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On the cohomology of biquadratic extensions

BRUNO KAHN

Introduction

Let F be a field of characteristic $\neq 2$ and M/F a biquadratic extension, that is a Galois extension with Galois group isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$. The three papers [T], [K1] and [MT] study relationships between the Galois cohomology of F, M and the three intermediate subfields L_1, L_2 and L_3 of M. The aim of the present paper is to further this study and in particular to extend results of [MT].

More specifically, in [MT] Merkurjev and Tignol introduce two series of seven-term complexes S_n and S^n ($n \geq 0$):

$$H^nM \oplus (H^nF)^3 \xrightarrow{\alpha_n} \bigoplus_{i=1}^{3} H^nL_i \xrightarrow{\beta_n} X \otimes H^nF \xrightarrow{\gamma_n} H^{n+1}F \xrightarrow{\text{res}}$$

$$H^{n+1}M \xrightarrow{\delta_n} \bigoplus_{i=1}^{3} H^{n+1}L_i \xrightarrow{\varepsilon_n} H^{n+1}M \oplus X \otimes H^{n+1}F \oplus H^{n+2}F \qquad (S_n)$$

$$H^{n+1}M \oplus (H^{n+1}F)^3 \xleftarrow{\alpha^n} \bigoplus_{i=1}^{3} H^{n+1}L_i \xleftarrow{\beta^n} G \otimes H^{n+1}F \xleftarrow{\gamma^n}$$

$$H^nF \xleftarrow{\text{cor}} H^nM \xleftarrow{\delta^n} \bigoplus_{i=1}^{3} H^nL_i \xleftarrow{\varepsilon^n} H^nM \oplus G \otimes H^nF \oplus H^{n-1}F, \qquad (S^n)$$

and prove their exactness in some cases. Here and everywhere in this paper, we denote for any field K by H^nK the Galois cohomology group $H^n(K, \mathbf{Z}/2)$. Our main result is the following:

THEOREM 1. *Sequences* S_n *and* S^n *are exact in the following cases*:

(1) *The Milnor conjecture holds for* F *and all its finite extensions in degree* $\leq n$ *and*

(a) $F = F^2 + F^2$

or

(b) M *is pythagorean*;

(2) $n = 2$.

[Part 2 of th. 1 is not new: see below. Also, the exactness of S_n and S^n for $n \leq 1$ is proven in [T] and [MT].]

Note that condition (a) of th. 1 holds in particular if -1 is a square in F. Recall that a field K is *pythagorean* if the set of squares of K is closed under addition, and that the Milnor conjecture in degree n for K predicts that the natural homomorphism $K_n^M(K)/2 \to H^nK$ is an isomorphism, where $K_n^M(K)$ is the n-th Milnor K-group of K (see [M]).

The Milnor conjecture has been proven for all fields and for $n = 2$ by Merkurjev, for $n = 3$ independently by Rost and Merkurjev–Suslin; a proof for $n = 4$ has been announced by Rost. It is also known in all degrees for special classes of fields, *eg* Kato's higher local fields. Part (1) of theorem 1 applies in all these cases.

Part (2) of theorem 1 is proven in [K1, §4] by using results of Suslin on torsion in K_2 of fields to reduce to the number field case, and in [MT] by using quadratic form theory and function fields of quadrics. It was also proven by Markus Rost by using function fields of quadrics [R]. We give here a new, more elementary proof, using results of [K2]. In [R] and [MT], the exactness of S_3 and S^3 at certain spots is proven unconditionally.

This paper is organised as follows. In section 1 we recall from [MT] the definition of the maps in S_n and S^n, as well as a general result relating their homology groups, for the convenience of the reader. In section 2 we introduce some functors which 'lift' Galois cohomology modulo 2: this is the main tool we use here. In section 3 we extend S_n on the left and S^n on the right; truncating the parts involving ε_n and α^n, we get new 7-term complexes S'_n and S'^n and exhibit *quasi-isomorphisms* $S'_n \to S'^n$: this can be viewed as a "duality" statement, extending a remark at the end of [MT, §1]. In section 4 we construct a partial homomorphism of complexes $S'_n \to S'_{n-1}$ [3], which induces homomorphisms between their homology groups, and prove an 'excision' property for the latter, using section 2. In section 5 we prove some technical results in Galois cohomology, which are consequences of the Milnor conjecture. In section 6, we prove theorem 1. Finally in section 7 we make a few further observations.

I thank Jean-Pierre Tignol for suggesting part (1b) of theorem 1. It is likely that one can in fact derive the exactness of sequences S_n and S^n from the Milnor conjecture, or at least Hilbert theorem 90 for higher Milnor K-theory in quadratic extensions, without extra hypotheses like (a) and (b). Actually (a) and (b) are those extreme cases when cup-product by (-1) in cohomology modulo 2 is respectively 0 or injective (see prop. 4); it is quite conceivable that one does not need such hypotheses to deduce proposition 5 from the Milnor conjecture. Unfortunately I haven't been able to do this, except in degree 2.

