Codimension one foliations without compact leaves.

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Abstract. A smooth closed connected manifold with Euler characteristic zero and dimension greater than three has a C^1 codimension one foliation with no compact leaf.

Novikov's celebrated compact leaf theorem ([N], see also [HH], [G] or [CL]) states that every foliation of certain three-dimensional manifolds, including S^3, by surfaces, has a compact leaf. Various authors have shown that many closed three-dimensional manifolds have the property that every codimension one foliation (of differentiability class C^0, C^1, or C^2, depending on the manifold) must have a compact leaf (Rosenberg, [P1], p. 352; Plante [P1], Cor. 7.4 and [P2]; Thurston [Th1]; Levitt [Lev]). Kneser [Kn] has shown the same result for C^0 foliations of the Klein bottle. We show that there is no such theorem for C^1 codimension one foliations in dimensions greater than three.

THEOREM. *Let M be a smooth (C^∞) closed manifold of dimension $n \geq 4$ with a C^∞ codimension one foliation F_0. Then M has a codimension one foliation F_1 with no compact leaves, locally defined by a C^1 differential 1-form and such that all the leaves are C^∞ submanifolds of M. Furthermore, F_1 can be chosen C^1-concordant to F_0 and such that their tangent plane fields TF_0 and TF_1 are homotopic as subbundles of the tangent bundle TM.*

The Theorem also holds for manifolds with boundary provided that the foliations are taken to be transverse to the boundary of M. The concordance of F_0 and F_1 implies that the underlying $(C^1$, codimension one) Haefliger Γ_1^1-structures [H2] are homotopic. In view of Thurston's existence theorem for codimension one foliations [Th3], the Theorem immediately implies the following Corollary.

COROLLARY. *Every smooth closed connected manifold of dimension $n \geq 4$ with vanishing Euler characteristic has a codimension one foliation with no compact leaves locally defined by a C^1-form and with all leaves smooth. Such a foliation exists in every homotopy class of tangent plane fields and of Haefliger Γ_1^1-structures.*

A weaker version of this result, a C^0 construction on manifolds of dimension five or more, was announced in [Sch2] and [Sch3]. B. Raymond [Ray] has also constructed C^0 codimension one foliations of spheres of odd dimension greater than four with no compact leaves with a Lipschitz tangent bundle and all leaves smooth.

Solodov [So] and Hector and Hirsch [HH] have proven Novikov's compact leaf theorem for C^0 foliations, and some of Plante's compact leaf theorems also hold for C^0 foliations. While for every r, $0 \leq r < \infty$, there exist C^r foliations that are not homeomorphic to C^{r+1} foliations in codimension two [Har1] and even in codimension one ([CC], [CC1], [Ts]), the most significant qualitative difference seems to occur between codimension one foliations of classes C^1 and C^2. Since our construction is C^1 but definitely not C^2 (see Remark 4.5), the following interesting question remains open.

Question 1. Does there exist a closed connected smooth manifold of dimension greater than three that admits a codimension one foliation – for example S^5 or $S^1 \times S^3$ – on which every codimension one foliation of class C^2 (or C^∞) has a compact leaf?

On a compact manifold any foliation must have a **minimal set**, that is, a closed nonempty union of leaves which contains no other such set. This follows from Zorn's lemma and the fact that the intersection of a nested family of nonempty compact sets is nonempty. (On the other hand, there are foliations on noncompact manifolds with no minimal sets, such as foliation (1a) of [Hec].) A minimal set of a codimension one foliation can be a closed leaf, the whole manifold (when every leaf is dense), or an **exceptional minimal set**, characterized by meeting closed transversal curves in Cantor sets. Thus the preceding question is related to the following one.

Question 2. Which compact manifolds have minimal codimension one foliations of class C^r (for example, $r = 0$, 2, or ∞)? Does S^5 or $S^1 \times S^3$ have such a foliation?

One can construct many minimal codimension one foliations, such as linear foliations of the torus T^2 with irrational slope, or foliations of a T^2 fiber bundle transverse to the fiber and inducing such a foliation on each fiber.

Plante has shown an important connection between compact leaves and transverse invariant measures for transversely oriented C^0 codimension one foliations of a compact manifold M. If $H_1(M; \mathbb{R})$ has dimension 0 or 1, then every leaf in the support of such a measure is compact ([P1], Th. 6.3). Such a measure exists if any leaf has nonexponential growth ([P1], Th. 4.1). It follows that if $\dim H_1(M; \mathbb{R}) \leq 1$, then either every leaf has exponential growth (as in our construction) or there exists a compact leaf. Sullivan [Sul] has reformulated and extended Plante's work in terms of foliation cycles.

For C^r foliations of codimension greater than one there are local constructions which produce a foliation with the given differentiability and no compact leaves, $0 \leq r \leq \infty$. The case of codimension greater than two has been known for some time [Sch1]. In codimension two the original C^1 plug [Sch1], which gave a negative response to Seifert's question about the existence of flows on S^3 with no periodic orbit, was improved to a C^2 plug by Jenny Harrison, using a delicate C^2 construction of a diffeomorphism of the plane preserving a fractal circle [Har2]. Recently K. Kuperberg found a remarkable C^∞ plug [Ku] (also see [KK], [KuG]), thus giving a local C^∞ construction for opening compact leaves of any smooth codimension two foliation.

The key idea of our construction is to insert an exceptional minimal set and another noncompact leaf which together form a compact set C which separates the manifold. Then nearby leaves are made to spiral in towards C from opposite "sides" in different directions, by analogy to the Reeb foliation of S^3, in which two solid tori foliated as **Reeb components** (see Figure 1) are glued together using a diffeomorphism of the bounding tori which interchanges meridians and longitudes, so that the noncompact leaves spiral in towards the torus in one direction on one side, and in the orthogonal direction on the other side. This phenomenon of leaves
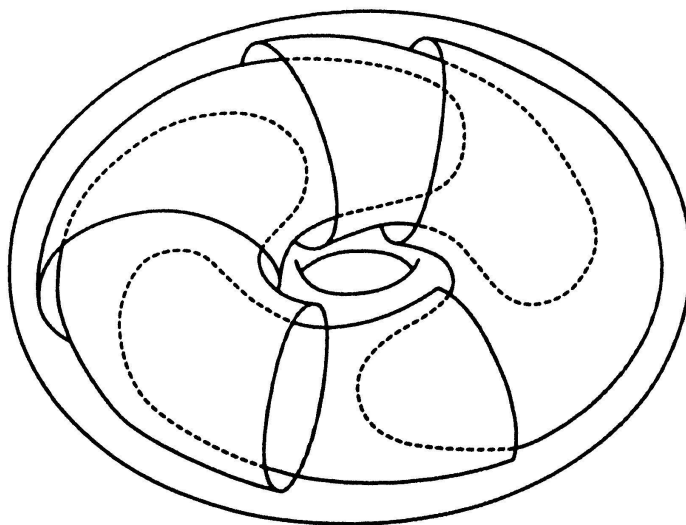

Figure 1. A Reeb component.

spiralling in towards a two-sided compact leaf in different directions on opposite sides of the leaf occurs frequently in explicit constructions of codimension one foliations (see [L]). The process which produces such spiralling around an exceptional minimal set – a process which we call **tilting** – is described in §4 and is shown to be C^1 in §5. The construction of the exceptional minimal set, described in §3, depends essentially on B. Raymond's remarkable construction of a C^∞ foliation of S^3 with an exceptional minimal set [Ray], modified by using an exceptional minimal set derived from a different group of diffeomorphisms of S^1 due to Hector [Hec].

To begin the construction modifying the given foliation F_0, in §1 we exploit the two-sided holonomy of an isolated compact leaf to obtain a "tube" $S^1 \times D^{n-1}$ with convenient properties. Then in §2 we modify F_0 inside the tube to a new C^∞ foliation F_1 on the tube minus a certain number of "holes", each diffeomorphic to $S^1 \times \text{Int } D^2 \times S^{n-3}$. The proof of the Theorem is completed in §4 by filling in the holes by using Propositions 4.1 and 4.2. At the end of §4 we also give a direct construction of foliations without compact leaves on $S^1 \times S^{n-1}$ ($n \geq 4$) using Proposition 4.1 but not using the constructions of §1 and §2.

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§1. Tubes with convenient holonomy

Notation and conventions. Throughout this paper M^n will be a smooth compact connected manifold of dimension $n \geq 4$, possibly with boundary ∂M. We shall assume that all foliations and plane fields have codimension one, unless the context clearly indicates the contrary (as in the case of a transverse one dimensional foliation). In the first three sections all structures will be smooth (which means of class C^∞). Let $D^k(r)$ denote the closed ball with center 0 and radius r in Euclidean k-dimensional space $\mathbb{R}^k$, and $S^{k-1}(r)$ or $\partial D^k(r)$ its boundary $(k-1)$-sphere; when $r = 1$ we write D^k and $\partial D^k = S^{k-1}$. The interior and boundary of a manifold (possibly with corners) N are denoted by $\text{Int } N$ and ∂N. By a **tube** we mean compact manifold (with boundary and possibly with corners) whose interior is diffeomorphic to $S^1 \times \text{Int } D^{n-1}$. On the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ let $F(t)$ denote the linear foliation obtained as the quotient of the foliation of the plane by parallel lines of slope $t \in \mathbb{R}$. Recall that a smooth mapping of manifolds $f: M \to N$ which is transverse to a foliation F on N induces a foliation $f^{-1}(F)$ on M, the **pullback** of F, whose leaves are the components of $f^{-1}(L)$ for each leaf L of F. Let $\pi : A = S^1 \times D^2 \times S^{n-3} \to S^1 \times D^2$ be the obvious projection. We can now state the main result of the first two sections.

PROPOSITION 1.1. *Let F_0 be a (smooth, codimension one) foliation of M transverse to ∂M, $\dim M \geq 4$. Then there exist finitely many disjoint embeddings*

$$g_i : A = S^1 \times D^2 \times S^{n-3} \to M, \qquad i = 1, \ldots, k,$$

and a (smooth) foliation F_1 of

$$M_1 = M - \bigcup_i \operatorname{Int} g_i(A)$$

transverse to ∂M_1 such that

(i) *Every compact leaf of F_1 meets some boundary component $g_i(\partial A)$;*

(ii) *For each i there exists $t_i \in \mathbb{R} - \{0\}$ such that*

$$(g_i|_{\partial A})^{-1} F_1 = (\pi|_{\partial A})^{-1} F(t_i).$$

Furthermore there are finitely many disjoint tubes $T_j \subset \operatorname{Int} M, j = 1, \ldots, l$, such that

$$\bigcup_i g_i(A) \subset \bigcup_j \operatorname{Int} T_j$$

and F_0 and F_1 coincide on $M - \bigcup_j T_j$, and there non-singular vector fields X_0 and X_1 on $\bigcup_j T_j$, respectively transverse to F_0 on $\bigcup_j T_j$ and to F_1 on $\bigcup_j T_j - \bigcup_i g_i(A)$, such that:

(iii) *X_0 is homotopic to X_1 by a homotopy through non-singular vector fields with support in $\bigcup_j \operatorname{Int} T_j$;*

(iv) *For each i, $X_1|_{g_i(A)} = g_{i*}(\partial/\partial s)$ where $\partial/\partial s$ is the vector field that generates the action of S^1 on the first factor of $A = S^1 \times D^2 \times S^{n-3}$.*

The vector fields X_0 and X_1 will be used to control the homotopy class of the tangent plane fields to the foliations.

We shall suppose that not all the leaves of F_0 are compact for if they are, we can modify F_0 by the usual "turbulization" construction to insert a Reeb component inside a tube T containing a transverse circle meeting all the leaves, producing a foliation with only one compact leaf and a tangent plane field in the same homotopy class as that of F_0. The constructions below can then be carried out inside the tube T.

The proof of Proposition 1.1 will be carried out in five steps. Steps 2, 3 and 4 deal with auxiliary foliations on 3-dimensional tubes.

Step 1. Construct loops γ_j with two-sided holonomy and disjoint tubes around them.

Step 2. Cut holes in each tube and modify the foliation.

Step 3. Control the homotopy class.

Step 4. Linearize the foliation on the boundary of each hole.

Step 5. Return to dimension $n > 3$.

Step 1. *Construct loops γ_j with two-sided holonomy and disjoint tubes around them*

Fix a Riemannian metric on M and let $F_0^\perp$ be the one-dimensional foliation orthogonal to F_0. The goal of Step 1 will follow easily from the following construction (see Figure 3A).

PROPOSITION 1.2. *There exist finitely many (smooth) embeddings $f_j : S^1 \times D^1 \to M$ with disjoint images, $j = 1, \ldots, l$, satisfying the following conditions:*

(i) *Every compact leaf of F_0 meets $\bigcup_j f_j(S^1 \times D^1(\frac{1}{2}))$.*

(ii) *the curve $f_j(S^1 \times \{0\})$ lies entirely on some compact leaf L_j and is orientation preserving in both L_j and M.*

(iii) *Each arc $f_j(\{s\} \times D^1)$ lies in a leaf of $F_0^\perp$, $s \in S^1$.*

(iv) *Each foliation $f_j^{-1}(F_0)$ has constant nonzero slope in a neighborhood of each of the curves $S^1 \times \{\pm\frac{1}{2}\}$.*

To begin the construction, consider a leaf L of F_0 and a leaf J of $F_0^\perp$ which meets L at a point p. We recall the definition of the **holonomy** of a loop γ on L based at p, $\gamma : I = [0, 1] \to L$, $\gamma(0) = \gamma(1) = p$. For a sufficiently small interval $J_0 \subset J$ containing p in its interior, we can lift γ along leaves of $F_0^\perp$ to leaves of F_0 near L and thus get a smooth well-defined mapping (see Figure 2)
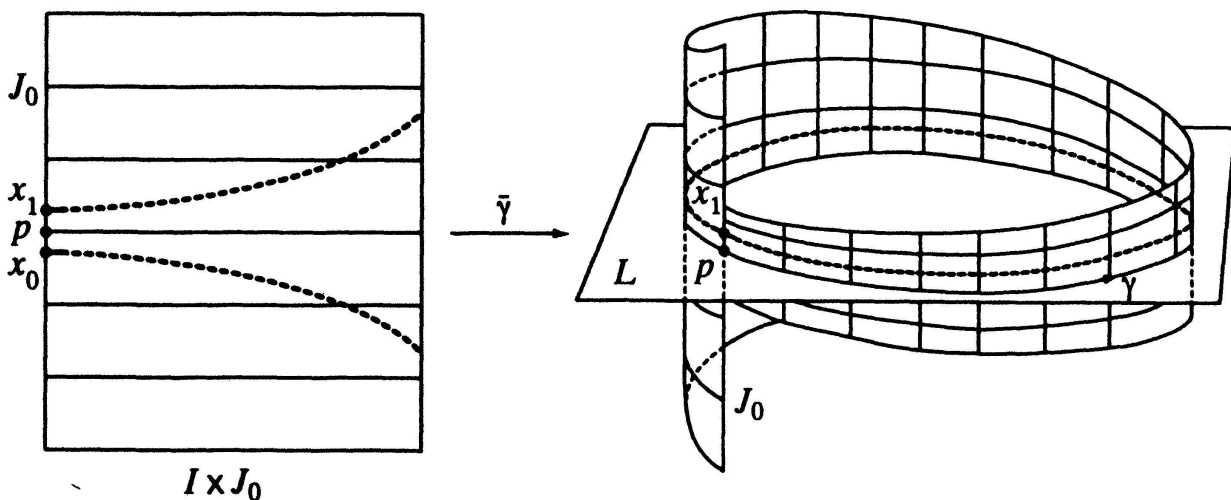

Figure 2. The holonomy mapping $\bar\gamma$ of the loop γ, showing the curves $\beta(I \times \{\pm 1/2\})$.

$$\bar{\gamma}: I \times J_0 \to M \tag{1.3}$$

such that for every $s \in I$ and $x \in J_0$, $\bar{\gamma}(s, x)$ lies on the same leaf of F_0 as x, and on the same leaf of $F_0^\perp$ as $\gamma(s)$. Then the holonomy mapping of γ is the diffeomorphism h_γ of J_0 into J defined by

$$h_\gamma(x) = \bar{\gamma}(1, x), \qquad x \in J_0. \tag{1.4}$$

Now let L be a compact leaf. Fix an orientation of J, thus ordering the interval J_0.

