n, 2)$), then by Lemma 2.6, $K^j \equiv L^j$ (mod$(n, 2)$). (1) now follows from Theorem 2.1. Parts (2) and (3) are proved similarly. □

COROLLARY 2.8. *Let $n$ be an integer greater than two. Let $A$ be a primitive $4n^{th}$ root of unity. Let $K$ and $L$ be oriented framed links in $S^3$.*

(1) *If $K \equiv L$ (mod$(n, 2)$), then $V_{K^j}(A^{-4}) = V_{L^j}(A^{-4})$ for all natural numbers $j$.*

(2) *If $K \equiv L$ (mod$(n, 1)$), then $V_{K^j}(A^{-4}) = V_{L^j}(A^{-4})$ for all even natural numbers $j$.*

(3) *If $K \equiv L$ (mod$(4n, 1)$), then $V_{K^j}(A^{-4}) = V_{L^j}(A^{-4})$ for all natural numbers $j$.*

*Proof.* Apply Lemma 2.6 and Theorem 2.2. □

COROLLARY 2.9. *Let $K$ and $L$ be oriented framed links in $S^3$. Suppose that there is some even natural number $j$ such that $K^j$ and $L^j$ have distinct Jones polynomials. Then, for any natural number $q$, $K \equiv L$ (mod$(n, q)$) for at most finitely many $n$.*

*Proof.* This is proved in the same way as Corollary 2.4, together with the observation from Lemma 2.6 (2) that $K \equiv L \pmod{(n, q)}$ implies that $K \equiv L \pmod{(n, 1)}$. □

Given the efficacy with which the Jones polynomial distinguishes links, the above corollary establishes the following conjecture in a large number of cases.

CONJECTURE 2.10. (cf. [2]) *If $K$ and $L$ are two different oriented links in $S^3$, then, for any non-negative integer $q$ and choice of framings on $K$ and $L$, $K \equiv L \pmod{(n, q)}$ for at most finitely many $n$.*

The following corollary relates the notion of crossing number to that of congruence classes.

COROLLARY 2.11. *Suppose that $K$ and $L$ are two knots with distinct Jones polynomials. Let $K$ and $L$ have crossing number $c(K)$ and $c(L)$ respectively. Then, for any framings on $K$ and $L$, $K \not\equiv L \pmod{(n, 2)}$ for any $n > 3\max\{c(K), c(L)\} + 1$.*

*Proof.* Throughout, we shall use the 'state-sum' terminology of [8]. Pick a diagram $D$ for the knot $K$, with $c(D)$ crossings. We do not insist that the framing of $K$ is the same as the blackboard framing due to $D$. A state for $D$ is a function $s: \{i \in \mathbb{N} : 1 \leq i \leq c(D)\} \to \{-1, 1\}$. A state $s$ gives a diagram $sD$ with the crossings of $D$ removed in a way determined by $s$. See [8] for more details. There it is shown that the Kauffman bracket $\langle D \rangle$ is a polynomial in $A$ with highest order $M\langle D \rangle$ satisfying

$$M\langle D \rangle \leq c(D) + 2|s_+(D)| - 2,$$

where $s_+$ is the state which sends all numbers to 1, and where $|s_+(D)|$ is the number of curves in the diagram $s_+(D)$. Now, a simple induction on $c(D)$ establishes that

$$|s_+(D)| \leq c(D) + 1,$$

and hence

$$M\langle D \rangle \leq 3c(D).$$

Similarly the lowest order $m\langle D \rangle$ satisfies the inequality

$$m\langle D \rangle \geq -3c(D).$$

Also, the writhe $w(D)$ of the diagram $D$ satisfies the inequality

$$|w(D)| \leq c(D).$$

Therefore, $V_K(t)$ is a polynomial in $t$ with lowest order $m(V_K(t))$ and highest order $M(V_K(t))$ satisfying