I also thank the referee for pointing out some stupid mistakes in the first version of this paper, which prompted a rewriting that will hopefully clarify its strategy.

In particular, it is thanks to his remarks that I realised that my construction of isomorphisms $\mathcal{H}_n(0) \xrightarrow{\cong} \mathcal{H}^n(4)$ and $\mathcal{H}_n(4) \xrightarrow{\cong} \mathcal{H}^n(0)$ was connected with an approach in [MT].

Notations. Throughout this paper, with one exception in §3, notation of [MT] is used. If A is an abelian group and m a nonzero integer, we denote by ${}_m\mathrm{A}$ (resp. by A/m) the kernel (resp. the cokernel) of multiplication by m in A.

1. Review of notation of [MT]

We recall here the definition of the maps appearing in (S_n) and (S^n), as defined in [MT]:

$$\alpha_n(u, (v_i)_{1 \le i \le 3}) = (\mathrm{N}_{\mathrm{M}/\mathrm{L}_i} u + (v_i)_{\mathrm{L}_i})_{1 \le i \le 3}$$

$$\beta_n((\ell_i)_{1 \le i \le 3}) = \sum_{i=1}^{3} (a_i) \otimes \mathrm{N}_{\mathrm{L}_i/\mathrm{F}} \ell_i$$

$$\gamma_n\left(\sum_{i=1}^{3} (a_i) \otimes f_i\right) = \sum_{i=1}^{3} (a_i) \cdot f_i$$

$$\delta_n(u) = (\mathrm{N}_{\mathrm{M}/\mathrm{L}_i} u)_{1 \le i \le 3}$$

$$\varepsilon_n((\ell_i)_{1 \le i \le 3}) = \left(\sum_{i=1}^{3} (\bar{\ell}_i)_{\mathrm{M}}, \beta_n((\ell_i)_{1 \le i \le 3}), \sum_{i=1}^{3} (a_i) \otimes \mathrm{N}_{\mathrm{L}_i/\mathrm{F}}((\sqrt{a_i}) \cdot \ell_i)\right);$$

$$\alpha^n((\ell_i)_{1 \le i \le 3}) = \left(\sum_{i=1}^{3} (\ell_i)_{\mathrm{M}}, (\mathrm{N}_{\mathrm{L}_i/\mathrm{F}} \ell_i)_{1 \le i \le 3}\right)$$

$$\beta^n\left(\sum_{i=1}^{3} \sigma_i \otimes u_i\right) = \left(\sum_{i=1}^{3} (\langle (a_i), \sigma_j \rangle u_j)_{\mathrm{L}_i}\right)_{1 \le i \le 3}$$

$$\gamma^n(u) = \sum_{i=1}^{3} \sigma_i \otimes (a_i) \cdot u$$

$$\delta^n((\ell_i)_{1 \le i \le 3}) = \sum_{i=1}^{3} (\ell_i)_{\mathrm{M}}$$

$$\varepsilon^n\left(u, \sum_{i=1}^{3} \sigma_i \otimes v_i, t\right) = (\overline{\mathrm{N}_{\mathrm{M}/\mathrm{L}_i}(u) + (\sqrt{a_i}) \cdot t_{\mathrm{L}_i}})_{1 \le i \le 3} + \beta^{n-1}\left(\sum_{i=1}^{3} \sigma_i \otimes u_i\right).$$

In these definitions, $\overline{}$ denotes conjugation by Galois involutions, the $\sqrt{a_i}$ are chosen such that $\Sigma\,(\sqrt{a_i})_{\mathrm{M}} = 0$ and $\langle , \rangle$ is the natural pairing $\mathrm{X} \times \mathrm{G} \to \mathbf{Z}/2$.

Merkurjev and Tignol set

$$\mathscr{H}_n(1) = \operatorname{Ker} \beta_n / \operatorname{Im} \alpha_n \qquad \mathscr{H}^n(1) = \operatorname{Ker} \alpha^n / \operatorname{Im} \beta^n$$

$$\mathscr{H}_n(2) = \operatorname{Ker} \gamma_n / \operatorname{Im} \beta_n \qquad \mathscr{H}^n(2) = \operatorname{Ker} \beta^n / \operatorname{Im} \gamma^n$$

$$\mathscr{H}_n(3) = \operatorname{Ker} \operatorname{res} / \operatorname{Im} \gamma_n \qquad \mathscr{H}^n(3) = \operatorname{Ker} \gamma^n / \operatorname{Im} \operatorname{cor}$$

$$\mathscr{H}_n(4) = \operatorname{Ker} \delta_n / \operatorname{Im} \operatorname{res} \qquad \mathscr{H}^n(4) = \operatorname{Ker} \operatorname{cor} / \operatorname{Im} \delta^n$$

$$\mathscr{H}_n(5) = \operatorname{Ker} \varepsilon_n / \operatorname{Im} \delta_n \qquad \mathscr{H}^n(5) = \operatorname{Ker} \delta^n / \operatorname{Im} \varepsilon^n$$

and prove:

THEOREM 0 ([MT], th. A; end of section 1). *For all* $n \geq 0$, *there are natural isomorphisms* $\mathscr{H}_n(2) \cong \mathscr{H}^n(2) \cong \mathscr{H}_n(4) \cong \mathscr{H}^n(4)$ *and* $\mathscr{H}_n(1) \cong \mathscr{H}^n(1) \cong \mathscr{H}_n(3) \cong \mathscr{H}^n(3)$. *Moreover, if* $\mathscr{H}_n(1) = \mathscr{H}^n(1) = \mathscr{H}_n(3) = \mathscr{H}^n(3) = 0$, *then* $\mathscr{H}_n(5) = \mathscr{H}^n(5) = 0$.

Remark. We shall not be concerned here by the maps ε_n and α^n, nor by the homology groups $\mathscr{H}_n(5)$ and $\mathscr{H}^n(5)$. In fact, we get rid of this part of complexes S_n and S^n in section 3.

2. An auxiliary functor

Fix an integer $n \geq 1$. We give ourselves, for any $i \leq n$, a functor $\tilde{\mathrm{H}}^i$: fields $\to$ abelian groups such that:

(1) $\tilde{\mathrm{H}}^i$ is a Mackey functor, *ie* is provided with a transfer for finite extensions having the usual properties (double coset formula);
(2) There is a natural transformation of Mackey functors $\rho : \tilde{\mathrm{H}}^i \to \mathrm{H}^i$ inducing an isomorphism $\tilde{\mathrm{H}}^i(\mathrm{K})/2 \xrightarrow{\cong} \mathrm{H}^i\mathrm{K}$ for any field K.

A trivial choice is $\tilde{\mathrm{H}}^i = \mathrm{H}^i$. Assuming Milnor's conjecture in degrees $\leq n$ for all fields considered, one may take for $\tilde{\mathrm{H}}^i$ one of the following functors:

(i) K_i^M;
(ii) $H^i(—, \mu_4^{\otimes i})$;
(iii) $M_i(—) = \mathbb{H}^0(—, G_m^{\overset{L}{\otimes} i})$ [K2] (denoted by $\check{M}_i(—)$ in [K2]).

In fact, we shall use here (ii) and (iii) rather than (i).

3. Extending S_n to the left (and S^n to the right)

(1) Define $\eta_n : \bigoplus_{i=1}^3 H^nL_i \oplus (H^{n-1}F)^3 \to H^nM \oplus (H^nF)^3$ by:

$$\eta_n((\ell_i)_{1 \le i \le 3}, (f_i)_{1 \le i \le 3}) = (\sum (\ell_i)_M, (a_1) \cdot f_1 + N_{L_2/F}\ell_2 + N_{L_3/F}\ell_3,$$

$$(a_2) \cdot f_2 + N_{L_1/F}\ell_1 + N_{L_3/F}\ell_3, (a_3) \cdot f_3 + N_{L_1/F}\ell_1 + N_{L_2/F}\ell_2).$$

Then $\alpha_n\eta_n = 0$. Set $\mathscr{H}_n(0) = \operatorname{Ker} \alpha_n / \operatorname{Im} \eta_n$. Adding this map to S_n and truncating it on the right, we get a new complex:

$$\bigoplus_{i=1}^{3} H^nL_i \oplus (H^{n-1}F)^3 \xrightarrow{\eta_n} H^nM \oplus (H^nF)^3 \xrightarrow{\alpha_n}$$

$$\bigoplus_{i=1}^{3} H^nL_i \xrightarrow{\beta_n} X \otimes H^nF \xrightarrow{\gamma_n} H^{n+1}F \xrightarrow{\text{res}} H^{n+1}M \xrightarrow{\delta_n} \bigoplus_{i=1}^{3} H^{n+1}L_i \qquad (S'_n)$$

with homology groups $\mathscr{H}_n(0), \mathscr{H}_n(1), \mathscr{H}_n(2), \mathscr{H}_n(3), \mathscr{H}_n(4)$.