LEMMA 1.5. *The loop γ on L, the interval $J_0 \subset J$ and points $x_0, x_1 \in J_0$, $x_0 < p < x_1$, can be chosen so that γ is orientation preserving in L and in M and*

$$h_\gamma(x_i) \neq x_i, \qquad i = 0, 1. \tag{1.6}$$

Proof. (See Fig. 2.) We recall from the theory of codimension one foliations on compact manifolds that the union of all compact leaves is closed and hence compact ([HH], Part B, p. 96). If follows that J must meet some noncompact leaf, for otherwise its **saturation** Sat(J) (by definition, the union of all leaves that meet J) would be both open and closed, contradicting the hypothesis that M is connected and that not all leaves of F_0 are compact.

Now suppose that L is two-sided in M. We show that it is possible to choose γ_1, J_0 and $x_1 \in J_0$ so that $p < h_{\gamma_1}(x_1) < x_1$. Otherwise there must exist $y > p$ on J such that for all loops γ, $h_\gamma|_{[p, y]}$ is the identity mapping, and it follows by Reeb stability that Sat$[p, y]$ is a one-sided neighborhood of L foliated as a product by F_0 and $F_0^\perp$ restricted to Sat$[p, y]$ (see [HH], Part B, p. 83). Let $y_1 \in J$ be the supremum of all such points y. Then the leaf L_{y_1} of F_0 through y_1 is compact and Sat$[p, y_1]$ is a maximal such one-sided product foliated neighborhood. Since the product structure cannot be extended beyond y_1, there must exist a loop γ^* on L_{y_1} based at y_1, an open interval $J_0 \subset J$ containing y_1, and a point $x_1 > y_1$ in J_0 such that $h_{\gamma^*}(x_1) < x_1$. Enlarge J_0, if necessary, so that $[p, x_1] \subset J_0$. Project γ^* along $F_0^\perp$ to a loop γ_1 on L based at p, so that $\gamma^*(s) = \bar{\gamma}_1(s, y_1)$, where $\bar{\gamma}_1$ is defined as in (1.3). Then $h_{\gamma_1}(x_1) = h_{\gamma^*}(x_1) < x_1$, as claimed.

Applying this argument on the left side of L we obtain a loop γ_0 and a point x_0 in J_0 (possibly extending J_0 to the left) such that $x_0 < h_{\gamma_0}(x_0) < p$. We may suppose that γ_0 and γ_1 are orientation preserving in L, by replacing γ_i by γ_i^2 if necessary. Now if $h_{\gamma_1}(x) \neq x$ for some $x < p$, then replace x_0 by x and $\gamma = \gamma_1$ satisfies (1.6). Use $\gamma = \gamma_0$ analogously if $h_{\gamma_0}(x) \neq x$ for some $x > p$. If no such x exists, then h_{γ_1}

fixes all points in J_0 to the left of p and h_{γ_0} fixes all points to the right, so consequently $\gamma = \gamma_0\gamma_1$ satisfies (1.6).

If the leaf L is one-sided in M then an analogous argument can be applied to a tubular neighborhood of the two-sided double cover of L. $\qquad\qquad\square$

Proof of Proposition 1.2. Given a compact leaf L, the preceding Lemma provides a loop γ on L with two-sided holonomy as in (1.6). We may suppose that $h_\gamma^{-1}(x_i)$ is defined and lies on J_0 for $i = 0, 1$, by replacing x_i by $h_\gamma(x_i)$ if necessary. By a small perturbation of γ we may also suppose that γ determines a smooth embedding of $S^1 = [0, 1]/(0 \sim 1)$ into L. We shall extend γ to a smooth embedding $f : S^1 \times D^1 \to M$ satisfying properties (ii), (iii) and (iv) of Proposition 1.2 with $f_j = f$ and $L_j = L$, by requiring that the diagram

$$
\begin{array}{ccc}
I \times D^1 & \xrightarrow{\ \beta\ } & I \times J_0 \\
{\scriptstyle q \times \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \bar\gamma} \\
S^1 \times D^1 & \xrightarrow{\ f\ } & M
\end{array}
$$

commute, where $q : I \to S^1$ is the quotient mapping, $\bar\gamma$ is the mapping of (1.3), and β is a diffeomorphism of the form

$$\beta(s, t) = (s, \beta_s(t)), \qquad (s, t) \in I \times D^1,$$

onto a certain subset of $I \times J_0$, such that $\beta_0 = \beta_1$, $\beta_0(-\tfrac{1}{2}) = x_0$, $\beta_0(\tfrac{1}{2}) = x_1$, and $\beta_s(0) = p$ for all $s \in I$. By a careful choice of β we may make the curves $\beta(I \times \{\pm\tfrac{1}{2}\})$ (indicated by dotted curves in Fig. 2) transverse to the foliation $I \times \{\text{point}\}$ of $I \times J_0$, and then make the slope of the leaves of $f^{-1}(F_0)$ constant in a neighborhood of each of the curves $S^1 \times \{\pm\tfrac{1}{2}\}$ (see Figure 3A). In this way we obtain an embedding $f_\alpha : S^1 \times D^1 \to M$ for every compact leaf L_α of F_0. Since the union of all the compact leaves is compact, it is covered by a finite number of the open sets $\mathrm{Sat}\, f_\alpha(S^1 \times (-\tfrac{1}{2}, \tfrac{1}{2}))$, say for the compact leaves $L_1, \ldots, L_l$, so that (i) is satisfied. Finally a small perturbation of the embeddings f_j will eliminate any intersections that may occur, in view of (iii) and the fact that the leaves have dimension greater than or equal to three. $\qquad\qquad\square$

To finish Step 1 of the proof of Proposition 1.1, it remains to construct the tubes T_j. In view of (1.2.ii) the surface $f_j(S^1 \times D^1)$ has a trivial normal bundle in M, so that there is a diffeomorphism
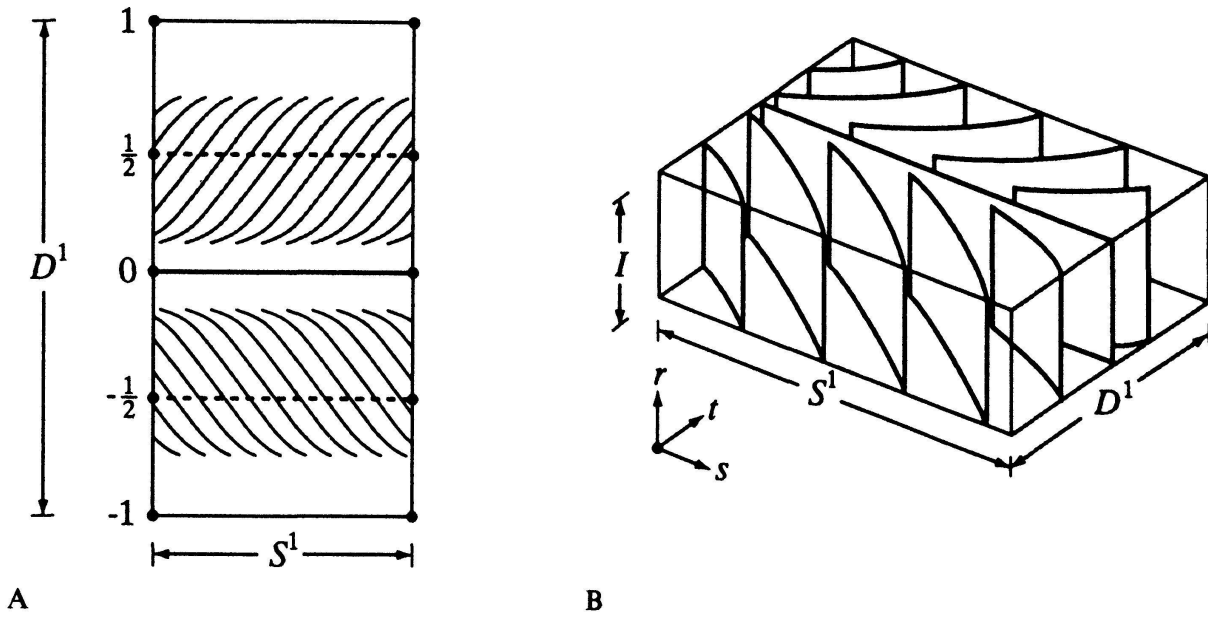
$$S^1 \times D^1 \times D^{n-2} \to N_j(\varepsilon) \tag{1.7}$$

Figure 3. A. $f_j^{-1}(F_0)$ on $S^1 \times D^1$ (cut open). B. $f_j^{-1}(F_0) \times I$ on $S^1 \times D^1 \times I$.

given by $f_j \times \mathrm{Id}_{D^{n-2}}$ followed by the trivialization and multiplication by ε, where $N_j(\varepsilon)$ is the subbundle of normal vectors to $f_j(S^1 \times D^1)$ of length at most ε. By (1.2.iii), $f_j(S^1 \times D^1)$ is orthogonal to F_0. Consequently the vectors of $N_j(\varepsilon)$ are tangent to the leaves of F_0, and so the exponential mapping *in the leaves* of F_0 defines smooth maps

$$\exp_{F_0} : N_j(\varepsilon) \to M, \tag{1.8}$$

which are disjoint embeddings, $j = 1, \ldots, l$, provided that ε is sufficiently small. Composing the diffeomorphisms (1.7) and (1.8) gives embeddings

$$\bar{f}_j : S^1 \times D^1 \times D^{n-2} \to M$$

whose disjoint images $T_j = \bar{f}_j(S^1 \times D^1 \times D^{n-2})$ are the desired tubes. Note that because we have used the exponential mapping in the leaves, rather than in M, the induced foliation coincides with the product of the foliation $f_j^{-1}(F_0)$ on $S^1 \times D^1$ by the disk D^{n-2}, that is,

$$\bar{f}_j^{-1}(F_0) = f_j^{-1}(F_0) \times D^{n-2}. \tag{1.9}$$

§2. Cutting open compact leaves

The diffeomorphisms $\bar{f}_j : S^1 \times D^1 \times D^{n-2} \to T_j$ permit us to work on the product foliation $f_j^{-1}(F_0) \times D^{n-2}$ on $S^1 \times D^1 \times D^{n-2}$, since by (1.9) it is the pullback of F_0

under $\bar{f}_j$. In view of this product structure, Steps 2, 3 and 4 of the Proof of Proposition 1.1 can be carried out in dimension three, where they are easier to understand, and then in Step 5 the resulting foliation will be transposed to the original dimension $n > 3$. As already mentioned, all structures in this section will be smooth.

Step 2. Cut holes in each tube and modify the foliation

Let N be $D^1 \times I$ less the two open disks E_0 and E_1 bounded by the ellipses centered at $(0, \frac{1}{4})$ and $(0, \frac{3}{4})$ with horizontal semiaxis $\frac{1}{2}$ and vertical semiaxis $\frac{1}{8}$, as shown in Figure 4A, and let $\partial_0 N = \partial \bar{E}_0 \cup \partial \bar{E}_1$. We shall modify the restriction F_0' of the foliation $f_j^{-1}(F_0) \times I$ (depicted in Figure 3B) to $S^1 \times N$, so as to obtain a foliation F_1' on $S^1 \times N$ such that

$$F_1' \text{ is transverse to the circles } \{s\} \times \partial \bar{E}_p, \qquad s \in S^1, p = 0, 1. \tag{2.1}$$

We first construct $F_1^+ = F_1'|_{S^1 \times N^+}$, where $N^+ = N \cap (0, 1] \times I$. Using the standard coordinates $s \in \mathbb{R}/\mathbb{Z} = S^1$ and $(t, r) \in [-1, 1] \times [0, 1]$ on N, recall from (1.2.iv) that near $S^1 \times \{\frac{1}{2}\} \times I$, F_0' is defined by a linear equation $s = at$ with $a \neq 0$, say $a > 0$.

