Upper bound for the first eigenvalue of algebraic submanifolds.

Comment. Math. Helvetici 69 (1994) 199–207 0010-2571/94/020199-09$1.50 + 0.20/0

Upper bound for the first eigenvalue of algebraic submanifolds

JEAN-PIERRE BOURGUIGNON, PETER LI, AND SHING TUNG YAU

1. Statement of results

Let M be a compact manifold endowed with a Riemannian metric. The spectrum of the Laplacian, Δ, acting on functions form a discrete set of the form $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty\}$. In 1970, Joseph Hersch [5] gave a sharp upper bound for the first non-zero eigenvalue λ_1 for any Riemannian metric on the 2-sphere in terms of its volume alone. Similar estimates for λ_1 on any compact oriented surfaces were derived by Yang and Yau [7]. The second and the third authors [6] studied the non-orientable surfaces and pointed out the relationship of λ_1 and the conformal class of the surface. In fact, their estimates were applied to study the Willmore problem. Another application of these types of upper bounds was found by Choi and Schoen [3] in relation to the set of all minimal surfaces in a compact 3-manifold of positive Ricci curvature.

The purpose of this paper is to prove a higher dimensional generalization of the above results. It was pointed out by Marcel Berger [1] that Hersch's theorem fails in higher dimensional spheres. In view of the relationship between λ_1 and the conformal structure of a surface as indicated by Li and Yau [6], we were thus motivated to study the complex category.

THEOREM. *Let M^m be an m-dimensional compact complex manifold admitting a holomorphic immersion $\Phi : M \to \mathbb{CP}^N$. Suppose that Φ is full in the sense that $\Phi(M)$ is not contained in any hyperplane of $\mathbb{CP}^N$. Then, for any Kähler metric ω on M, the first non-zero eigenvalue $\lambda_1(M, \omega)$ satisfies*

$$\lambda_1(M, \omega) \leq 4m \frac{N+1}{N} d([\Phi], [\omega]).$$

Research of the first author was partially supported by a grant from the European Union and of the second and the third authors by their NSF grants.

Here, $d([\Phi], [\omega])$ *is the holomorphic immersion degree, a homological invariant defined by*

$$d([\Phi], [\omega]) = \frac{\int_M \Phi^*(\sigma) \wedge \omega^{m-1}}{\int_M \omega^m}$$

where σ *is the Kähler form of* $\mathbb{CP}^N$ *with respect to the Fubini–Study metric, and* [ ] *denotes the homology or cohomology class defined by the enclosed object.*

Use of this estimate can be found in [3].

Let us remark that the totally geodesic subvarieties $\mathbb{CP}^m$ endowed with the Fubini–Study metric σ have $\lambda_1(\mathbb{CP}^m, \sigma) = 4(m+1)$. In particular, if we apply the Theorem to a fully immersed algebraic submanifold with the induced Fubini–Study metric, then we have an interesting gap estimate.

COROLLARY. *Let* $\Phi : M^m \to \mathbb{CP}^N$ *be a full immersion of an m-dimensional complex manifold into* $\mathbb{CP}^N$ *(with* $N > m$*). Then, the first eigenvalue of M with respect to the metric induced by the Fubini–Study metric* σ *on* $\mathbb{CP}^N$ *has an upper bound given by*

$$\lambda_1(M, \Phi^*(\sigma)) \leq 4m\frac{N+1}{N} < \lambda_1(\mathbb{CP}^m, \sigma).$$

The authors thank Jean Benoît Bost and the referee for helpful comments on a first version of the paper.

2. Proof of the theorem

a) A convenient embedding of $\mathbb{CP}^N$

Let us first review a well known construction concerning the complex projective space $\mathbb{CP}^N$. As we shall see later, it is related to its spectral properties.

Let $\Psi : \mathbb{CP}^N \to \mathcal{M}_{N+1}(\mathbb{C}) \cong \mathbb{C}^{(N+1)^2}$ be the (real analytic) embedding mapping a point in $\mathbb{CP}^N$ – a 1-dimensional vector subspace of $\mathbb{C}^{(N+1)}$ – to the orthogonal projection onto this line.