(2) "Dually", define $\eta^n : H^{n+1}M \oplus (H^{n+1}F)^3 \to \bigoplus_{i=1}^3 H^{n+1}L_i \oplus (H^{n+2}F)^3$ by:

$$\eta^n(u, (v_i)) = (N_{M/L_1}u + (v_2)_{L_1} + (v_3)_{L_1}, N_{M/L_2}u + (v_1)_{L_2} + (v_3)_{L_2},$$

$$N_{M/L_3}u + (v_1)_{L_3} + (v_2)_{L_3}, ((a_i) \cdot v_i)_{1 \le i \le 3}).$$

Then $\alpha^n\eta^n = 0$. Set $\mathscr{H}^n(0) = \operatorname{Ker} \eta^n / \operatorname{Im} \alpha^n$. Adding this map to S^n and truncating it on the left, we get a new complex:

$$\bigoplus_{i=1}^{3} H^{n+1}L_i \oplus (H^{n+2}F)^3 \xleftarrow{\eta^n} H^{n+1}M \oplus (H^{n+1}F)^3 \xleftarrow{\alpha^n}$$

$$\bigoplus_{i=1}^{3} H^{n+1}L_i \xleftarrow{\beta^n} G \otimes H^{n+1}F \xleftarrow{\gamma^n} H^nF \xleftarrow{\text{cor}} H^nM \xleftarrow{\delta^n} \bigoplus_{i=1}^{3} H^nL_i, \qquad (S'^n)$$

with homology groups $\mathscr{H}^n(4), \mathscr{H}^n(3), \mathscr{H}^n(2), \mathscr{H}^n(1), \mathscr{H}^n(0)$.

We now define a homomorphism of complexes $S'_n \to S'^n$ as follows:

$$\begin{array}{ccccccccccccc}
\bigoplus_{i=1}^{3} H^nL_i \oplus (H^{n-1}F)^3 & \to & H^nM \oplus (H^nF)^3 & \to & \bigoplus_{i=1}^{3} H^nL_i & \to & X \otimes H^nF & \to & H^{n+1}F & \to & H^{n+1}M & \to & \bigoplus_{i=1}^{3} H^{n+1}L_i \\
\downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \Delta_n} & & \downarrow{\scriptstyle \nu_n} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} \\
\bigoplus_{i=1}^{3} H^nL_i & \to & H^nM & \to & H^nF & \to & G \otimes H^{n+1}F & \to & \bigoplus_{i=1}^{3} H^{n+1}L_i & \to & H^{n+1}M \oplus (H^{n+1}F)^3 & \to & \bigoplus_{i=1}^{3} H^{n+1}L_i \oplus (H^{n+2}F)^3
\end{array}$$

where pr_1 denotes the projection on the first factor, i_1 the inclusion of the first factor and μ_n, Δ_n and ν_n are defined as follows:

$$\mu_n((\ell_i)_{1 \le i \le 3}) = \sum \ell_i$$

$$\Delta_n((a_1) \otimes f) = \sigma_2 \otimes (a_2) \cdot f + \sigma_3 \otimes (a_3) \cdot f, \text{ etc.}$$

$$\nu_n(f) = (f_{L_i})_{1 \le i \le 3}$$

Note that we just extended the morphism of complexes appearing at the end of [MT, §1]. It is a tedious but easy exercise to check that these maps indeed define a morphism of complexes (*ie*, that the above diagram commutes).

PROPOSITION 1. *The homomorphism* $S'_n \to S'^n$ *defined above is a quasi-isomorphism, defining natural isomorphisms* $\mathscr{H}_n(i) \xrightarrow{\cong} \mathscr{H}^n(4-i)$, $0 \le i \le 4$.

Proof. Again this is a tedious but straightforward exercise, involving the exact sequences attached to the various quadratic extensions involved in M/F. We only give the most difficult (since it does not appear in [MT]):

Proof at Δ_n: Let $x = \sigma_1 \otimes f_1 + \sigma_2 \otimes f_2 \in G \otimes H^{n+1}F$ map to 0 by β^n. Then $(f_1)_{L_2} = (f_2)_{L_1} = (f_1 + f_2)_{L_3} = 0$, hence $f_2 = (a_1) \cdot g_1, f_1 = (a_2) \cdot g_2, f_1 + f_2 = (a_3) \cdot g_3$. Check that $x = \gamma^n(\Sigma g_i) + \Delta_n(\Sigma (a_i) \otimes g_i)$ and conclude that the map $\mathscr{H}_n(2) \to \mathscr{H}^n(2)$ induced by Δ_n is surjective. Similarly, let $x = (a_1) \otimes f_1 + (a_2) \otimes f_2 \in X \otimes H^nF$ be such that $\gamma_n(x) = (a_1) \cdot f_1 + (a_2) \cdot f_2 = 0$ and $\Delta_n(x) = \gamma^n(u)$ for some $u \in H^nF$. Then $\Delta_n(x) = \sigma_1 \otimes (a_2) \cdot f_1 + \sigma_2 \otimes (a_1) \cdot f_2$, hence $(a_1) \cdot (u + f_2) = (a_2) \cdot (u + f_1) = 0$, hence $f_1 = u + N\ell_2$ and $f_2 = u + N\ell_1$ for $\ell_i \in H^nL_i$ $(i = 1, 2)$ and $x = (a_1) \otimes N\ell_1 + (a_2) \otimes N\ell_2 + (a_3) \otimes u$. But then $(a_3) \cdot u = 0$, $u = N\ell_3$ and $x = \beta_n(N\ell_1, N\ell_2, N\ell_3)$, showing the injectivity of $\mathscr{H}_n(2) \to \mathscr{H}^n(2)$.