Let F^+ be the foliation on $S^1 \times N^+$ defined by $s = at$ on $S^1 \times B$, where $B = N^+ \cap (0, \frac{1}{2}] \times [\frac{1}{4}, \frac{3}{4}]$, and which coincides with F_0' on $S^1 \times (N^+ - B)$ (see Figure 5A). To transform F^+ into the foliation F_1^+ satisfying (2.1), the "tongue" of each leaf of F^+ is bent backwards by rotating it in the S^1 direction through a varying

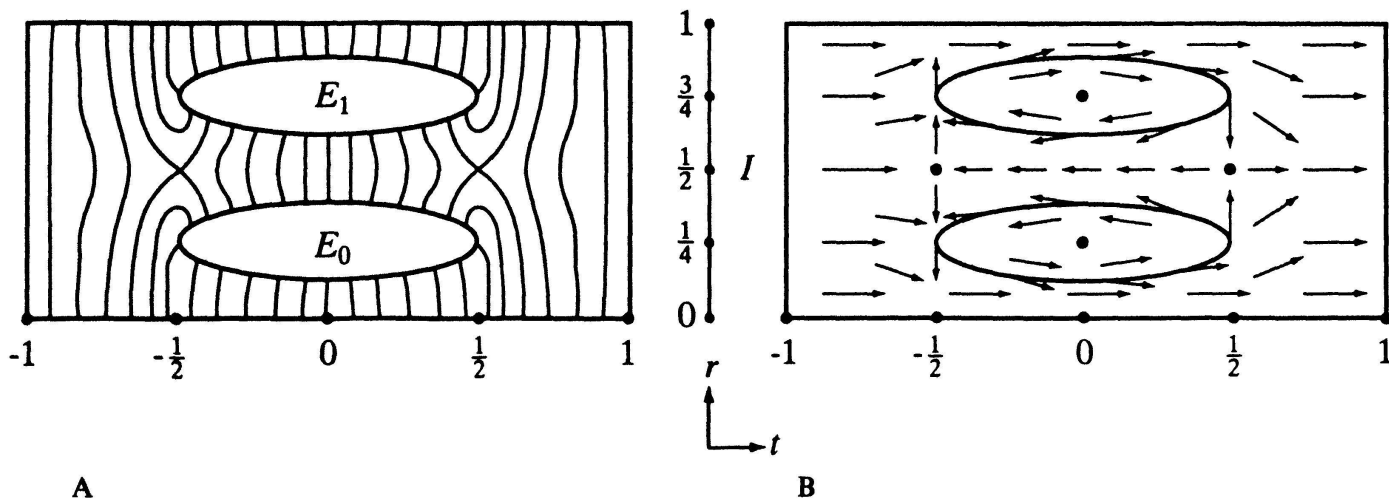A   B

Figure 4. A. The trace of F_1' on a slice $\{point\} \times N$. B. The transverse vector field X projected onto a slice $\{x\} \times D^1 \times I$.
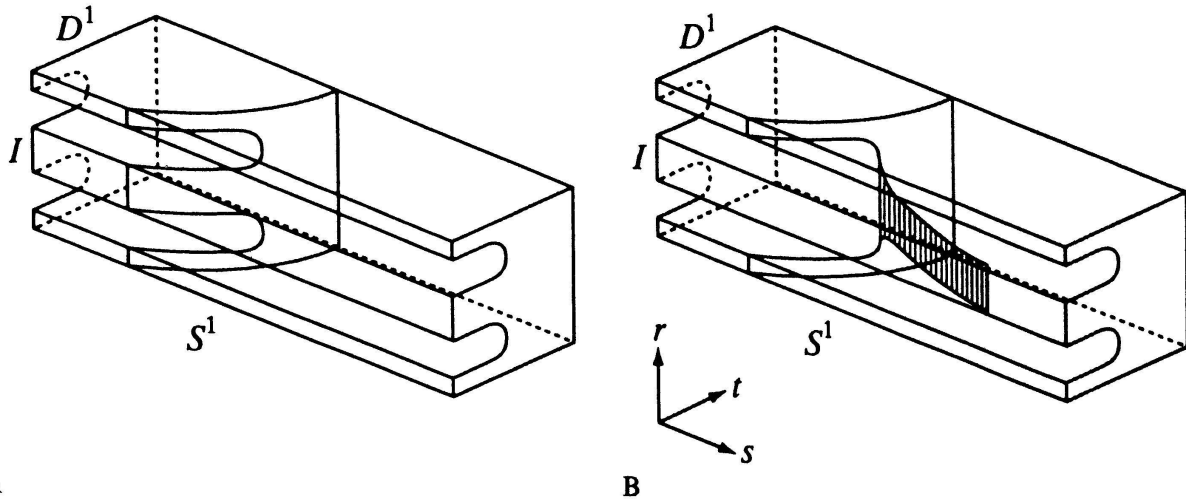
Figure 5. A. A leaf of the foliation F^+ on $S^1 \times N^+$. B. A leaf of the foliation F_1^+ on $S^1 \times N^+$.

distance (see Figure 5B). Explicitly, let $F_1^+ = g^{-1}(F^+)$ where $g : S^1 \times N^+ \to S^1 \times N^+$ is a diffeomorphism of the form

$$g(s, t, r) = (s - Kg_0(t, r), t, r), \qquad (s, t, r) \in S^1 \times N^+, \tag{2.2}$$

where $K > 0$ is a large constant and $g_0 : N^+ \to [0, \infty)$ is a function supported in a small neighborhood of B such that

$$g_0|_{B \cap \partial_0 N} \text{ has nonvanishing gradient} \tag{2.3}$$

$$g_0(t, r) = e^{1/t} \qquad \text{for } (t, r) \in B, \qquad t < \tfrac{1}{4}. \tag{2.4}$$

An analogous construction produces a foliation F_1^- on $S^1 \times N^-$ where $N^- = N \cap [-1, 0) \times I$. Let F_1' be the foliation which coincides with $F_1^\pm$ on $S^1 \times N^\pm$ and has as its remaining leaves the components of $S^1 \times (((\{0\} \times I) \cap N)$. Then F_1' is clearly smooth on $S^1 \times N^\pm$ and, using (2.4), can easily be verified to be smooth where $t = 0$. Take K large enough so that F_1' satisfies (2.1), which is possible in view of (2.3). In particular, F_1' is transverse to $S^1 \times \partial_0 N$. For a convenient choice of g_0 and K, the trace of F_1' on each slice $\{s\} \times N$ will have the form depicted in Figure 4A, with exactly two singularities, both of saddle type, at the points $(s, \pm\tfrac{1}{2}, \tfrac{1}{2})$.

Step 3. Control the homotopy class

The choice of g_0 and K just mentioned makes it possible to construct a unit vector field X on $S^1 \times D^1 \times I$ which is transverse to F_1' on $S^1 \times N$ and whose

projection on each slice $\{s\} \pm D^1 \times I$ has the form depicted in Figure 4B. We shall choose X to agree with $\partial/\partial t$ near $\partial(S^1 \times D^1 \times I)$ and with properties that permit us to show that

$$X \text{ is homotopic to } \partial/\partial t \text{ modulo } \partial(S^1 \times D^1 \times I). \tag{2.5}$$

It is not difficult to verify that, with a convenient choice of g_0 and K, there is a unit vector field

$$X(s, t, r) = S(t, r) \, \partial/\partial s + T(t, r) \, \partial/\partial t + R(t, r) \, \partial/\partial r \tag{2.6}$$

on $S^1 \times D^1 \times I$, transverse to F_1' on $S^1 \times N$, invariant under the S^1 action on the first factor, and such that

$$S \text{ is nowhere zero on } S^1 \times (E_0 \cup E_1), \tag{2.7}$$

$$\text{on } S^1 \times \partial_0 N, X \text{ is tangent to } \{\text{point}\} \times \partial_0 N, \tag{2.8}$$

$$X(s, t, r) = \pm\partial/\partial s \iff (t, r) \in \{(\pm\tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{4}), (0, \tfrac{3}{4})\}, \tag{2.9}$$

and

$$\text{near } \partial(S^1 \times D^1 \times I), X \text{ coincides with } \partial/\partial t. \tag{2.10}$$

To show (2.5), in view of S^1 invariance, it suffices to show that X and $\partial/\partial t$ are homotopic on a slice $\{s_0\} \times D^1 \times I$ modulo boundary, or equivalently that the restriction $X : (\{s_0\} \times D^1 \times I, \{s_0\} \times \partial(D^1 \times I)) \to (S^2, \partial/\partial t)$ has the same degree as the constant mapping $\partial/\partial t$, which has degree zero. We may calculate the degree of X by adding the degrees at the points of $X^{-1}(\partial/\partial s)$. We choose the signs of S on $S^1 \times E_i$ ($i = 0, 1$) by setting $S(0, \tfrac{1}{4}) = S(\tfrac{1}{2}, \tfrac{1}{2})$ and $S(0, \tfrac{3}{4}) = S(-\tfrac{1}{2}, \tfrac{1}{2})$. Then $X^{-1}(\partial/\partial s)$ will contain the same number of saddle points (of degree -1) as centers (of degree $+1$), and thus the degree of X is zero, establishing (2.5).

Step 4. *Linearize the foliation on the boundary of each hole*

In this step we shall partially fill in each hole $S^1 \times E_p$, $p = 0, 1$, leaving two smaller holes in each, such that the foliations induced on the boundary are linear, as required in (1.1.ii). The construction is a straightforward application of an idea of Thurston ([Th2], pp. 316–317).

In view of the simplicity of the group $G = \mathrm{Diff}^{\infty}_{+}(S^1)$ of orientation preserving diffeomorphisms of S^1, it is generated by conjugates of rotations R_a, $a \in \mathbb{R}/\mathbb{Z}$. In fact, any diffeomorphism $g \in G$ can be expressed as

$$g = R_a g' R_b g'^{-1} \tag{2.11}$$

for some $g' \in G$ and nontrivial $a, b \in \mathbb{R}/\mathbb{Z}$. Choose b to be a **Herman number**, i.e. such that any element $\bar{g} \in G$ whose rotation number is $b \in \mathbb{R}$ is smoothly conjugate to R_b (see [Her], p. 8). As a makes a complete circuit of the circle, the rotation number of $R_{-a}g$ also makes a complete circuit, varying continuously, so for some $a \in \mathbb{R}$, $R_{-a}g$ will have rotation number b, and then $R_{-a}g = g' R_b g'^{-1}$ for some $g' \in G$, establishing (2.11).

Now identify the annulus $I \times S^1$ with a neighborhood of $\partial \bar{E}_p$ in $\bar{E}_p$ so that $\{1\} \times S^1$ is identified with $\partial \bar{E}_p$. Using variables (s, r, u) on $S^1 \times I \times S^1$, observe in view of (2.1) that as s makes a positive circuit of the first factor S^1_s, the foliation F'_1 restricted to $S^1_s \times \{1\} \times S^1_u$ determines a holonomy mapping $g \in \mathrm{Diff}^{\infty}_{+}(S^1_u)$. Let D be a small disk in $\mathrm{Int}(S^1_s \times I)$ and let $D_2 = S^1 \times I - \mathrm{Int}\, D$ (see Figure 6). Since $\pi_1(D_2)$ is free on two generators, we can suspend the diffeomorphisms R_a and $g' R_b g'^{-1}$ given by (2.11) to obtain a foliation F'_2 of $D_2 \times S^1_u$ (see Figure 7) which coincides with F'_1 on $S^1_s \times \{1\} \times S^1_u$, is transverse to the factor S^1_u, restricts to the foliation $F(a)$ on $S^1_s \times \{0\} \times S^1_u$, and such that its pullback under the mapping $\partial \bar{g} = \partial g'' \times g' : S^1 \times S^1_u \to \partial D \times S^1_u$ is $(\partial \bar{g})^{-1}(F'_2) = F(b)$, for some diffeomorphism $\partial g'' : S^1 \to \partial D$ which is the restriction of a diffeomorphism $g'' : D^2 \to D$.

By this method we construct a foliation $F'_2(p)$ on the submanifold $D_2(p) \times S^1_u$ of $S^1 \times \bar{E}_p$ (where we have added the index $p = 0, 1$ to the notation). Let F'_3 be the
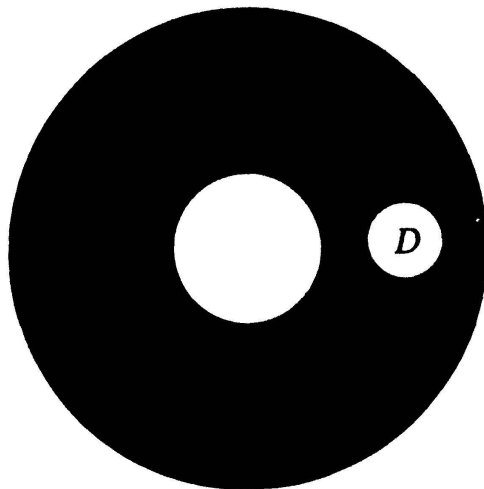

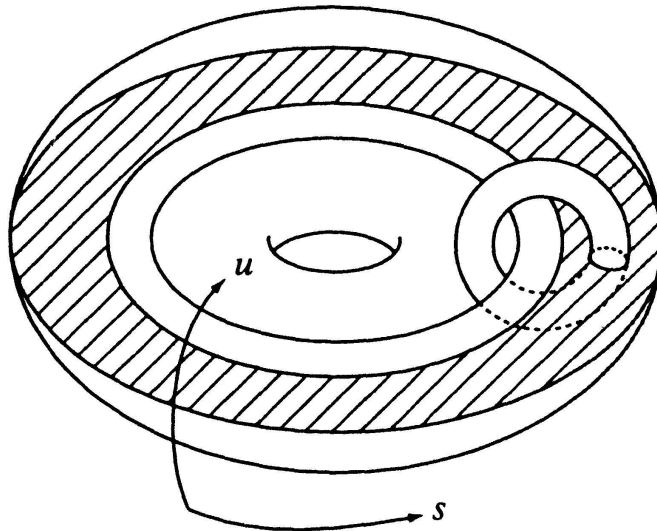Figure 6. $D_2 = S^1 \times I - \mathrm{Int}\, D$.

Figure 7. $D_2 \times S^1_u$ showing one factor $D_2 \times \{u\}$.

foliation on

$$N' = S^1_s \times N \cup D_2(0) \times S^1_u \cup D_2(1) \times S^1_u \qquad (2.12)$$

which coincides with F'_1 on $S^1_s \times N$ and with $F'_2(p)$ on $D_2(p) \times S^1_u$. Letting $D_1(p)$ be the disk $E_p - \mathrm{Int}(I \times S^1)$, observe that $S^1 \times D^1 \times I - \mathrm{Int}\, N'$ consists of four solid tori $S^1_s \times D_1(p)$ and $D(p) \times S^1_u$, $p = 0, 1$. We thus have four disjoint embeddings

$$\bar{g}_\alpha : S^1 \times D^2 \to S^1 \times D^1 \times I \qquad (2.13)$$

such that the restriction $\partial \bar{g}_\alpha : S^1 \times \partial D^2 \to N'$ satisfies

$$(\partial \bar{g}_\alpha)^{-1} F'_3 = F(t_\alpha) \qquad (2.14)$$

for some $t_\alpha \in \mathbb{R} - \{0\}$, $\alpha = 1, 2, 3, 4$.