Consider homogeneous coordinates $[Z] = [z_0; z_1; \ldots; z_N]$ on $\mathbb{CP}^N$. One then has

$$\Psi([Z]) = \frac{ZZ^*}{Z^*Z} = \left(\frac{z_i \bar{z}_j}{\sum_{k=0}^{N} |z_k|^2}\right)_{0 \le i,j \le N},$$

where Z^* denotes the conjugate transpose of Z. Clearly,

$$\mathscr{C} := \Psi(\mathbb{CP}^N) = \{A \in \mathscr{M}_{N+1}(\mathbb{C}) \mid A = A^2 = A^*, \text{rank } A = 1\}.$$

It easily follows from the diagonalizability of Hermitian matrices in a unitary basis that

$$\mathscr{C} = \{A \in \mathscr{M}_{N+1}(\mathbb{C}) \mid A = A^*, \text{rank } A = 1, \text{trace } A = 1\},$$

that the real convex hull of $\mathscr{C}$ in $\mathscr{M}_{N+1}(\mathbb{C})$ is

$$\mathscr{H} := \{A \in \mathscr{M}_{N+1}(\mathbb{C}) \mid A = A^*, A \ge 0, \text{trace } A = 1\}.$$

and that the boundary of the compact convex set $\mathscr{H}$ (sitting in the affine space $\{A \in \mathscr{M}_{N+1}(\mathbb{C}) \mid A = A^*, \text{trace } A = 1\}$ of real dimension $N(N+2)$) is

$$\partial\mathscr{H} = \{A \in \mathscr{M}_{N+1}(\mathbb{C}) \mid A = A^*, A \ge 0, \text{trace } A = 1, \text{and rank } A \le N\},$$

i.e., that $\mathscr{C}$ is the set of extremal points of $\mathscr{H}$.

b) The action of the group of holomorphic transformations

We can now relate this embedding to the homogeneous structure of the complex projective space by considering the action of holomorphic transformations.

The group of holomorphic transformations acting on $\mathbb{CP}^N$ with respect to its homogeneous coordinates is given by

$$\mathrm{PGl}_{N+1}(\mathbb{C}) = \{B \in \mathrm{Gl}_{N+1}(\mathbb{C}) \mid \det B = 1\}/\text{center}.$$

Let $[Z] = [z_0; z_1; \ldots; z_N] \in \mathbb{CP}^N$. The action of $B \in \mathrm{Gl}_{N+1}(\mathbb{C})$ on $\mathbb{CP}^N$ is given by $[Z] \mapsto [BZ]$. When described in terms of elements in $\mathscr{C} = \Psi(\mathbb{CP}^N)$, B maps the projector ZZ^*/Z^*Z to $BZZ^*B^*/Z^*(B^*B)Z$.

The isometric transformations of $\mathbb{CP}^N$ coincide with the elements of the quotient group SU_{N+1} modulo its center. The coset space of $\mathrm{PGl}_{N+1}(\mathbb{C})$ by this subgroup can be interpreted as the set of "non-isometric actions", and can be identified with $\mathrm{Sl}_{N+1}(\mathbb{C})/\mathrm{SU}_{N+1}$. Any class of the coset space has a representative in

$$\mathscr{S} := \{B \in \mathrm{Sl}_{N+1}(\mathbb{C}) \mid B = B^* \text{ and } B > 0\}.$$

Notice that the real dimension of $\mathscr{S}$ is $N(N+2)$, as is the real dimension of $\mathscr{H}$.

Since we will need to work with the boundary of $\mathscr{S}$, having a constraint on the determinant is not so convenient. Therefore, we introduce the cone

$$\mathscr{S}' := \{B \in \mathrm{Gl}_{N+1}(\mathbb{C}) \mid B = B^* \text{ and } B > 0\},$$

of which $\mathscr{S}$ is a basis, i.e., can be obtained by dividing out $\mathscr{S}'$ by the usual projective equivalence relation of multiplication by a positive real scalar, namely

$$\mathscr{S} = \mathscr{S}'/\{B \sim \alpha B, \alpha \in \mathbb{R}_+^*\}.$$

The set $\mathscr{S}'$ has a topological boundary given by

$$\partial\mathscr{S}' = \{B \in \mathrm{Gl}_{N+1}(\mathbb{C}) \mid B = B^*, B \geq 0, \text{ and rank } B \leq N\}.$$