4. Two maps $\phi : \mathscr{H}_n(0) \to \mathscr{H}_{n-1}(3)$ and $\theta : \mathscr{H}_n(1) \to \mathscr{H}_{n-1}(4)$

There is a commutative diagram:

$$\begin{array}{ccccccc}
(\mathrm{H}^{n-1}\mathrm{F})^3 \oplus \bigoplus_{i=1}^{3} \mathrm{H}^n\mathrm{L}_i & \xrightarrow{\eta_n} & \mathrm{H}^n\mathrm{M} \oplus (\mathrm{H}^n\mathrm{F})^3 & \xrightarrow{\alpha_n} & \bigoplus_{i=1}^{3} \mathrm{H}^n\mathrm{L}_i & \xrightarrow{\beta_n} & \mathrm{X} \otimes \mathrm{H}^n\mathrm{F} \\
\zeta_n \downarrow & & \phi_n \downarrow & & \theta_n \downarrow & & \lambda_n \downarrow \\
\mathrm{X} \otimes \mathrm{H}^{n-1}\mathrm{F} & \xrightarrow{\gamma_{n-1}} & \mathrm{H}^n\mathrm{F} & \xrightarrow{\mathrm{res}} & \mathrm{H}^n\mathrm{M} & \xrightarrow{\delta_{n-1}} & \bigoplus_{i=1}^{3} \mathrm{H}^n\mathrm{L}_i
\end{array}$$

where

$$\zeta_n((f_i)_{1 \le i \le 3}, (\ell_i)_{1 \le i \le 3}) = \sum (a_i) \otimes f_i, \qquad \phi_n((u, (v_i)) = \mathrm{N}_{\mathrm{M/F}} u + \sum v_i,$$

$$\theta_n((\ell_i)_{1 \le i \le 3}) = \sum_{i=1}^{3} (\ell_i)_{\mathrm{M}} \quad \text{and}$$

$$\lambda_n\left(\sum (a_i) \otimes f_i\right) = ((f_2 + f_3)_{\mathrm{L}_1}, (f_1 + f_3)_{\mathrm{L}_2}, (f_1 + f_2)_{\mathrm{L}_3}).$$

It induces homomorphisms:

$$\phi_n : \mathscr{H}_n(0) \longrightarrow \mathscr{H}_{n-1}(3);$$

$$\theta_n : \mathscr{H}_n(1) \longrightarrow \mathscr{H}_{n-1}(4).$$

For simplicity, we write ϕ and θ instead of ϕ_n and θ_n.

Remark. There is presumably a similar commutative diagram involving S^n and S^{n-1}, but we don't care to write it down.

Denote by $\tilde{\mathrm{Z}}_n(0)$ the kernel of $\tilde{\alpha}_n : \tilde{\mathrm{H}}^n\mathrm{M} \oplus (\tilde{\mathrm{H}}^n\mathrm{F})^3 \to \bigoplus_{i=1}^3 \tilde{\mathrm{H}}^n\mathrm{L}_i$, where $\tilde{\alpha}_n$ is defined analogously to α_n (see [MT]), and similarly by $\tilde{\mathrm{Z}}_n(1)$ the group $\{(\tilde{\ell}_i)_{i \le i \le 3} \in \bigoplus \tilde{\mathrm{H}}^n(\mathrm{L}_i) \mid \mathrm{N}_{\mathrm{L}_1/\mathrm{F}}\tilde{\ell}_1 = \mathrm{N}_{\mathrm{L}_2/\mathrm{F}}\tilde{\ell}_2 = \mathrm{N}_{\mathrm{L}_3/\mathrm{F}}\tilde{\ell}_3\}$ (apparently there is no slick way to describe the latter group as the kernel of a map like β_n). The map ρ of section 2 induces homomorphisms $\rho : \tilde{\mathrm{Z}}_n(0) \to \mathscr{H}_n(0)$ and $\rho : \tilde{\mathrm{Z}}_n(1) \to \mathscr{H}_n(1)$.

PROPOSITION 2. (a) *In the sequence* ${}_2\tilde{\mathrm{Z}}_n(0) \xrightarrow{\rho} \mathscr{H}_n(0) \xrightarrow{\phi} \mathscr{H}_{n-1}(3)$, *one has* $\operatorname{Ker}(\phi) \subseteq \operatorname{Im}(\rho)$.

(b) *In the sequence* ${}_2\tilde{\mathrm{Z}}_n(1) \xrightarrow{\rho} \mathscr{H}_n(1) \xrightarrow{\theta} \mathscr{H}_{n-1}(4)$, *one has* $\operatorname{Ker}(\theta) \subseteq \operatorname{Im}(\rho)$.