Finally observe that by (2.8) the vector field X constructed in Step 3 is transverse to the factor $S^1_s \times I$ of $S^1_s \times I \times S^1_u$ provided that $I \times S^1$ is identified with a small enough neighborhood of $\partial \bar{E}_p$. By homotoping X towards $\partial/\partial u$, while maintaining the nonvanishing (2.7) of the coefficient of $\partial/\partial s$, we may make X transverse to $F'_2(p)$ and thus to F'_3. We also want

$$\bar{g}_{\alpha *}(\partial/\partial s) = \pm X \qquad \text{on } \bar{g}_\alpha(S^1 \times D^2), \qquad (2.15)$$

where $\partial/\partial s$ is the vector field tangent to the first factor of $S^1 \times D^2$. This is easy to accomplish by a homotopy of X in a small neighborhood of $D(p) \times S^1_u$, but may be impossible on $S^1_s \times D_1(p)$. Nevertheless we may change the sign of a, and thus the

direction of spiralling of the leaves on $S^1_s \times \partial D_1(p)$, by adding an integer to a (and subtracting it from b), and in view of (2.7), with one sign or the other there will exist a homotopy of X in a small neighborhood of $S^1_s \times D_1(p)$ to a vector field still transverse to F'_2 and satisfying (2.15). We take the sign in (2.15) to be positive, possibly replacing $\bar{g}_\alpha$ by its composition with a diffeomorphism of $S^1 \times D^2$ which reverses the orientation of S^1.

Step 5. Return to dimension $n > 3$

For each index j we have modified the foliation $f_j^{-1}(F_0) \times I$ on $S^1 \times D^1 \times I$ to a foliation $F'_3 = F'_3(j)$ on N', which is $S^1 \times D^1 \times I$ with four open solid tori $\bar{g}_\alpha(S^1 \times \mathrm{Int}\, D^2)$ removed.

Define embeddings $g_{j,\alpha} : S^1 \times D^2 \times S^{n-3} \to T_j$ by setting

$$g_{j,\alpha}(u, x, y) = \bar{f}_j(s, t, ry),$$

where $g_\alpha(u, x) = (s, t, r)$. Notice that

$$f_j \varphi^{-1}(N') = T_j - \bigcup_\alpha g_{j,\alpha}(\mathrm{Int}\, A)$$

for $1 \leq \alpha \leq 4$, where N' is given by (2.12), $A = S^1 \times D^2 \times S^{n-3}$, and

$$\varphi : S^1 \times D^1 \times D^{n-2} \to S^1 \times D^1 \times I$$

is defined $\varphi(s, t, y) = (s, t, |y|)$, $(s, t, y) \in S^1 \times D^1 \times D^{n-2}$.

Define F_1 on $T_j - \bigcup g_{j,\alpha}(\mathrm{Int}\, A)$ to be $(\bar{f}_j^{-1} \circ \varphi)^{-1} F'_3(j)$. Then (ii) of Proposition (1.1) follows from (2.14) and (i) follows from (1.2.i) and the construction of $F'_3(j)$.

Lift the vector field $X = X(j)$ on $S^1 \times D^1 \times I$ to $X'(j)$ on $S^1 \times D^1 \times D^{n-2}$ by requiring that $\varphi_*(X'(j)) = X(j)$ and that the component of $X'(j)$ on the factor D^{n-2} be radial. Then the homotopy of $X(j)$ to $\partial/\partial t$ on $S^1 \times D^1 \times I$ also lifts to a homotopy of $X'(j)$ to $\partial/\partial t$ on $S^1 \times D^1 \times D^{n-2}$ modulo boundary, so that $X_0 = \bar{f}_{j*}(\partial/\partial t)$ and $X_1 = \bar{f}_{j*}(X'(j))$ are homotopic on T_j modulo ∂T_j, and (iii) is satisfied. Clearly X_0 is transverse to F_0 and X_1 is transverse to F_1, and (iv) follows from (2.15) with the positive sign. This completes the Proof of Proposition 1.1. □

In order to prove that the concordance class can be kept the same, we need the following result.
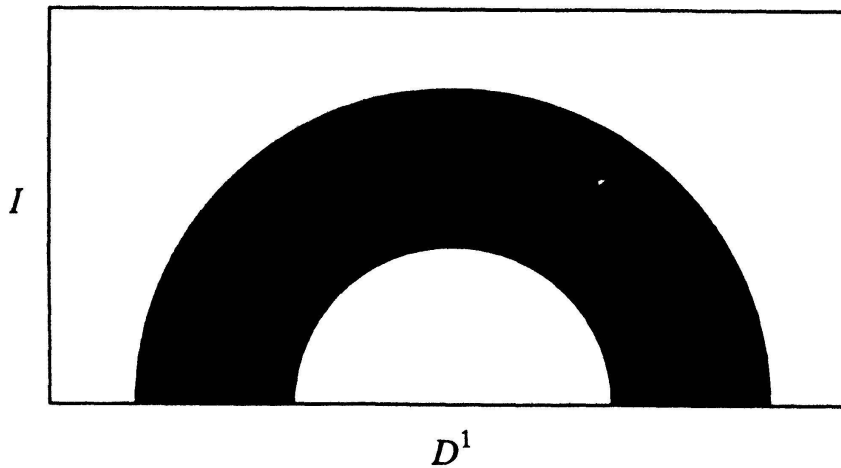
I

D^1

Figure 8. $D_+ = D^2 \cap D^1 \times I$ showing the semiannulas where $F' \neq F_0$.

PROPOSITION 2.16. *Let F_0 be a foliation of a closed manifold N and let F_+ be a foliation of $N \times D^1$ which agrees with the product foliation $F_0 \times D^1$ outside a compact subset of $N \times (0, 1)$. Then the foliation F which agrees with F_+ on $N \times I$ and is invariant under the involution $(x, t) \mapsto (x, -t)$ is concordant to F_0 by a concordance supported on a compact subset of $N \times (-1, 1) \times I$. The concordance is as differentiable as F_0 and F_+.*

Proof. Consider the upper half disk $D_+ = D^2 \cap D^1 \times I$ as in Figure 8. The mapping $\varphi : N \times D_+ \to N \times I$, $\varphi(x, t, u) = (x, \sqrt{t^2 + u^2})$ for $x \in N$, $(t, u) \in D_+$, is a submersion except where $(t, u) = (0, 0)$, so the foliation $\varphi^{-1} F_+$ is well defined. On $N \times D^1 \times I$ the foliation F' which agrees with $\varphi^{-1} F_+$ on $N \times D_+$ and with $F_0 \times D^1 \times I$ on the rest of $N \times D^1 \times I$ is the required concordance. $\square$

We remark that the foliation F inherits certain properties if F_+ has them, such as having no compact leaves, or having smooth leaves.

§3. Raymond's foliation of S^3 and Hector's exceptional minimal set

The goal of this section is the following Proposition, which will be used in the next two sections. Recall that all structures in this section are supposed smooth. Let N be a compact connected k-manifold ($k \geq 3$) whose boundary is separated into two closed sets $\partial_0 N$ and $\partial_1 N$. The main case of interest is $N = B \times I$, and then we set $\partial_i N = B \times \{i\}$, $i = 0, 1$. To simplify the notation, throughout this section S^1 is identified with $\mathbb{R}/12\mathbb{Z}$.

PROPOSITION 3.1. *There exist a transversely oriented foliation F_U of an open set $U \subset \text{Int } N$, a compact saturated set $C \subset U$ containing no compact leaves, a*

submersion $\beta : U - C \rightarrow (0, 1)$ *constant on each leaf of* F_U, *a* 1-*dimensional foliation* $F_U^\perp$ *on* U *transverse to* F_U, *and a diffeomorphism* τ *of* $S^1 = \mathbb{R}/12\mathbb{Z}$ *onto a leaf of* $F_U^\perp$ *that meets every leaf of* F_U, *with the following properties.*

   (i) *On each leaf* T *of* $F_U^\perp|_{U-C}$, $\beta|_T$ *is an oriented diffeomorphism into* $(0, 1)$, *and if* T *has a lower endpoint* $x_0 \in \bar{T} \cap C$ (*resp., an upper endpoint* $x_1 \in \bar{T} \cap C$), *then*

$$\lim_{\substack{x \rightarrow x_0 \\ x \in T}} \beta(x) = 0 \quad \left(resp., \lim_{\substack{x \rightarrow x_1 \\ x \in T}} \beta(x) = 1 \right).$$

   (ii) *There exists a constant* $K > 0$ *such that for every* $t \in B = S^1 - \tau^{-1}(C)$, $|1/\varphi'(t)| \leq K$ *and* $|(1/\varphi')'(t)| \leq K$, *where* $\varphi = \beta \circ \tau|_B$.

   (iii) $N - C = A_0 \cup A_1$ *where the* A_i *are disjoint open sets such that* $\partial_i N \subset A_i$, $i = 0, 1$.

Note that it follows from (3.1.i) that if $\bar{T}$ has two endpoints in C, then $\beta|_T$ is surjective. It will be shown below that the general case of the Proposition follows easily from the following special case.

LEMMA 3.2. *Proposition* 3.1 *holds for* $N = S^2 \times I$.

The standard argument using Zorn's lemma shows that the compact saturated set C, which is nonempty since it separates N, must contain a **minimal set** C_1, i.e. a closed saturated nonempty set which contains no such proper subset. For codimension one foliations there are only three types of minimal sets – a single compact leaf, the whole manifold (when every leaf is dense), and the third type, called **exceptional**, characterized by intersecting a transverse closed curve in a Cantor set. Since the conditions of Proposition 3.1 exclude the first two types, C_1 must be an exceptional minimal set. While the earliest known exceptional minimal set, Denjoy's example on the torus, cannot occur in a C^2 foliation ([D], [Sc]), Sacksteder [Sack1] has constructed a C^∞ foliation of a 3-manifold with an exceptional minimal set related to an action of the group $G_0 = \mathbb{Z}/2 * \mathbb{Z}/3$ on the circle that has an exceptional minimal set (defined by **invariant** rather than saturated sets).

There are two significant difficulties in proving Lemma 3.2 – the exceptional minimal set must fit inside S^3 (since $S^2 \times I$ does) and the holonomy on each complementary interval must be trivial so that (3.1.i) can be satisfied. The first difficulty is resolved by Bernard Raymond's remarkable construction of a foliation of S^3 with an exceptional minimal set.

THEOREM 3.3 (B. Raymond [Ray]). *For certain actions of the group* $G_0 = \mathbb{Z}/2 * \mathbb{Z}/3$ *on* S^1 *with an exceptional minimal set* $C_0 \subset S^1$ *(for example Sacksteder's action* [Sack 1]) *there exist a* C^∞ *foliation* F *of* S^3 *and a mapping* $\tau : S^1 \to S^3$ *transverse to* F *such that*

(i) *The set* $C_1 = \mathrm{Sat}\ \tau(C_0)$ *is an exceptional minimal set of* F,

(ii) *The holonomy pseudogroup induced on* S^1 *by* F *under* τ *is generated by the elements of* G_0.

The second difficulty is resolved by an interesting exceptional minimal set constructed by Hector. I thank him for suggesting its use.

THEOREM 3.4 (Hector [Hec]). *There exist an action of* $G_0 = \mathbb{Z}/2 * \mathbb{Z}/3$ *on* S^1 *with an exceptional minimal set* C_0, *a constant* $K > 0$, *and an orientation preserving local diffeomorphism* $\psi : S^1 - C_0 \to (-1, 1)$, *constant on each orbit and surjective on each component of* $S^1 - C_0$, *such that*

$$|1/\psi'(t)| \le K \quad and \quad |(1/\psi')'(t)| \le K \tag{3.5}$$

for every $t \in S^1 - C_0$.

It does not seem to be possible to complete Raymond's construction of the foliation F on S^3 using Hector's action of G_0 on S^1, but it can be carried far enough to prove Lemma 3.2. We shall describe Hector's action and Raymond's construction, and then prove Lemma 3.2 and Proposition 3.1.

Hector's exceptional minimal set. In [Hec], pp. 252–255 and 260–262, G. Hector introduced a group Φ of compactly supported diffeomorphisms of $\mathbb{R}$ with two generators f and χ and with an exceptional minimal set. Let f, $\chi \in \mathrm{Diff}^\infty_+(S^1)$ be the diffeomorphisms obtained by restricting the generators to [0, 12] and passing to the quotient $S^1 = \mathbb{R}/12\mathbb{Z}$. The diffeomorphism f is a rotation of order 3 defined by

$$f(t) = t + 4, \qquad t \in \mathbb{R}/12\mathbb{Z}.$$

The diffeomorphism $g = \chi \circ f^2$ (see Figure 9) has order 2, interchanges $[-1, 1]$ and $[3, 5]$ by rotations, and is a strictly expanding diffeomorphism $(g'(t) > 1)$ from $(1, 3)$ to $(5, 11)$. Then g and f generate a group $G_0 \approx \mathbb{Z}/2 * \mathbb{Z}/3$ that acts on S^1. Note that χ is the identity on $[0, 1] \cup [11, 12]$, and hence, by periodicity, on $[-1, 1]$.