It will hereafter be convenient to represent points in the boundary $\partial\mathscr{S}$ of $\mathscr{S}$ by lines in $\partial\mathscr{S}'$. (Indeed, notice that going to infinity in $\mathscr{S}$ implies, because of the determinant constraint, that at least one eigenvalue of the linear transformation goes to 0, hence that the linear transformation approaches the topological boundary of $\mathscr{S}'$.)

c) *Taking centers of mass with respect to special measures*

If μ is a non zero positive measure defined on $\mathbb{CP}^N$, we can define the center of mass with respect to μ through Ψ by

$$G(\mu) := \left(\int_{\mathbb{CP}^N} d\mu([Z])\right)^{-1} \left(\int_{\mathbb{CP}^N} \frac{ZZ^*}{Z^*Z}\, d\mu([Z])\right) \in \mathscr{H}.$$

The action of B induces an action on the center of mass through Ψ by

$$B \mapsto \left(\int_{\mathbb{CP}^N} d\mu([Z])\right)^{-1} \left(\int_{\mathbb{CP}^N} \frac{BZZ^*B^*}{Z^*(B^*B)Z}\, d\mu([Z])\right),$$

which also takes its values in $\mathscr{H}$. We will denote by ψ the mapping induced on $\mathscr{S}$ (i.e., $\psi : \mathscr{S} \to \mathscr{H}$), and by $\hat{\psi}$ the map induced on $\mathscr{S}'$. These maps are obviously continuous.

We claim that ψ maps any sequence (B_n) in $\mathscr{S}$ going to its boundary to a sequence which can only accumulate at points of $\partial\mathscr{H}$. Indeed, to any point of accumulation of (B_n) is associated $B \in \partial\mathscr{S}'$ (hence having a non trivial kernel ker B), such that, for a subsequence (B_{n_i}) tending to B, $B_{n_i} = \alpha_{n_i}(B + \epsilon_{n_i})$ where ϵ_{n_i} tends to 0 in the space of non negative Hermitian matrices. It is then clear that the sequence $(\psi(B_{n_i}))$ is made of matrices which are integrals of perturbations of matrices all having ker B as kernel.

We will now derive a sufficient condition on the measure μ under which the mapping $\hat{\psi} : \bar{\mathscr{S}}' - \{0\} \to \mathscr{H}$ is continuous. The mapping $B \mapsto BZZ^*B^*/Z^*B^*BZ$ is not necessarily continuous for a fixed Z. The points of discontinuity are contained in the set given by the vanishing of the denominator Z^*B^*BZ. This occurs when Z is a null vector of B^*B, and hence a null vector of B itself. If we now fix B, the set of discontinuous points is $\{[Z] \mid Z \in \ker B\}$. However, this set describes a standard $\mathbb{CP}^{N-k} \subset \mathbb{CP}^N$.

Since the integrand in the definition of ψ is uniformly bounded, by the dominated convergence lemma, it follows that the map $\psi : \mathscr{S} \to \mathscr{H}$ has a continuous extension to the boundary if the support of μ intersects each $\mathbb{CP}^{N-k}$ at a set of μ-measure zero. This is in particular the case for the measure μ, image under a full immersion $\Phi : M \to \mathbb{CP}^N$ of the Riemannian measure of a complex Kähler manifold M of (complex) dimension m, since μ can be viewed as a measure in $\mathbb{CP}^N$ supported in $\Phi(M)$ which intersects any totally geodesic $\mathbb{CP}^{N-k}$ along a subvariety of dimension (strictly) less than m, hence in a set of μ-measure 0.

We now restrict our attention to this family of measures, which is the setting of the theorem.

d) Achieving any point in $\mathscr{H}$ as a center of mass

We now claim that the map $\psi : \mathscr{S} \to \mathscr{H}$ is onto.

We just proved that ψ maps sequences accumulating at points in $\partial\mathscr{S}$ to sequences accumulating to points in $\partial\mathscr{H}$ and that, in our case, ψ has a continuous extension to the boundary. If we can show that $\psi : \partial\mathscr{S} \to \partial\mathscr{H}$ has a non-trivial degree, then the fixed point theorem will imply the surjectivity of ψ.

We take the viewpoint that the measure μ is a measure defined on $\mathbb{CP}^N$ with support in the image $\Phi(M)$.