Proof of (a). Let $(u, (v_i)) \in \operatorname{Ker} \phi$. Lift u and the v_i into $\tilde{u} \in \tilde{\mathrm{H}}^n\mathrm{M}$, $\tilde{v}_i \in \tilde{\mathrm{H}}^n\mathrm{F}$. By assumption:

$$\sum \tilde{v}_i = 2\tilde{v} \tag{1}$$

$$\mathrm{N}_{\mathrm{M/L}_i}\tilde{u} = (\tilde{v}_i)_{\mathrm{L}_i} + 2'\tilde{x}_i \qquad (1 \le i \le 3) \tag{2}$$

for suitable $\tilde{v}$, $\tilde{x}_i$. Here, contrary to [MT], we denote by $x \mapsto {}'x$ the conjugation by the generator of $\mathrm{Gal}(L_i/F)$ acting on $T(L_i)$ for any functor T defined over fields (same notation for $i = 1, 2, 3$). This is just a trick to get better formulas at the end.

Up to modifying $\tilde{v}_1$ into $\tilde{v}_1 - 2\tilde{v}$, we may even assume, and we do assume, that $\tilde{v} = 0$ in (1). Further, up to replacing $\tilde{H}^n$ by $\mathbf{Z}[\frac{1}{3}] \otimes_{\mathbf{Z}} \tilde{H}^n$, we may, and do, assume that multiplication by 3 is invertible in $\tilde{H}^n$. Then, taking norms of (2), we get:

$$\begin{aligned} N_{M/F}\tilde{u} &= 2(\tilde{v}_1 + N_{L_1/F}\tilde{x}_1) = 2(\tilde{v}_2 + N_{L_2/F}\tilde{x}_2) = 2(\tilde{v}_3 + N_{L_3/F}\tilde{x}_3) \\ &= \tfrac{2}{3}(N_{L_1/F}\tilde{x}_1 + N_{L_2/F}\tilde{x}_2 + N_{L_3/F}\tilde{x}_3), \end{aligned} \tag{3}$$

hence

$$\tilde{v}_1 = \tfrac{1}{3}(N_{L_2/F}\tilde{x}_2 + N_{L_3/F}\tilde{x}_3) - \tfrac{2}{3}N_{L_1/F}\tilde{x}_1 + \tilde{v}'_1,$$

$$\tilde{v}_2 = \tfrac{1}{3}(N_{L_1/F}\tilde{x}_1 + N_{L_3/F}\tilde{x}_3) - \tfrac{2}{3}N_{L_2/F}\tilde{x}_2 + \tilde{v}'_2,$$

$$\tilde{v}_3 = \tfrac{1}{3}(N_{L_1/F}\tilde{x}_1 + N_{L_2/F}\tilde{x}_2) - \tfrac{2}{3}N_{L_3/F}\tilde{x}_3 + \tilde{v}'_3,$$

with $2\tilde{v}'_1 = 2\tilde{v}'_2 = 2v'_3 = 0$.

Next, extend scalars on (2). We get:

$$(1 + \sigma_i)\tilde{u} = (\tilde{v}_i)_M + 2({}'\tilde{x}_i)_M \qquad (1 - i \le 3).$$

Adding up gives:

$$2\tilde{u} + (N_{M/F}\tilde{u})_M = 2\sum ({}'\tilde{x}_i)_M.$$

Using (3) again (to substitute $N_{M/F}\tilde{u}$), we get:

$$\begin{aligned} 2\tilde{u} &= 2\sum ({}'\tilde{x}_i)_M - \tfrac{2}{3}[(\tilde{x}_1 + {}'\tilde{x}_1)_M + (\tilde{x}_2 + {}'\tilde{x}_2)_M + (\tilde{x}_3 + {}'\tilde{x}_3)_M] \\ &= -\tfrac{2}{3}\sum (\tilde{x}_i)_M + \tfrac{4}{3}\sum ({}'\tilde{x}_i)_M, \end{aligned}$$

hence

$$\tilde{u} = -\tfrac{1}{3}\sum (\tilde{x}_i)_M + \tfrac{2}{3}\sum ({}'\tilde{x}_i)_M + \tilde{u}', \ 2\tilde{u}' = 0.$$