Lemmas 2, 3 and 5 of [Hec], p. 254, interpreted in terms of G_0 acting on $S^1 = \mathbb{Z}/12\mathbb{Z}$, state that there is a unique exceptional minimal set C_0 and that $(-1, 1)$
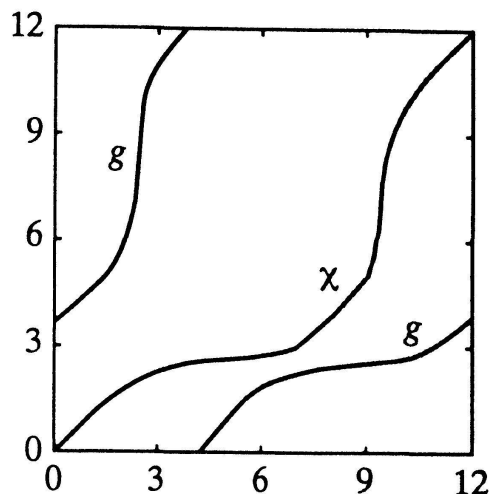
Figure 9. The functions g and χ on $S^1 = \mathbb{R}/12\mathbb{Z}$.

is a component of $S^1 - C_0$ such that

$$\text{Sat}(-1, 1) = S^1 - C_0 \qquad\qquad (3.6)$$

If $h(-1, 1) = (-1, 1)$ for some $h \in G_0$ then $h|_{(-1, 1)} = \text{Id}$. $\qquad (3.7)$

LEMMA 3.8. *Given any component (a, b) of $(1, 3) - C_0 \cap (1, 3)$, there is a unique element $h = g_k g_{k-1} \cdots g_1 \in G_0$ such that*

(i) *Each $g_i = f^{\varepsilon_i} g$ for some $\varepsilon_i = \pm 1$.*

(ii) *For $i < k$, $g_i g_{i-1} \cdots g_1 (a, b) \subset (1, 3)$.*

(iii) *$h(a, b) = (-1, 1)$.*

Proof. Observe that $g(1, 3) = (5, 11)$ and that $g'(t) > 1$ if $t \in (1, 3)$. Suppose that (i) and (ii) hold for $1 \le i < p$ for some $p \ge 1$. Then $g_{p-1} \cdots g_1(a, b) \subset (1, 3)$ and consequently $S = g \, g_{p-1} \cdots g_1(a, b) \subset (5, 11)$. Since the orbit of 1 contains all odd integers and is contained in C_0, S must be contained in $(5, 7)$, $(7, 9)$ or $(9, 11)$. In the first and third cases, choose $\varepsilon_p = \pm 1$ so that $f^{\varepsilon_p}(S) \subset (1, 3)$, set $g_p = f^{\varepsilon_p} g$, and continue the induction. In the second case, set $k = p$ and $g_k = fg$ so that (iii) follows. Let $h_p = g_p \cdots g_1$ and note that

the intervals $h_p(a, b)$ are expanding and disjoint, $p = 0, 1, \ldots,$ $\qquad (3.9)$

so the process must terminate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

It is fairly easy to adapt this reasoning to prove (3.6) and (3.7) and also show that the closure of the orbit of 1 is the unique exceptional minimal set of the action. (See [Hec], pp. 252–255 for more details.)

Proof of Theorem 3.4. Let G_0 be the group of diffeomorphisms of S^1 just described and C_0 its exceptional minimal set. In view of (3.6) and (3.7) we may define $\psi : S^1 - C_0 \to (-1, 1)$ by setting $\psi|_{(-1, 1)} = \mathrm{Id}$ and requiring that $\psi \circ h|_{S^1 - C_0} = \psi$ for every $h \in G_0$. Then ψ is clearly smooth and takes each component of $S^1 - C_0$ diffeomorphically onto $(-1, 1)$.

The second inequality of (3.5) is proved by a Denjoy-type argument (cf. [Sack2], p. 81). Choose constants $\bar{K}$, $\theta > 0$ such that for $t \in S^1 = \mathbb{R}/12\mathbb{Z}$

$$\bar{K}^{-1} \le |g'(t)| \le \bar{K} \quad \text{and} \quad |g''(t)| \le \theta/\bar{K}. \tag{3.10}$$

Let (a, b) be a component of $S^1 - C_0$ contained in $(1, 3)$ and take $g_1, \ldots, g_k, h_1, \ldots, h_k \in G_0$ as in (3.8) and (3.9). We claim that

$$h_p'(u) \le e^{2\theta} h_p'(v), \quad \text{for any } u, v \in (a, b) \text{ and } p = 1, \ldots, k. \tag{3.11}$$

In fact, by the Chain Rule and the Mean Value Theorem, there are points $t_i \in (h_{i-1}(u), h_{i-1}(v))$, $i = 1, \ldots, p$, such that

$$\left| \log(h_p'(u)/h_p'(v)) \right| \le \sum_{i=1}^{p} \left| \log g_i'(h_{i-1}(u)) - \log g_i'(h_{i-1}(v)) \right|$$

$$= \sum_{i=1}^{p} \left| g_i'(t_i)^{-1} g_i''(t_i) \right| \left| h_{i-1}(u) - h_{i-1}(v) \right|$$

$$\le \theta \sum_{i=1}^{p} \left| h_{i-1}(u) - h_{i-1}(v) \right| \qquad \text{by (3.10)}$$

$$\le 2\theta \qquad \text{by (3.9) and (3.8ii).}$$

Since ψ and h agree on (a, b), we have, for $t \in (a, b)$,

$$\log \psi'(t) = \sum_{i=1}^{k} \log g_i'(h_{i-1}(t)).$$

Differentiating this equation yields

$$\left| \psi''(t) \psi'(t)^{-1} \right| \le \sum_{i=1}^{k} \left| g_i'(h_{i-1}(t))^{-1} g_i''(h_{i-1}(t)) \right| h_{i-1}'(t)$$

$$\le \theta \sum_{i=0}^{k-1} h_i'(t).$$

Now $h_i(b) - h_i(a) = (b - a)h_i'(\bar{t}_i)$ for some $\bar{t}_i \in (a, b)$, and $2 = 1 - (-1) = (b - a)\psi'(\bar{t})$ for some $\bar{t} \in (a, b)$ so using (3.11) twice in the last inequality we obtain

$$\left| \psi''(t)\psi'(t)^{-2} \right| \leq \theta\, e^{4\theta}\psi'(\bar{t})^{-1} \sum_{i=0}^{k-1} h_i'(\bar{t}_i)$$

and then

$$\left| (1/\psi')'(t) \right| \leq \theta\, e^{4\theta}$$

since $\sum_i (h_i(b) - h_i(a)) \leq 2$ by (3.9). We have just proven the second inequality of (3.5) for $K = \theta\, e^{4\theta}$ and $t \in (1, 3) - C_0 \cap (1, 3)$, but it also holds for $f^{\pm 1}(t)$ (since $(\psi \circ f)'(t) = \psi'(t)$) and for $t \in (p - 1, p + 1)$, $p = 0, 4, 8$ (where $\psi'(t) = 1$), and this covers all points of $S^1 - C_0$. Finally we have $\psi'(t) \geq 1$ for all $t \in S^1 - C_0$ so that the first inequality of (3.5) holds with $K = 1$. $\square$

Next we recall Raymond's construction briefly, in order to describe certain details that we need. (See [Ray] for further details.) Let $G_0 \approx \mathbb{Z}/2 * \mathbb{Z}/3$ be a group of diffeomorphisms of the circle $S_t^1 = \mathbb{R}/12\mathbb{Z}$ with an exceptional minimal set $C_0 \subset S^1$. We suppose that we can choose generators $g, f \in G_0$ and covering diffeomorphisms $\tilde{g}, \tilde{f} \in \text{Diff}_+^\infty(\mathbb{R})$ so that $\tilde{g}^2(t) = \tilde{f}^3(t) = t + 12$. This is possible in the case of Hector's group, as is easily verified (see Figure 9). Over the two-holed disk $D_2 = S_s^1 \times I - \text{Int } D$ (see Figure 6 and §2) we suspend the action of G_0 to obtain a foliation F_0 of $D_2 \times S_t^1$ transverse to the second factor, such that the foliation induced by F_0 on the covering space $D_2 \times \mathbb{R}$ satisfies

The holonomy over $S_s^1 \times \{0\}$ (resp. $S_s^1 \times \{1\}$) is $\tilde{g}$ (resp. $\tilde{f}$).   (3.12)

Explicitly, the fundamental group $\pi = \pi_1(D_2)$ acts canonically on the universal cover $\tilde{D}_2$ and also acts on S_t^1 so that a loop freely homotopic to $S_s^1 \times \{0\}$ (resp. $S_s^1 \times \{1\}$) acts by g (resp. f), and then the foliation F_0 on $D_2 \times S_t^1 \approx \tilde{D}_2 \times_\pi S_t^1$ is the quotient of the foliation $\tilde{D}_2 \times \{\text{point}\}$ on $\tilde{D}_2 \times S_t^1$. One easily checks, in view of (3.12), that the foliations induced by F_0 on $S_s^1 \times \{0\} \times S_t^1$ and $S_s^1 \times \{1\} \times S_t^1$ are by circles of homotopy classes $(2, 1)$ and $(3, 1)$ respectively. Now

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

acts on $T^2 = \mathbb{R}^2/12\mathbb{Z}^2$ by left multiplication and therefore on $S_s^1 \times I \times S_t^1 \approx T^2 \times I$. The pullback $A^{-1}F_0$ under the diffeomorphism A is a foliation on $S_s^1 \times I \times S_t^1$
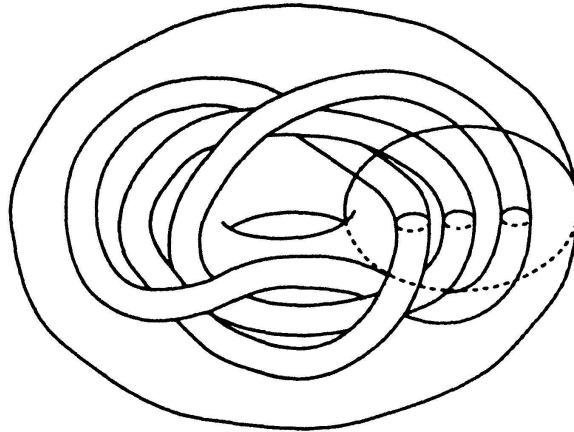
Figure 10. The manifold N is the 3-sphere with a tubular neighborhood of the trefoil knot removed.

minus an open solid torus. On $S^1_s \times \{0\} \times S^1_t$ and $S^1_s \times \{1\} \times S^1_t$ the foliations by circles induced by $A^{-1}F_0$ now are of types $(1, 0)$ and $(0, 1)$ respectively, so that after an isotopy near the boundary, we can attach two solid tori foliated by disks along these boundary components to obtain S^3 minus the interior of a solid torus $h : S^1 \times D^2 \to S^3$,

$$N = S^3 - h(S^1 \times \text{Int } D^2)$$

$$= \{S^1_s \times I \times S^1_t - h(S^1 \times \text{Int } D^2)\} \cup D^2_s \times \{0\} \times S^1_t \cup S^1_s \times \{1\} \times D^2_t,$$

foliated by a foliation F_1 which restricts to $A^{-1}F_0, D^2_s \times \{0\} \times \{\text{point}\}$, and $\{\text{point}\} \times \{1\} \times D^2_t$ on the three subspaces. One can verify that the hole $h(S^1 \times \text{Int } D^2)$ is a tubular neighborhood of the trefoil knot. (See Figure 10, where the hole and the tori $S^1_s \times \{0\} \times S^1_t$ and $S^1_s \times \{1\} \times S^1_t$ are depicted.) Now define $\tau : S^1 \to N$ by setting $\tau(t) = (x_0, t)$ for some point $x_0 \in \text{Int } D_2$ and note that $C' = \text{Sat}_{F_1}\tau(C_0)$ is an exceptional minimal set for F_1, but it meets ∂N.

By an ingenious trick Raymond gets the exceptional minimal set into the interior of the manifold: he doubles N along a cylinder $B = h(J_1 \times \partial D^2) \subset \partial N$, for some closed interval $J_1 \subset S^1$ such that $C' \cap \partial N \subset \text{Int } B$. Such an interval J_1 need not exist in general, but in the case of Hector's action of G_0, the diffeomorphisms f, g and their covering diffeomorphisms $\tilde{f}, \tilde{g} \in \text{Diff}^\infty_+(R)$ all agree on $[-1, 1]$, so that the holonomy $\tilde{g}^{-1} \circ \tilde{f}$ on ∂N is the identity on $[-1, 1] \subseteq R/12Z$. After an isotopy near ∂N, the foliation $F_1|_{\partial N}$ will have leaves $h(\{s\} \times \partial D^2)$ for $s \in [-1, 1]$. Since $(0, 1)$ and C_0 are disjoint, any choice of J_1 as a closed interval containing $S^1 - (0, 1)$ in its interior will suffice to guarantee that $C' \cap \partial N \subset \text{Int } B$ holds.

Now introduce corners on ∂N along ∂B (see Figure 11), so that B is part of a round sphere, and N is the (outside) ball less the hole. Reflect N through this sphere, and obtain the double N' of N along B as the union of N and its reflected
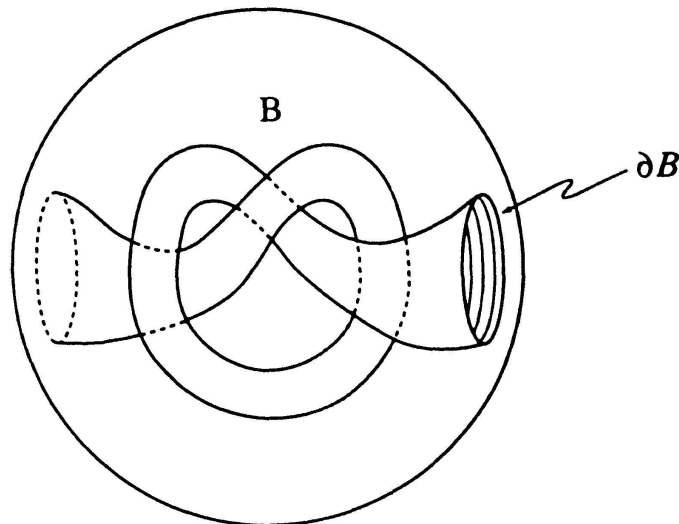
Figure 11. The manifold N with corners introduced prior to doubling along the cylinder B.

image. Now the set C_1 consisting of C' and its reflected image is an exceptional minimal set of the doubled foliation F' on N' such that $C_1 \subset \text{Int } N'$. Raymond carefully chooses the interval J_1, for Sacksteder's action and others with similar properties, so that the foliation on the boundary öf the doubled hole (a tubular neighborhood of the square knot, which is the double of the trefoil) is given by circles transverse to the disk factor. Hence one can complete Raymond's foliation F by inserting a Reeb component and making the leaves spiral around it along $\partial N'$.

Proof of Lemma 3.2. In the case of Hector's action of G_0 on S^1, it seems to be impossible to complete Raymond's construction by inserting a Reeb component since there will be singularities on $\partial N'$, but it suffices to get a foliation on an open set $U \subset S^3$.