$$m(V_K(t)) \geq \frac{-3c(K)}{2},$$

$$M(V_K(t)) \leq \frac{3c(K)}{2}.$$

Thus, $V_K(t) - V_L(t)$ is a polynomial in $t$ with breadth $B(V_K(t) - V_L(t))$ satisfying

$$B(V_K(t) - V_L(t)) \leq 3 \max\{c(K), c(L)\}.$$

CLAIM. *If* $K \equiv L$ (mod($n$, 2)) *and* $\theta$ *is an* $n^{th}$ *root of unity other than* $-1$, *then* $V_K(\theta) = V_L(\theta)$.

If $\theta$ is an $n^{\text{th}}$ root of unity, then it is a primitive $n_1^{\text{th}}$ root of unity for some natural number $n_1$ which divides $n$. If $n_1 = 1$, then $\theta = 1$ and it is well known that $V_K(1) = V_L(1)$. If $n_1 = 2$, then $\theta = -1$ which is contrary to assumption. Hence, we may assume that $n_1 > 2$. It is not hard to find a primitive $4n_1^{\text{th}}$ root of unity $A$ such that $\theta = A^{-4}$. If $K \equiv L$ (mod($n$, 2)), then by Lemma 2.6 (2), $K \equiv L$ (mod($n_1$, 2)) and hence by Theorem 2.2, $V_K(\theta) = V_L(\theta)$. This proves the Claim.

Thus, if $K \equiv L$ (mod($n$, 2)), then

$$B(V_K(t) - V_L(t)) \geq n - 1.$$

and hence

$$n \leq 3 \max\{c(K), c(L)\} + 1.$$ □

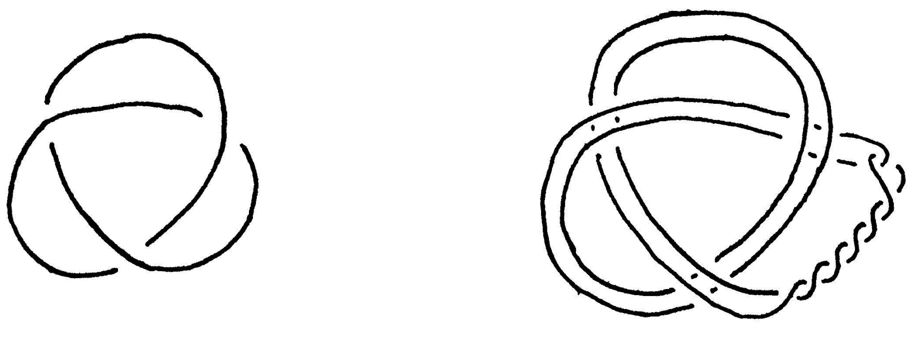

Figure 6

*Examples* 2.12. Let $K$ be a knot with a reduced alternating diagram $D$. Let $DK$ be an untwisted double of $K$. Then $DK$ has a diagram with $4c(D)+2|w(D)|+2$ crossings, where $c(D)$ is the number of crossings of $D$ and $w(D)$ is its writhe. See Figure 6. It is shown in [8] that $DK$ has non-trivial Jones polynomial. Thus, Corollary 2.11 gives that $DK$ is not congruent modulo $(n, 2)$ to the unknot for any $n$ greater than $12c(D)+6|w(D)|+7$. In fact, sharper bounds may be deduced, using the fact that one of the knots in Corollary 2.11 is the unknot. Since, $DK$ has trivial Alexander module, classical methods (for example [10]) could not have given this result.

## 3. A generalisation – congruence of manifolds

The kernel of the proof of Theorem 2.1 was that changing the framing on a certain unknotted surgery curve by a multiple of $n$ did not change the quantum-$SU(2)$ invariants at certain roots of unity. The point of this section is to show that similar results hold when the surgery curve is arbitrary. Thus we investigate the following equivalence relation on closed connected oriented 3-manifolds.