Note that

$$(\tilde v'_1)_{L_1} - N_{M/L_1}\tilde u' = (\tilde v_1)_{L_1} - \tfrac{1}{3}(N_{L_2/F}\tilde x_2 + N_{L_3/F}\tilde x_3)_{L_1} + \tfrac{2}{3}(\tilde x_1 + {}'\tilde x_1)$$
$$- \left\{N_{M/L_1}\tilde u + \tfrac{1}{3}N_{M/L_1}\left[\sum (\tilde x_i)_M\right] - \tfrac{2}{3}N_{M/L_1}\left[\sum ({}'\tilde x_i)_M\right]\right\}$$
$$= (\tilde v_1)_{L_1} - \tfrac{1}{3}(N_{L_2/F}\tilde x_2 + N_{L_3/F}\tilde x_3)_{L_1} + \tfrac{2}{3}(\tilde x_1 + {}'\tilde x_1)$$
$$- \{N_{M/L_1}\tilde u + \tfrac{1}{3}[2\tilde x_1 + (N_{L_2/F}\tilde x_2 + N_{L_3/F}\tilde x_3)_{L_1}]$$
$$- \tfrac{2}{3}[2'\tilde x_1 + (N_{L_2/F}\tilde x_2 + N_{L_3/F}\tilde x_3)_{L_1}]\}$$
$$= (\tilde v_1)_{L_1} - N_{M/L_1}\tilde u + 2'\tilde x_1 = 0 \qquad \text{by (4)},$$

and similarly $(\tilde v'_2)_{L_2} = N_{M/L_2}\tilde u'$, $(\tilde v'_3)_{L_3} = N_{M/L_3}\tilde u'$, hence $(\tilde u', (\tilde v'_i)) \in {}_2\tilde{Z}_n(0)$. This proves (a).

Proof of (b). Let $(\ell_i)_{1\le i\le 3} \in \operatorname{Ker}\theta$. Choose $\tilde\ell_i \in \tilde H^n(L_i)$ lifting ℓ_i. By assumption:

$$\sum_{i=1}^{3} (\tilde\ell_i)_M = 2\tilde u \tag{4}$$

$$N_{L_i/F}\tilde\ell_i = \tilde\ell + 2\tilde x_i \qquad (1 \le i \le 3) \tag{5}$$

for some $\tilde u, \tilde\ell, \tilde x_i$.

Take N_{M/L_1} of (4) and get (using (5)):

$$2\tilde\ell_1 + (2\tilde\ell + 2\tilde x_2 + 2\tilde x_3)_{L_1} = 2N_{M/L_1}\tilde u,$$

ie

$$\tilde\ell_1 + (\tilde\ell + \tilde x_2 + \tilde x_3)_{L_1} = N_{M/L_1}\tilde u + \tilde\ell'_1, \qquad \text{with } 2\tilde\ell'_1 = 0.$$

Similarly,

$$\tilde\ell_2 + (\tilde\ell + \tilde x_1 + \tilde x_3)_{L_2} = N_{M/L_2}\tilde u + \tilde\ell'_2, \qquad \text{with } 2\tilde\ell'_2 = 0;$$

$$\tilde\ell_3 + (\tilde\ell + \tilde x_1 + \tilde x_2)_{L_3} = N_{M/L_3}\tilde u + \tilde\ell'_3, \qquad \text{with } 2\tilde\ell'_3 = 0.$$

Note that $N_{L_1/F}\tilde\ell'_1 = N_{L_2/F}\tilde\ell'_2 = N_{L_3/F}\tilde\ell'_3 = 4\tilde\ell + 2(\tilde x_1 + \tilde x_2 + \tilde x_3) - N_{M/F}\tilde u$, so that $(\tilde\ell'_1, \tilde\ell'_2, \tilde\ell'_3) \in {}_2\tilde{Z}_n(1)$. This completes the proof of prop. 2.

5. Cohomological properties and permanence properties

For any field K, denote by ε the class of -1 in H^1K via Kummer theory.

PROPOSITION 3. *Let the Milnor conjecture hold for* K *in degrees* $\leq n$. *Then, for all* $i \leq n$, *there is an exact sequence*:

$$0 \longrightarrow H^{i-1}K \xrightarrow{\sigma} H^{i-1}(K, \mu_4^{\otimes i}) \xrightarrow{\rho} H^{i-1}K$$

$$\xrightarrow{\cdot\varepsilon} H^iK \xrightarrow{\sigma} H^i(K, \mu_4^{\otimes i}) \xrightarrow{\rho} H^iK \longrightarrow 0.$$

Moreover, $\sigma\rho$ *is multiplication by* 2 *on* $H^{i-1}(K, \mu_4^{\otimes i})$ *and* $H^i(K, \mu_4^{\otimes i})$.

Proof. Consider the long cohomology exact sequence associated to the short exact sequence of coefficients $0 \to \mathbf{Z}/2 \to \mu_4^{\otimes i} \to \mathbf{Z}/2 \to 0$:

$$\cdots \xrightarrow{\partial^{j-2}} H^{j-1}K \xrightarrow{\sigma^{j-1}} H^{j-1}(K, \mu_4^{\otimes i}) \xrightarrow{\rho^{j-1}} H^{j-1}K$$

$$\xrightarrow{\partial^{j-1}} H^jK \xrightarrow{\sigma^j} H^j(K, \mu_4^{\otimes i}) \xrightarrow{\rho^j} H^jK \xrightarrow{\partial^j} \cdots$$

Note that for i even $\mu_4^{\otimes i} \cong \mathbf{Z}/4$, and for i odd $\mu_4^{\otimes i} \cong \mu_4$ as Galois modules. Therefore, the Milnor conjecture implies that σ^j (resp. ρ^j, ∂^j) is injective (resp. surjective, 0) when i and j have different (resp. the same) parity. For $j = i$, this gives an exact sequence as in the statement of prop. 3; the identification of ∂^{i-1} to cup-product by ε follows for example from [K3, lemma 1]. Finally, the claim about $\sigma\rho$ is trivial.