Since the mapping $\psi : S^1 - C_0 \to (-1, 1)$ given by (3.4) is constant on each orbit, it determines a mapping $\bar\psi : N' - C_1 \to (-1, 1)$ constant on each leaf of F' such that $\bar\psi\tau(t) = \psi(t)$ for $t \in S^1 - C_0$. We choose the interval J_1 mentioned above so that $S^1 - \text{Int } J_1 \subset (0, 1)$ (using the parametrization $S^1 = R/12Z$). Since χ is the identity on $(0, 1)$, this is possible. Then the set $C = C_1 \cup \bar\psi^{-1}(0)$ is a compact set in Int N'. Furthermore C separates N' into two open sets $\bar\psi^{-1}(-1, 0)$ and $\bar\psi^{-1}(0, 1)$. Since $S^3 - \text{Int } N'$ meets only one of the closures of these two sets, C separates S^3. Choose a 1-dimensional foliation $F^\perp$ transverse to F' with $\tau(S^1)$ as a leaf. Let γ be an arc in a leaf of $F^\perp$ with endpoints p_0 and p_1 in distinct components of $S^3 - C - \tau(S^1)$, and remove small open balls centered at p_0 and p_1 with closures contained in these components. Choose the balls and identify their complement with $S^2 \times I$ so that $\gamma \cap (S^2 \times I) = \{x_1\} \times I$ (for some $x_1 \in S^2$) and so that for a small

closed disk neighborhood D of x_1, $D \times I$ is product foliated, i.e.,

$$F'|_{D \times I} = \{D \times \{t\}\}$$

$$F^{\perp}|_{D \times I} = \{\{x\} \times I\}.$$

(3.13)

Let A_i be disjoint open sets such that $S^2 \times I - C = A_0 \cup A_1$ and $S^2 \times \{i\} \subset A_i$, $i = 0, 1$. Let $U = S^2 \times (0, 1) \cap \operatorname{Sat} \tau(S^1)$, $F_U = F'|_U$ and $F_U^{\perp} = F^{\perp}|_U$.

We now define the submersion $\beta : U - C \to (0, 1)$ by setting $\beta(x) = \bar{\psi}(x)$ if $\bar{\psi}(x) \in (0, 1)$ and $\beta(x) = \bar{\psi}(x) + 1$ if $\bar{\psi}(x) \in (-1, 0)$. One easily checks that β satisfies (3.1.i), and (3.5) implies (3.1.ii). □

Proof of Proposition 3.1.

Case 1. $N = S^{k-1} \times I$, $k \geq 3$. For $k = 3$ this is just Lemma 3.2. For $k > 3$ it suffices to multiply $S^2 \times I$, U, C, and F_U on the left by S^{k-3}, and then embed $S^{k-3} \times S^2 \times I$ in $S^{k-1} \times I$ so that it separates $S^{k-1} \times \{0\}$ from $S^{k-1} \times \{1\}$. In view of (3.13) we may choose the embedding so that there exist an open arc $J \subset (0, 1)$, a point $x_1 \in S^{k-1}$ such that $C \cap (\{x_1\} \times I) \subset \{x_1\} \times J$, and a product neighborhood $D_1 \times J$ of $\{x_1\} \times J$ foliated as a product by the foliations $S^{k-3} \times F_U$ and $\{\text{point}\} \times F_U^{\perp}$, where D_1 is a closed disk neighborhood of x_1 in S^{k-1}. The remainder of the proof in this case is a straightforward transposition of the properties from $S^2 \times I$ to the product $S^{k-3} \times S^2 \times I$.

Case 2. $N = B \times I$ where B is a closed connected $(k-1)$-manifold. We start with the result of Case 1 for $S^{k-1} \times I$ and change it to $B^{k-1} \times I$ by a kind of surgery. Let $C \subset U \subset S^{k-1} \times I$, the foliations F_U and $F_U^{\perp}$, the closed disk D_1 and the arc J be as in Case 1. If B_0 is B with the interior of a closed disk removed, then B is diffeomorphic to $B_0 \cup (S^{k-1} - \operatorname{Int} D_1)$, with the boundaries identified. Let F_{U_B} be the foliation on

$$U_B = B_0 \times J \cup (U - D_1 \times I),$$

which coincides with $B_0 \times \{\text{point}\}$ on $B_0 \times J$ and with F_U on $U - D \times I$. The compact saturated set

$$C_B = B_0 \times C_J \cup C - (D_1 \times I)$$

separates the two boundary components, where C_J is the subset of J such that $\{x_1\} \times C_J = C \cap (\{x_1\} \times I)$. The remaining details are routine.

The general case follows easily from Case 2 by setting $B = \partial_0 N$ and identifying $B \times I$ with a collar neighborhood of $\partial_0 N$ in N. □

§4. Tilting the leaves to fill the holes

According to Proposition 1.1, the original foliation F_0 can be modified to a smooth foliation F_1 defined on M_1 which is obtained from the original manifold M by removing a finite number of "holes" $g_i(\text{Int } A)$ with disjoint closures $g_i(A)$, where $A = S^1 \times D^2 \times S^{n-3}$. The foliation F_1 is transverse to ∂M_1 and on the boundary of the ith hole the induced foliation $(g_i|_{\partial A})^{-1} F_1$ is the product of a linear foliation on the torus $S^1 \times \partial D^2$ by S^{n-3}. In this section we shall construct a foliation G_i of A without any compact leaves for each i, so that G_i is transverse to ∂A and $G_i|_{\partial A} = (g_i|_{\partial A})^{-1} F_1$. Then gluing these foliations into the holes will enable us to complete the proof of the Theorem stated in the Introduction.

Henceforth we fix i. To begin the construction of G_i, we represent the disk D^2 as $D^2(\frac{1}{2}) \cup S^1 \times I$ by identifying $S^1 \times I$ with the annulus $\frac{1}{2} \le |z| \le 1$ by the mapping $(z, t) \mapsto (2t - 1)z$. Define the restriction of G_i to $S^1 \times D(\frac{1}{2}) \times S^{n-3}$ to be the foliation whose leaves are $\{\text{point}\} \times D(\frac{1}{2}) \times S^{n-3}$. The remaining "hole" still to be foliated is now the interior of $S^1 \times S^1 \times I \times S^{n-3}$, and its two boundary components are foliated as follows. On $S^1 \times S^1 \times \{0\} \times S^{n-3}$ the foliation (just defined) has leaves $\{\text{point}\} \times S^1 \times \{0\} \times S^{n-3}$, and on $S^1 \times S^1 \times \{1\} \times S^{n-3}$ the foliation is $(g_i|_{\partial A})^{-1} F_1 = F(t_i) \times \{1\} \times S^{n-3}$, $t_i \ne 0$. Extend these two foliations to product foliations, denoted F_0' and F_1' respectively, on $S^1 \times S^1 \times I \times S^{n-3}$, by multiplying each by I. Note that both are transverse to the first factor S^1 and (in view of their linearity on $S^1 \times S^1$) invariant under the S^1-action on this factor. We shall complete the definition of G_i by constructing a foliation F' on $S^1 \times S^1 \times I \times S^{n-3}$ by modifying F_0' and F_1' in a way analogous to the behavior of the Reeb foliation of S^3 near the torus leaf T^2: the other leaves spiral in asymptotically towards T^2 from opposite sides in two different directions. (It is essentially this change of directions across T^2 which makes it impossible to avoid the torus leaf on S^3.) We shall replace T^2 by a compact saturated set C' (the union of an exceptional minimal set and a single extra non-compact leaf) contained in $\text{Int}(S^1 \times S^1 \times I \times S^{n-3})$ and separating the two boundary components. The remaining leaves of F' will start as leaves of F_0' and F_1' near the respective boundary components, but will be modified so as to spiral asymptotically around C' from opposite sides.

The two following Propositions, to be proven in this section and the next, respectively, will complete the construction of F' and prove the Theorem. We consider a smooth principal S^1 fiber bundle $\pi : N' \to N$ where N is a compact connected manifold whose boundary has two components $\partial_0 N$ and $\partial_1 N$. Thus $\partial N' = \partial_0 N' \cup \partial_1 N'$, where $\partial_i N' = \pi^{-1}(\partial_i N)$.

PROPOSITION 4.1. *Let F_0' and F_1' be smooth foliations on N' transverse to the fiber S^1 and invariant under its principal action. If $n = \dim N' \ge 4$ then there is a*

foliation F' *of* N' *transverse to* $\partial N'$ *with no compact leaves, such that*

(i) *The leaves of* F' *are* C^∞ *submanifolds;*

(ii) *Near* $\partial_i N'$ *the foliations* F_i' *and* F' *agree for* $i = 0, 1$;

(iii) *There is a vector field* Y' *on* N' *simultaneously transverse to* F_0', F_1' *and* F' *and coinciding near* $\partial N'$ *with the vector field* Y *on* N' *that generates the principal* S^1*-action.*

PROPOSITION 4.2. *The construction of* F' *in Proposition* 4.1 *can be carried out so that* F' *is locally defined by a* C^1 1-*form on* N'.

Proof of the Theorem. Let $N = S^1 \times I \times S^{n-3}$, take the product fibration $\pi : N' = S^1 \times N \to N$ with the two foliations F_0' and F_1' on N' defined above, and let $\partial_i N = S^1 \times \{i\} \times S^{n-3}$, $i = 0, 1$. From Propositions 4.1 and 4.2 we obtain the foliation F' with the indicated properties. Gluing together smoothly F' on $N' = S^1 \times S^1 \times I \times S^{n-3}$, with the foliation $\{\text{point}\} \times D^2(\frac{1}{2}) \times S^{n-3}$ on $S^1 \times D^2(\frac{1}{2}) \times S^{n-3}$ yields the desired foliation G_i of $A = S^1 \times D^2 \times S^{n-3}$, which we transport to the "hole" $g_i(A) \subset M$ by the diffeomorphism g_i. Thus we extend F_1 to a foliation F on M such that $g_i^{-1}(F) = G_i$ for each i.

The foliation G_i has no compact leaves because F' has none. Thus F has no compact leaves, for every compact leaf of F_1 on $M_1 = M - \bigcup_i g_i(A)$ meets some boundary component $g_i(\partial A)$, where it is glued to a noncompact leaf of $g_i(G_i)$. Clearly F is locally defined by a C^1-differential 1-form and all its leaves are C^∞ submanifolds of M, because F' and each G_i have these properties, provided that we smooth the corners along the boundaries $g_i(\partial A)$.

In order to check that the plane fields TF_0 and TF are homotopic on each tube T_j modulo ∂T_j, it suffices in view of (1.1.iii) to exhibit a vector field X on $\bigcup_j T_j$ transverse to TF, coinciding with X_1 on $\bigcup_j T_j - \bigcup_i g_i(A)$, and homotopic to X_1 on $g_i(A)$ modulo boundary for each i. For a fixed index i let Y' be the vector field on N' given by (4.1.iii) extended to be equal to Y (the vector field on A which generated the S^1 action on the first factor) on $A - \text{Int } N'$. Now Y and Y' are both transverse to F_0' on N', so they are homotopic on A modulo ∂A. By (1.1.iv), $X_1|_{g_i(A)} = g_{i*}(Y)$ so it suffices to define $X|_{g_i(A)} = g_{i*}(Y')$.

Now if the foliations F_0 and F are not C^1-concordant, we can change F inside each tube T_j, essentially doubling the modification from F_0 to F by adding the mirror image of the modification inside each tube alongside the original modification, as explained in Proposition 2.16, so that the resulting foliation has all the properties of F already verified and in addition is C^1-concordant to F_0. To apply 2.16, let $N = S^1 \times D^1 \times D^{n-3}$, identify D^{n-2} with $D^{n-3} \times D^1$, and let F_+ be the pullback $f_j'^{-1}F$ where $f_j' = \bar{f}_j \circ (\text{Id}_N \times \psi) : N \times D^1 \to T_j$, ψ is a diffeomorphism of D^1 which pushes all the changes into the interval $(0, 1)$, and $\bar{f}_j : S^1 \times D^1 \times D^{n-2} \to T_j$ is the diffeomorphism defined at the end of §1. $\qquad\square$

Proof of Proposition 4.1. Since the base N of the principal circle bundle $\pi : N' \to N$ has dimension at least three, we can apply Proposition 3.1 to obtain a foliation F_U of an open set $U \subset \text{Int } N$, a transverse 1-dimensional foliation $F_U^\perp$, and a compact saturated set $C \subset U$ containing no compact leaves such that $N - C$ is the disjoint union of open sets A_0 and A_1, with the properties stated there. Given two smooth foliations F_0' and F_1' on N' transverse to the fibration and invariant under the principal S^1 action $\alpha : S^1 \times N' \to N'$, we must construct a new foliation F' and a vector field Y' on N' satisfying the conclusion of (4.1).

We foliate $C' = \pi^{-1}(C)$ by the inverse images under π of the leaves of F_U in C. On $A_i' = \pi^{-1}(A_i)$ we shall deform the leaves of $F_i'|_{A_i'}$ so that they spiral around the leaves in C', for $i = 0, 1$. The spiralling will be produced by the action α and a smooth proper function $f : N - C \to \mathbb{R}$, called the tilting function, which determines the direction and extent of the spiralling. More precisely, we define a diffeomorphism

$$f' : N' - C' \to N' - C', \qquad f'(x') = \alpha(-f(\pi(x')), x'), \tag{4.3}$$

where by an abuse of notation we write $f(\pi(x'))$ for its image $f(\pi(x'))$ modulo 1 in $S^1 = \mathbb{R}/\mathbb{Z}$, and set

$$F'|_{A_i'} = (f')^{-1}(F_i')|_{A_i'}, \qquad i = 0, 1. \tag{4.4}$$

We shall choose the **tilting function** f to vanish in a neighborhood of $N - U$ so that F' will agree with F_i' in a neighborhood of $A_i' - A_i' \cap U'$ which contains $\partial_i N'$, $i = 0, 1$, where $U' = \pi^{-1}(U)$. We also choose f to be strictly increasing on the part of each leaf T of $F_U^\perp|_{U - C}$ that lies in a certain neighborhood of C. Since f is proper, $f(x)$ tends to $+\infty$ (respectively, $-\infty$) as x tends to C along T in the positive (respectively, negative) direction (see Figure 12). The leaves are tilted more and more by f' as they approach C', thus causing them to spiral around C' (see Figure 13). One can check that $F_U^\perp$ can be lifted to a 1-dimensional foliation on U' transverse to the leaves of F' near C'. This proves that F' has bidistinguished neighborhoods near C' and is therefore a foliation. (This fact will follow independently from the construction, in §5, of a C^1 1-form defining F' locally.)