Let us denote by μ_0 the canonical measure on $\mathbb{CP}^N$ induced by the Fubini–Study metric σ normalized to have the same total mass as μ. We can define a 1-parameter family of mappings $\psi_t : \mathcal{S} \to \mathcal{H}$ given by

$$\psi_t(B) = t\psi(B) + (1-t)\psi_0(B)$$

where

$$\psi_0(B) = \left(\int_{\mathbb{CP}^N} d\mu_0([Z])\right)^{-1}\left(\int_{\mathbb{CP}^N} \frac{BZZ^*B^*}{Z^*(B^*B)Z}\, d\mu_0([Z])\right)$$

is the center of mass computed with respect to the measure μ_0. (Note that $\psi_t(B)$ can also be viewed as the center of mass computed with respect to the measure $\mu_t = t\mu + (1-t)\mu_0$.) Since both ψ and ψ_0 are continuous, ψ_t is a continuous 1-parameter family of maps between $\mathcal{S}$ and $\mathcal{H}$. Moreover, by the above argument, ψ_t maps $\partial\mathcal{S}$ into $\partial\mathcal{H}$. Hence, it suffices to check that the degree of $\psi_0 : \partial\mathcal{S} \to \partial\mathcal{H}$ is non-trivial. On the other hand, by the equivariant property of the Fubini–Study metric w.r.t. the action of SU_{N+1}, one readily checks that ψ_0 is a diffeomorphism between PGl_{N+1}/PSU_{N+1} and $\mathcal{H}$, hence has a non-trivial degree on the boundary.

This implies that the map ψ is surjective.

e) *Estimating the lowest eigenvalue by appropriate test functions*

We are now ready to estimate the first non-zero eigenvalue $\lambda_1(M, \omega)$. Again, let us first discuss the Laplacian Δ with respect to the Fubini–Study metric of $\mathbb{CP}^N$. It is known ([2], page 172) that the real and imaginary parts of the functions $\Psi(Z)_{ij}$ given by $\{\Re e(\Psi(Z)_{ij}), \Im m(\Psi(Z)_{ij})\}$ for $0 \le i, j \le N$ span the real subspace generated by the constant functions and the first eigenspace of Δ. In fact, the complex Laplacian $\Box$ when applied to the function $\Psi(Z)_{ij}$ gives

$$\Box\Psi(Z)_{ij} = (N+1)\Psi(Z)_{ij}$$

when $i \neq j$. Also, the diagonal elements satisfy

$$\Box\left(\Psi(Z)_{ii} - \frac{1}{N+1}\right) = (N+1)\left(\Psi(Z)_{ii} - \frac{1}{N+1}\right).$$

Hence, using the fact that $\Box = \frac{1}{4}\Delta$, the first eigenvalue of $\mathbb{CP}^N$ is given by $4(N+1)$.

The identity

$$\sum_{i,j=0}^{N} |\Psi(Z)_{ij}|^2 = 1$$

implies that the image $\mathscr{C}$ of $\mathbb{CP}^N$ under Ψ is contained in the real unit sphere $\mathbb{S}^{2(N+1)^2-1}$ in $\mathbb{C}^{(N+1)^2}$. Moreover, $\mathscr{C}$ sits in the real affine subspace F of real dimension $N(N+2)$. By the orthogonality to constants of eigenfunctions for a non zero value (or by symmetry), the center of mass

$$\psi_0(\mathbf{I}) = \left(\int_{\mathbb{CP}^N} d\mu_0\right)^{-1}\left(\int_{\mathbb{CP}^N} \Psi([Z])\, d\mu_0\right)$$

is given by the matrix $(1/(N+1))\mathbf{I}$. Hence, it is contained in F. Moreover, the normal vector to the hyperspace described by trace $\Psi([Z]) = 1$ is also given by $1/(N+1)\mathbf{I}$. Therefore

$$\mathscr{C} \subset F \cap \mathbb{S}^{2(N+1)^2-1} = \mathscr{P}$$

where $\mathscr{P}$ is the affine sphere of dimension $N^2 + 2N - 1$ with radius $\sqrt{N}/\sqrt{N+1}$ centered at the point given by the diagonal matrix $1/(N+1)\mathbf{I}$. The convex hull $\mathscr{H}$ of $\mathscr{C}$ is then contained in $\mathscr{D}$, the ball contained in F of boundary $\mathscr{P}$. In fact, one readily checks that the embedding