DEFINITION 3.1. Let $n$ be a non-negative integer. Let $M$ and $M'$ be closed connected oriented 3-manifolds. Then $M$ and $M'$ are said to be **congruent modulo** $n$, written $M \equiv M' \pmod{n}$, if there are links $L_1, \ldots, L_m$ in $S^3$ and framings $F_i$ and $F'_i$ on each $L_i$ satisfying the following conditions.

(1) The framings on each component of $L_i$ which arise from $F_i$ and $F'_i$ differ by a multiple of $n$.
(2) Surgery on $L_i$ with framing $F'_i$, and surgery on $L_{i+1}$ with framing $F_{i+1}$ both yield the same oriented manifold.
(3) Surgery on $L_1$ with framing $F_1$ yields $M$, and surgery on $L_m$ with framing $F'_m$ yields $M'$.

Note that, by [5], there is a framed link in $S^3$ surgery along which yields a given closed connected oriented 3-manifold $M$. When $n$ is not divisible by 4, there is a quantum invariant defined for $M$ together with a specified class in $H^1(M; \mathbb{Z}_2)$. The existence of this invariant was first noted by Turaev in [14]. Lickorish has exhibited a skein-theoretic version in [7]. We denote the invariant associated with the zero cohomology class by $\mathcal{I}^0_A(M)$.

THEOREM 3.2. *Let $n$ be an integer greater than two, and let $A$ be a primitive $4n^{th}$ root of unity. Let $M$ and $M'$ be closed connected oriented 3-manifolds.*

(1) *If* $M \equiv M'$ *(mod* $4n$*), then* $|\mathscr{I}_A(M)| = |\mathscr{I}_A(M')|$.
(2) *If* $n$ *is not divisible by* 4, *and* $M \equiv M'$ (mod $n$), *then* $|\mathscr{I}_A^0(M)| = |\mathscr{I}_A^0(M')|$.

*Proof.* It suffices to prove the theorem when $m = 1$ in Definition 3.1. Let $D$ be a diagram of $L_1$ with framing $F_1$, and let $D'$ be the same diagram, but with the framings altered so as coincide with $F'_1$. Then, in the terminology of [7, p. 185],

$$\mathscr{I}_A(M) = \langle \mu\omega, \ldots, \mu\omega \rangle_D \langle \mu\omega \rangle_{U_-}^{\sigma} \langle \mu\omega \rangle_U^{-1}.$$

Here, $U_-$ is a diagram of the unknot with framing $-1$. (Similarly, $U_+$ is a diagram of the unknot with framing 1.) Also, $\mu$ is a real number satisfying $\mu^{-2} = \langle \omega \rangle_{U_+} \langle \omega \rangle_{U_-}$, and $\sigma$ is the signature of the linking matrix associated with $D$. Now, $|\langle \mu\omega \rangle_{U_-}| = 1$, since $\langle \mu\omega \rangle_{U_+} = \langle \mu\omega \rangle_{U_-}^{-1}$, and $\langle \mu\omega \rangle_{U_+}$ and $\langle \mu\omega \rangle_{U_-}$ are complex conjugate. Therefore, in Case (1),

$$\begin{aligned} |\mathscr{I}_A(M)| &= |\langle \mu\omega, \ldots, \mu\omega \rangle_D| \, |\langle \mu\omega \rangle_U^{-1}| \\ &= |\langle \mu\omega, \ldots, \mu\omega \rangle_{D'}| \, |\langle \mu\omega \rangle_U^{-1}| \\ &= |\mathscr{I}_A(M')|. \end{aligned}$$

A similar argument, with $\omega$ replaced throughout by $\omega_0$, establishes Case (2). □

*Remark* 3.3. Note that under the conditions of Theorem 3.2 (1), we can deduce that the Turaev-Viro invariants of $M$ and $M'$ associated with the complex number $A$ are the same. See [12] for instance.

## Acknowledgements

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