The restriction of F' to each A_i' is clearly smooth. The remaining leaves are in C' and are C^∞ by construction. They are noncompact since they are fibered over the (noncompact) leaves of C. The leaves in each A_i' are noncompact because they are covering spaces of the components of A_i which must have points of C in their closure, since C separates A_0 from A_1 in N which is connected.
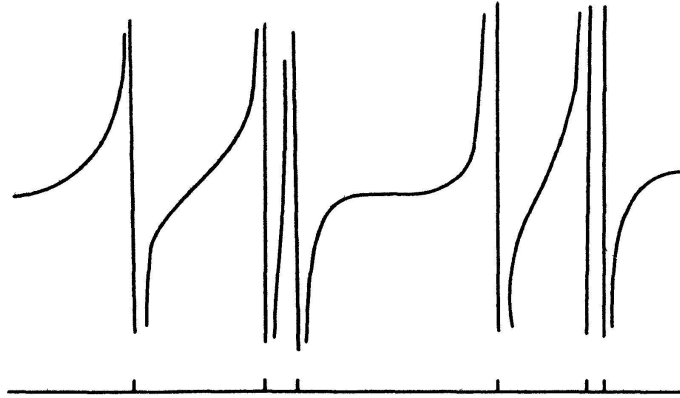
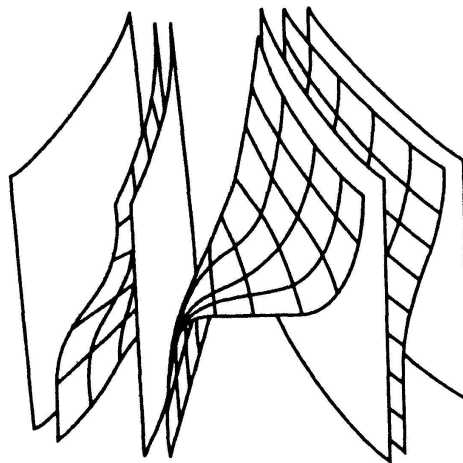Figure 12. The tilting function f along a leaf of $F_U^{\perp}$.

Figure 13. The foliation F'.

Lastly we construct the vector field

$$Y' = \begin{cases} Y - \varepsilon\lambda Z & \text{on } U' \\ Y & \text{on } N' - U', \end{cases}$$

where Z is a vector field on U' tangent to the lift of $F_U^{\perp}$ with the same orientation and $\lambda : N' \to [0, 1]$ is a smooth function which is 0 near $N' - U'$ and 1 near C'. Clearly Y' and Y coincide near $\partial N'$. We claim that for any sufficiently small $\varepsilon > 0$, Y' is simultaneously transverse to the foliations F_0', F_1' and F'. Indeed, Y is transverse to F_0' and F_1' by hypothesis and consequently transverse to F' on $N' - C'$, although tangent to F' along C'. The small perturbation makes Y' transverse to F' along C' and (because of the choice of sign) preserves the transversality to F' on $N' - C'$.   □

Remark 4.5. The foliation F' is C^1 by (4.2), but it cannot be C^2. Suppose that F' were defined by C^2 distinguished neighborhoods. The same is true of the

foliation F_U on $U \subset N$. By Sacksteder's Theorem 1 [Sack2], there must be a loop α on a leaf in C with strongly contracting holonomy. Then $\pi^{-1}(\alpha(S^1))$ is a torus contained in some leaf of C' such that the holonomy of one generator of its fundamental group (the lift of α) is a contraction, while the holonomy of the other generator (the fiber S^1) has as fixed point set a Cantor set (the intersection of C' with a transverse arc). This contradicts Lemma 1 of Kopell [K], which asserts that if two C^2 diffeomorphisms of $\mathbb{R}$ fixing the origin commute, so that the first is a contraction and the second has other fixed points besides 0, then the second must be the identity mapping.

Remark 4.6. Propositions 4.1 and 4.2 yield a shorter construction (not using Prop. 1.1) of C^1 foliations without compact leaves on $S^1 \times S^{n-1}$ ($n \geq 4$) and similar manifolds. Let N be a product neighborhood of the equatorial $(n-2)$ sphere in S^{n-1}, and let $F_0' = F_1'$ be the product foliation $\{\text{point}\} \times S^{n-1}$ on $S^1 \times S^{n-1}$. Then (4.1) will modify the foliation F_0' inside $N' = S^1 \times N$ so that all compact leaves are eliminated, and (4.2) asserts that the result is C^1.

§5. The foliation is defined by a C^1 1-form

In this section we prove Proposition 4.2 by showing that the foliation F' constructed on N' in the proof of Proposition 4.1 is locally defined by a C^1 1-form. First we define the tilting function f and then we pass to a convenient coordinate chart where the coefficients of the defining 1-form can be explicitly calculated and shown to be C^1. The notation of Proposition 4.1 is maintained. According to Proposition 3.1 there exist a submersion $\beta : U - C \to (0, 1)$ constant on each leaf of F_U and a diffeomorphism τ of $S^1 = \mathbb{R}/\mathbb{Z}$ onto a leaf of $F_U^{\perp}$ that meets every leaf of F_U.

The tilting function. We define $f : N - C \to \mathbb{R}$ by

$$\begin{cases} f(x) = r(x)b(\beta(x)) & \text{for } x \in U - C \\ f(x) = 0 & \text{for } x \in N - U \end{cases} \tag{5.1}$$

in terms of smooth functions $r : N - C \to [0, +\infty)$ and $b : (0, 1) \to \mathbb{R}$ which will now be chosen. Fix a smooth Riemannian metric g on N such that $d(C, N - U) > 2$, where d is the distance function defined by g, and so that $F_U^{\perp}$ is orthogonal to F_U at every $x \in U$ such that $d(x, C) \leq 2$. For a point $x \in U - C$ let T_x be the leaf of the foliation $F_U^{\perp}|_{U-C}$ passing through x, and let $|T_x| \in (0, +\infty]$ be its

length. Define $r : U - C \to \mathbb{R}$ by setting

$$\begin{cases} r(x) = a(-\log|T_x|) & \text{if } d(x, C) \le 1 \\ r(x) = 0 & \text{if } d(x, C) \ge 2 \end{cases} \qquad (5.2)$$

and smoothly extending r over $U - C$, where $a : [-\infty, \infty) \to [1, \infty)$ is a smooth function such that $a(x) = 1$ for $x \le 1$ and $a(x) = x$ for $x \ge 2$. Note that r vanishes near $N - U$ so that (5.1) will give a smooth function f, and that

$$r(x) = -\log|T_x| \qquad \text{if } |T_x| \le e^{-2}. \qquad (5.3)$$

Next the smooth function $b : (0, 1) \to \mathbb{R}$ (see Figure 14) will be constructed so that $b' > 0$, the symmetry

$$b = b(u) = -b(1 - u), \qquad u \in (0, 1) \qquad (5.4)$$

holds, and

$$1/b, 1/b', (1/b')', b/b', (b/b')' \to 0 \qquad \text{as } u \to 0^+ \text{ or } u \to 1^-. \qquad (5.5)$$

These five functions are therefore bounded on $(0, 1)$ and $b \to \infty$ or $b \to -\infty$ according as $u \to 1^-$ or $u \to 0^+$. The stated properties of b are easily verified if we set

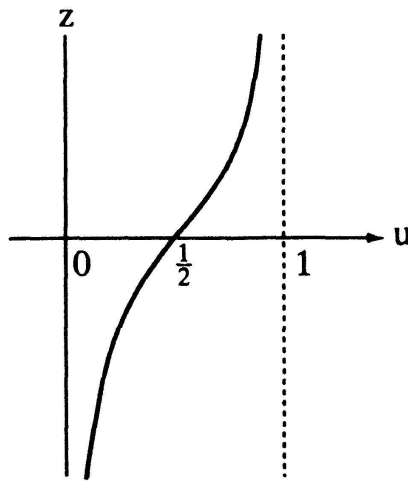$$b(u) = \rho(u) e^{1/(1-u)} - \rho(1 - u) e^{1/u}, \qquad u \in (0, 1),$$

Figure 14. The function $z = b(u)$.

where $\rho : [0, 1] \to [0, 1]$ is a smooth function such that $\rho(u) = 0$ for u near 0, $\rho(u) = 1$ for u near 1, $\rho'(u) \geq 0$ for all $u \in (0, 1)$, and $\rho'(\frac{1}{2}) > 0$.

A coordinate chart. Given an arbitrary point $x_0' \in C'$, take a bidistinguished coordinate chart ([HH], Part A, p. 19) ψ defined on a connected open neighborhood $U_0 \subset U$ of $x_0 = \pi(x_0')$,

$$\psi : U_0 \to J \times V \subset \mathbb{R} \times \mathbb{R}^{n-2} \tag{5.6}$$

relative to the foliations F_U and $F_U^\perp$. This means that for all $t \in J = (t_1, t_2)$ and $y \in V$, the sets $\psi^{-1}(\{t\} \times V)$ and $\psi^{-1}(J \times \{y\})$ are plaques of leaves of F_U and $F_U^\perp$ respectively. It follows that the sets A_i and C determine open sets $B_i \subset J$ and a relatively closed set $C_0 \subset J$ such that $B_i \times V = \psi(U_0 \cap A_i)$, $i = 0, 1$, and $C_0 \times V = \psi(U_0 \cap C)$. We choose U_0, V and J small enough so that ψ extends to a diffeomorphism on a compact set $\bar{U}_0 \subset U$. Since the circle $\tau(\mathbb{R}/\mathbb{Z})$ of Proposition 3.1 meets all the leaves of F_U there is a path γ on the leaf L_{x_0} containing x_0 from some point $\tau(t_0)$ ($t_0 \in \mathbb{R}$) to x_0. We choose the interval J and the diffeomorphism ψ so that

$$\psi h_\gamma \tau(t) = (t, y_0), \quad t \in J,$$

where we identify $J \subset \mathbb{R}$ with its image in $\mathbb{R}/\mathbb{Z}$ and h_γ is the holonomy mapping of γ, as in (1.4). It follows that if $x \in U_0 - C$ and $\psi(x) = (t, y)$ then $\beta(x) = \beta(h_\gamma \tau(t)) = \beta \tau(t)$ since x and $h_\gamma \tau(t)$ are in the same plaque and β is constant along the leaves of F_U. Therefore

$$\beta(x) = \varphi(t) \quad \text{where} \quad \psi(x) = (t, y), \quad x \in U_0 - C, \tag{5.7}$$

where φ is the mapping of (3.1.ii). By an abuse of notation we write r for $r \circ \psi^{-1}$ so that r becomes a function of $(t, y) \in (J - C_0) \times V$. We choose U_0 small enough so that $d(x, C) \leq 1$ for all $x \in U_0$, which implies by (5.2) that r is locally constant as a function of t.

The lifted coordinate charts. Let $U_0' = \pi^{-1}(U_0) \subset N'$ and let L_i' be the leaf of $F_i'|_{U_0'}$ containing x_0', $i = 0, 1$. Then $\pi_i = \pi|_{L_i'} : L_i' \to U_0$ is a covering map, and we choose U_0 sufficiently small to make each π_i a diffeomorphism. We define S^1-equivariant coordinate charts

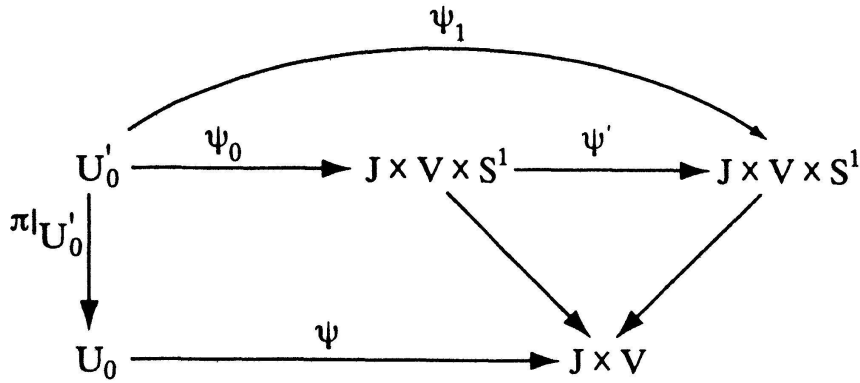$$\psi_i : U_0' \to J \times V \times S^1, \quad i = 0, 1$$

Figure 15. The coordinate charts.

by setting

$$\psi_i(x') = (\psi(x), z) \qquad \text{for } x' = \alpha(z, \pi_i^{-1}(x)) \in U_0', \tag{5.8}$$

where $x \in U_0, z \in S^1 = \mathbb{R}/\mathbb{Z}$, and $\alpha : S^1 \times N' \to N'$ is the S^1-action of the principal S^1 bundle $\pi : N' \to N$.

The foliation F'' on the coordinate space. We shall define a foliation F'' on $J \times V \times S^1$ such that $\psi_0^{-1}F'' = F'|_{U_0'}$. From (5.8), the equivariance of ψ_i and the invariance of F_1' under α, it follows that

$$\psi_i^{-1}F_h = F_i', \qquad i = 0, 1, \tag{5.9}$$

where F_h is the horizontal product foliation $J \times V \times \{\text{point}\}$ of $J \times V \times S^1$. Note that the S^1-equivariant diffeomorphism

$$\psi' = \psi_1 \circ \psi_0^{-1} : J \times V \times S^1 \to J \times V \times S^1 \tag{5.10}$$

has the form

$$\psi'(t, y, z) = (t, y, z + \lambda(t, y)) \tag{5.11}$$

for some smooth function $\lambda : J \times V \to \mathbb{R}/\mathbb{Z}$ which does not depend on z. From (5.9) and (5.10) it follows that

$$\psi_0^{-1}(\psi')^{-1}F_h = F_1'. \tag{5.12}$$

Parallel to (4.3) we define

$$\bar{f}' : (J - C_0) \times V \times S^1 \to (J - C_0) \times V \times S^1,$$

$$\bar{f}'(t, y, z) = (t, y, z - \bar{f}(t, y)),$$

(5.13)

where $\bar{f} = f \circ \psi^{-1} : (J - C_0) \times V \to \mathbb{R}$ with its values taken modulo 1, and set

$$F_h' = (\bar{f}')^{-1} F_h$$

(5.14)

on $(J - C_0) \times V \times S^1$. Thus F_h' is the foliation of $(J - C_0) \times V \times S^1$ obtained by modifying the horizontal foliation F_h so that its leaves spiral around $C_0 \times V \times S^1$.

DEFINITION 5.15. *Define the foliation F'' on $J \times V \times S^1$ to have the same leaves as F_h' on $B_0 \times V \times S^1$, the same leaves as $(\psi')^{-1} F_h'$ on $B_1 \times V \times S^1$, and the leaves $\{t\} \times V \times S^1$ ($t \in C_0$) on $C_0 \times V \times S^1$.*

LEMMA 5.16. $\psi_0^{-1} F'' = F'|_{U_0'}$.