$$\Psi : \mathbb{CP}^N \to \mathscr{P}$$

is the Veronese minimal embedding of $\mathbb{CP}^N$ given by its first eigenspace. The fact that $\psi_0(\mathbf{I}) = 1/(N+1)\mathbf{I}$ implies that it is contained in $\mathscr{H}$. By the surjectivity of ψ, there exists $B \in \mathscr{S}$ such that $\psi(B) = 1/(N+1)\mathbf{I}$. This implies that

$$\left(\int_M dV\right)^{-1}\left(\int_M {}^*\Psi([BZ(\Phi(p))])_{ij}\, dV(p)\right) = \frac{1}{N+1}\delta_{ij}.$$

By the Rayleigh principle, the first eigenvalue of M can be estimated by

$$\lambda_1 \int_M \left(\left(\Psi([BZ])_{ij} - \frac{1}{N+1}\delta_{ij}\right) \circ \Phi(p)\right)^2 dV(p)$$

$$\leq \int_M \left| \nabla\left(\left(\Psi([BZ])_{ij} - \frac{1}{N+1}\delta_{ij}\right) \circ \Phi(p)\right)\right|^2 dV(p)$$

$$= \int_M |\nabla(\Psi([BZ(\Phi(p))])_{ij})|^2 \, dV(p)$$

$$= \frac{4}{(m-1)!} \int_M \partial(\Psi([BZ \circ \Phi])_{ij}) \wedge \bar{\partial}(\Psi([BZ \circ \Phi])_{ij}) \wedge \omega^{m-1}$$

where ω is the Kähler form on M.

Summing over $0 \leq i, j \leq N$, and using the fact that $\mathscr{P}$ is an affine sphere with radius $\sqrt{N}/\sqrt{N+1}$, we conclude that

$$\lambda_1 \frac{N}{N+1} \int_M \frac{\omega^m}{m!} \leq 4 \int_M \sum_{i,j=0}^{N} (\partial(\Psi([BZ \circ \Phi])_{ij}) \wedge \bar{\partial}(\Psi([BZ \circ \Phi])_{ij})) \wedge \frac{\omega^{m-1}}{(m-1)!}.$$

The theorem follows by observing that, since Ψ is the Veronese embedding, the form

$$\sum_{i,j=0}^{N} \partial(\Psi([BZ \circ \Phi]))_{ij} \wedge \bar{\partial}(\Psi([BZ \circ \Phi]))_{ij} = (B \circ \Phi)^* \left(\sum_{i,j=0}^{N} \partial\Psi([Z])_{ij} \wedge \bar{\partial}\Psi([Z])_{ij} \right)$$
$$= \Phi^* B^* \sigma$$

where σ is the Kähler form on $\mathbb{CP}^N$ with respect to the Fubini–Study metric. Let us now observe that both σ and $B^*\sigma$ represent the same cohomology class in $\mathbb{CP}^N$. Hence

$$\int_M \Phi^* B^*(\sigma) \wedge \omega^{m-1} = \int_M \Phi^*(\sigma) \wedge \omega^{m-1},$$

and the theorem follows.

REFERENCES

[1] M. Berger, *Sur les premières valeurs propres des variétés riemanniennes*, Composito Math. *26* (1973), 129–149.

[2] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Math. (1973).

[3] H. I. Choi and R. Schoen, *The space of minimal embeddings of a surface into a three dimensional manifold of positive Ricci curvature*, Inventiones Math. *81* (1985), 387–394.

[4] M. Gromov, *Metric properties of Kähler manifolds*, Preprint I.H.E.S. (1993).

[5] J. HERSCH, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris *270* (1970), 1645–1648.
[6] P. LI and S. T. YAU, *A new conformal invariant and its application to the Willmore conjecture and the first eigenvalue of compact surfaces*, Inventiones Math. *69* (1982), 269–291.
[7] P. YANG and S. T. YAU, *Eigenvalues of the Laplacian on compact Riemannian surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa *7* (1980), 55–63.

Centre de Mathématiques
U.R.A. 169 du C.N.R.S.
Ecole Polytechnique
F-91128 Palaiseau Cedex
France

Department of Mathematics
University of California at Irvine
Irvine CA 92717
USA

and

Department of Mathematics
Harvard University
Cambridge MA 02138
USA

Received January 8, 1993