Proof. By (5.9), $\psi_0^{-1} F_h = F_0'$. Therefore the S^1-equivariant mapping ψ_0 transforms the construction (4.3)–(4.4) of $(f')^{-1} F_0'$ on $A_0' \cup U_0'$ into the construction (5.13)–(5.14) of $F_h' = (\bar{f}')^{-1} F_h$ on $B_0 \times V \times S^1$, that is,

$$\psi_0^{-1}(F_h'|_{B_0 \times V \times S^1}) = (f')^{-1} F_0'|_{A_0' \cap U_0'},$$

or equivalently, by the definitions of F'' and F',

$$\psi_0^{-1}(F''|_{B_0 \times V \times S^1}) = F'|_{A_0' \cap U_0'}.$$

Analogously the fact (5.12) that $\psi_0^{-1}(\psi')^{-1} F_h = F_1'$ implies that $\psi_0^{-1}(\psi')^{-1} F_h' = (f')^{-1} F_1'$ and consequently

$$\psi_0^{-1}(F''|_{B_1 \times V \times S^1}) = F'|_{A_1' \cap U_0'}.$$

Since C' is foliated by leaves of $F_U|_C$ multiplied by S^1, and $\psi_0^{-1}(\{t\} \times V \times S^1)$ is a plaque of such a leaf if $t \in C_0$, we get

$$\psi_0^{-1}(F''|_{C_0 \times V \times S^1}) = F'|_{C' \cap U_0'},$$

proving the Lemma. $\square$

The 1-*form* ω *that defines* F''. On $J \times V \times S^1$ we have the coordinates $t \in J$, $y = (y^1, \ldots, y^{n-2}) \in V$, and $z \in S^1 = \mathbb{R}/\mathbb{Z}$. As a prelude to calculating the coefficients of the 1-form

$$\omega = dt + \sum_{j=1}^{n-2} P_j(t, y) \, dy^j + Q(t, y) \, dz, \tag{5.17}$$

which defines the foliation F'' on $J \times V \times S^1$, we shall calculate the coefficients of the smooth 1-form

$$\omega_0 = dt + \sum_{j=1}^{n-2} p_j(t, y) \, dy^j + q(t, y) \, dz, \tag{5.18}$$

which defines the foliation F'_h on $(J - C_0) \times V \times S^1$. The function $\bar{f} = f \circ \psi^{-1} : (J - C_0) \times V \to \mathbb{R}$, is given by

$$\bar{f}(t, y) = r(t, y)b(\varphi(t)) \tag{5.19}$$

in view of (5.7) and (5.1). From the definition (5.13)–(5.14) of F'_h it follows that on $(J - C_0) \times V \times S^1$ the leaves of F'_h are given by

$$z = r(t, y)b(\varphi(t)) + \text{constant.}$$

Therefore their tangent plane field is the kernel of the 1-form

$$rb'(\varphi)\varphi' \, dt + \sum_j \partial_j rb(\varphi) \, dy^j - dz,$$

where the variables t and y are suppressed and $\partial_j = \partial/\partial y^j$. Consequently

$$\begin{cases} p_j = (rb'(\varphi)\varphi')^{-1}b(\varphi)\partial_j r \\ q = -(rb'(\varphi)\varphi')^{-1}. \end{cases} \quad \text{and} \tag{5.20}$$

In order to control the convergence of these coefficients to 0 as t approaches C_0, we introduce the following definition, where $J = (t_0, t_1) \subset \mathbb{R}$ is any open interval, $C_0 \subset J$ is a closed subset, and $V \subset \mathbb{R}^m$ is any open set. Call each component (s_0, s_1) of $J - C_0$ a **gap** and the set $S(s_0, s_1) = (s_0, s_1) \times V$ a **gap set**. For $\varepsilon > 0$ let S_ε be the union of the gap sets $S(s_0, s_1)$ for which $s_1 - s_0 < \varepsilon$.

DEFINITION 5.21. A continuous function $g : (J - C_0) \times V \to \mathbb{R}$ is **0-controlled** if

(i) For each gap set $S(s_0, s_1)$, $y_1 \in V$, and $i = 0, 1$, either $s_i = t_i$ or

$$\lim g(t, y) = 0 \quad as \quad (t, y) \to (s_i, y_1), \quad (t, y) \in S(s_0, s_1).$$

(ii) $\sup\{|g(t, y)| : (t, y) \in S_\varepsilon\} \to 0$ as $\varepsilon \to 0$.

A C^r $(1 \le r < \infty)$ function $g : (J - C_0) \times V \to \mathbb{R}$ is **r-controlled** if g and its partial derivatives of order at most r are 0-controlled.

We shall need the three following lemmas whose proofs will be given at the end of this section.

LEMMA 5.22. *Let* $g : (J - C_0) \times V \to \mathbb{R}$ *be r-controlled and define* $g(t, y) = 0$ *for* $(t, y) \in C_0 \times V$. *Then* $g : J \times V \to \mathbb{R}$ *is a* C^r *function.*

LEMMA 5.23. *Let* $g_0, g_1 : (J - C_0) \times V \to \mathbb{R}$ *be r-controlled functions. Then*

(i) $g_0 + g_1 : (J - C_0) \times V \to \mathbb{R}$ *is r-controlled,*

(ii) *If* $a : (J - C_0) \times V \to \mathbb{R}$ *is a* C^r *function with uniformly bounded* C^r *norm, then the product* $a \cdot g_0 : (J - C_0) \times V \to \mathbb{R}$ *is r-controlled,*

(iii) *If* $J - C_0 = B_0 \cup B_1$ *where* B_0 *and* B_1 *are disjoint open sets and* $g : (J - C_0) \times V \to \mathbb{R}$ *satisfies* $g|_{B_i \times V} = g_i|_{B_i \times V}$ *for* $i = 0, 1$, *then* g *is r-controlled.*

Returning to our specific situation, $J \times V = \psi(U_0)$, we have the following result.

LEMMA 5.24. *The derivatives* $\partial_j r$ *and* $\partial_j \partial_k r$ $(1 \le j, k \le n - 2)$ *are uniformly bounded on* $(J - C_0) \times V$.

Next we show that $q(t, y)$ *and* $p_j(t, y)$ are 1-controlled.

By (3.1.ii), $|1/\varphi'(t)| \le K$ and $|(1/\varphi')'(t)| \le K$ for some constant $K > 0$. Consequently (5.5) and (3.1.i) imply that the functions

$$\frac{1}{\varphi'(t)}\left(\frac{1}{b'} \circ \varphi\right)(t) \quad \text{and} \quad \frac{1}{\varphi'(t)}\left(\frac{b}{b'} \circ \varphi\right)(t) \tag{5.25}$$

and their first derivatives with respect to t,

$$
\begin{cases}
\left(\dfrac{1}{b'}\right)' \circ \varphi(t) + \left(\dfrac{1}{\varphi'}\right)'(t)\left(\dfrac{1}{b'} \circ \varphi\right)(t) & \text{and} \\[4mm]
\left(\dfrac{b}{b'}\right)' \circ \varphi(t) + \left(\dfrac{1}{\varphi'}\right)'(t)\left(\dfrac{b}{b'} \circ \varphi\right)(t)
\end{cases}
$$

are uniformly bounded on $(J - C_0) \times V$ and satisfy the condition (5.21.i). Now U_0 was chosen so that $d(x, C) \le 1$ for $x \in U_0$, so by (5.2), $r(x) \ge 1$ on U_0, which contains $\psi(J \times V)$. Furthermore the gap length $|T_x|$ tends to zero as $\varepsilon \to 0$, that is,

$$
\sup\{|T_x| : x \in \psi^{-1}(S_\varepsilon)\} \to 0 \qquad \text{as } \varepsilon \to 0,
$$

because the metric is bounded on $\bar{U}_0$ which is compact. Now by (5.3) $r(x) = -\log|T_x|$ when $|T_x|$ is sufficiently small, so the function $r(y)^{-1}$ and (by (5.24)) its derivatives $-r^{-2}\partial_j r$ are bounded and tend to zero uniformly for $(t, y) \in S_\varepsilon$ as $\varepsilon \to 0$. This implies that the products of the functions (5.25) by $r(t, y)^{-1}$ are 1-controlled. The first product is precisely $-q(t, y)$, and multiplying the second product by $\partial_j r$, which has bounded C^1 norm by (5.24), yields $p_j(t, y)$ which is therefore 1-controlled by (5.23.ii). (Note that the presence of the factor $r(x)$ in the definition (5.1) of $f(x)$ is essential here. If the factor $r(x)$ were removed, then the C^1-norms of the coefficients p_j and q would have approximately the same range of values on each gap, so that their first derivatives could not be continuous.)

From (5.18) and (5.11) it follows that the 1-form which defines $(\psi')^{-1}F'_h$ is

$$
\psi'^* \omega_0 = (1 + (q \circ \psi')\lambda_t)\, dt + \sum_j (p_j \circ \psi' + (q \circ \psi')\, \partial_j \lambda)\, dy^j + (q \circ \psi')\, dz \qquad (5.26)
$$

on $J \times V \times S^1$, where $\lambda_t = \partial\lambda/\partial t$. Now $q(t, y)$ is 1-controlled and hence by (5.22) extends to a C^1 function, also denoted q, on $J \times V$. Thus $g(t, y) = 1 + (q \circ \psi')\lambda_t$ is a C^1-function and (possibly replacing $J \times V$ by a smaller product neighborhood of $\psi(x_0)$) we may assume that $g = g(t, y) > 0$ and $1/g$ has bounded C^1 norm on $J \times V$. Since ψ' only changes the coordinate z and p_j and q do not depend on z, we have $q = q \circ \psi'$ and $p_j = p_j \circ \psi'$. Normalizing the coefficient of dt in (5.26) now yields

$$
\omega_1 = (1/g)\psi'^* \omega_0 = dt + \sum_j \bar{p}_j(t, y)\, dy^j + \bar{q}(t, y)\, dz, \qquad (5.27)
$$

where $\bar{p}_j = (1/g)(p_j + q\,\partial_j\lambda)$ and $\bar{q} = (1/g)q$, which are 1-controlled by (5.23). Finally from the definition (5.15) of F'' it follows that the coefficients in (5.17) satisfy

$$P_j\big|_{B_0 \times V} = p_j\big|_{B_0 \times V},\ P_j\big|_{B_1 \times V} = \bar{p}_j\big|_{B_1 \times V},\ P_j\big|_{C_0 \times V} = 0$$

and similarly for Q. Then (5.23.iii) and (5.22) imply that P_j and Q are 1-controlled on $(J - C_0) \times V$ and hence C^1 on $J \times V$. Thus ω is a C^1 1-form defining F'' which is diffeomorphic to F' in a neighborhood of x_0'. $\qquad\square$

Proof of Lemma 5.22. The proof for $r = 0$ is a straightforward exercise. For $r = 1$, observe that $\partial_j g = \partial g/\partial y^j = 0$ on $C_0 \times V$, where g vanishes identically. For the verification that $\partial g/\partial t = 0$ on $C_0 \times V$, we fix a point in V so that g becomes a function of t alone. Given $c \in C_0$ and t in the gap (s_0, s_1), say $c \le s_0$, the Mean Value Theorem on $[s_0, t]$ yields

$$\big|g(t) - g(c)\big| = \big|g(t) - g(s_0)\big| \le \big|g'(\lambda)\big| \cdot \big|t - s_0\big|$$

$$\le \big|g'(\lambda)\big| \cdot \big|t - c\big|$$

for some $\lambda \in (s_0, s_1)$, while $\big|g(t) - g(c)\big| = 0$ if $t \in C_0$. Since $g'(t)$ is 0-controlled on $(J - C_0) \times V$, this implies that $g'(c) = 0$, so $\partial g/\partial t = 0$ on $C_0 \times V$. Applying the case $r = 0$ to the derivatives $\partial_j g$ and $\partial g/\partial t$ shows that they are continuous, so g is C^1. The case $r > 1$ follows by an easy induction from the cases $r = 0$ and $r = 1$. $\qquad\square$

Proof of Lemma 5.23. Given r-controlled functions $g_0, g_1 : (J - C_0) \times V \to \mathbb{R}$, by (5.22) they extend to C^r functions on $J \times V$, and from the proof of (5.22) it is clear that their derivatives of order less than or equal to r vanish identically on $C_0 \times V$. The assertions (i), (ii) and (iii) now follow easily. $\qquad\square$

Proof of Lemma 5.24. Denote by g the metric induced on $J \times V$ by the restriction of the Riemannian metric g to U_0 under the diffeomorphism ψ. Since g extends to the compact set $\bar{J} \times \bar{V}$, there exists a constant $K > 0$ such that the coefficient $g_{00}(t, y) = g(\partial/\partial t, \partial/\partial t)(t, y)$ satisfies

$$g_{00} \ge K^{-2},\ \left|\frac{\partial g_{00}}{\partial y^j}\right| \le K \quad \text{and} \quad \left|\frac{\partial^2 g_{00}}{\partial y^j\,\partial y^k}\right| \le K$$

on $J \times V$. From (5.3) we obtain

$$e^{-r(y)} = \int_{s_0}^{s_1} \sqrt{g_{00}(t, y)}\, dt \ge K^{-1}(s_1 - s_0) \tag{5.28}$$

on each gap set $S(s_0, s_1)$ that is narrow enough to have $|T_x| \le e^{-2}$ for every $x \in \psi^{-1} S(s_0, s_1)$. Differentiating the equation (5.28) with respect to y^j yields

$$e^{-r}|\partial_j r| = \left| \int_{s_0}^{s_1} \frac{1}{2} g_{00}^{-1/2} \, \partial_j g_{00} \, dt \right| \le K^2 (s_1 - s_0)/2,$$

which together with (5.28) shows that $|\partial_j r| \le K^3/2$. Differentiating (5.28) twice yields $|\partial_j \partial_k r| \le K^3(K^3 + 1)/2$ by a similar argument.

There remain at most finitely many gap sets $S(s_0, s_1)$ containing points with $|T_x| > e^{-2}$. For each such gap set the first and second derivatives of r are uniformly bounded on a slice $\{t\} \times \bar{V}$, $t \in (s_0, s_1)$. Since r and its derivatives are locally constant in t, this bound holds on all of $S(s_0, s_1)$. $\square$

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