COROLLARY. *With the notations and hypotheses of prop.* 3, *one has*

$$\rho({}_2H^i(K, \mu_4^{\otimes i})) = \varepsilon \cdot H^{i-1}K \subseteq H^iK. \quad \square$$

PROPOSITION 4. (a) *In case* (1a) *of theorem* 1, *cup-product by* $\varepsilon : H^iF \to H^{i+1}F$ *is identically* 0 *for* $i \leq n$. *Moreover, the same holds for any* 2*-extension of* F (*in particular* M *and the* L_i).

(b) *In case* (1b) *of theorem* 1, *cup-product by* $\varepsilon : H^iM \to H^{i+1}M$ *is injective for* $i \leq n$. *Moreover, the same holds for* F *and the* L_i.

Recall that an extension K′/K is called a 2*-extension* if it is contained in a Galois extension of K whose Galois group is a (pro-) 2-group.

Proof. In case (a), the claim is obvious if ε itself is 0. Otherwise, let $F = F(\sqrt{-1})$. Then the hypothesis on F means that the norm $N_{F/F}: F'^* \to F^*$ is surjective. This in turn is equivalent to $\varepsilon \cdot a = 0$ for all $a \in H^1F$, *eg* by the exact sequence associated to the quadratic extension F′/F [MT, (1)]. The Milnor conjecture implies that H^*F is multiplicatively generated by H^1F up to degree n, hence cup-product by $\varepsilon : H^iF \to H^{i+1}F$ is identically 0 for all $i \leq n$.

To see that the same holds for any 2-extension of F, it is enough to deal with a quadratic extension E of F, and to see that E satisfies condition (a) of theorem 1. But this is a consequence of the vanishing of $\mathscr{H}'_1(1)$ for the biquadratic extension $E(\sqrt{-1})/F$ (see lemma 2 below).

In case (b), let $M' = M(\sqrt{-1})$. By the long exact sequence associated to the quadratic extension M′/M, injectivity of the cup-product by ε on H^iM is equivalent to the vanishing of $N_{M'/M}: H^iM' \to H^iM$. For $i = 1$, this is hypothesis (b) of theorem 1. For $i \geq 2$, H^iM' is by assumption additively generated by cup-products of elements of H^1M', and even by cup-products of the form $u_{M'} \cdot x$, where $u \in H^{i-1}M$ and $x \in H^1M'$ [BT, cor. 5.3]. For such an element,

$$N_{M'/M}(u_{M'} \cdot x) = u \cdot N_{M'/M}x = 0.$$

Finally, if M is pythagorean so is any subfield E of M such that $[M : E] < +\infty$ [L, th. 5.14]; in particular, F and the L_i are pythagorean, hence the last claim of prop. 4.

6. Proof of theorem 1

To prove theorem 1, the key step is:

PROPOSITION 5. *Under the hypotheses of theorem* 1, *there are inclusions*:

$$\operatorname{Ker}(\theta) \subseteq \varepsilon \cdot \mathscr{H}_{n-1}(1)$$

$$\operatorname{Ker}(\phi) \subseteq \varepsilon \cdot \mathscr{H}_{n-1}(0)$$

where, as above, ε *denotes the class of* -1 *via Kummer theory and the maps* θ *and* ϕ *are defined in section* 4.

In view of th. 0 and prop. 1, Proposition 5 obviously implies th. 1 by induction on n. (In case 2 of theorem 1, we use Tignol's theorem that S_1 and S^1 are exact [T], [MT].)

Proof of proposition 5 *in case* 1. We choose $\tilde{H}^i = H^i(—, \mu_4^{\otimes i})$. By prop. 4 and the corollary to prop. 3, in case (1a), $\mathrm{Im}\,({}_2\tilde{H}^n \to H^n) = 0$, hence $\rho({}_2\tilde{Z}_n(1)) = 0$ and $\rho({}_2\tilde{Z}_n(0)) = 0$ in prop. 2. Therefore prop. 5 holds trivially. In case (1b), for $x \in \bigoplus_{i=1}^3 \tilde{H}^{n-1}L_i$, we have with obvious notations:

$$\varepsilon \cdot \beta_{n-1}(x) = \beta_n(\varepsilon \cdot x).$$

Therefore, prop. 4 implies that $(\mathrm{Ker}\,\beta_n) \cap (\varepsilon \cdot \bigoplus_{i=1}^3 \tilde{H}^{i-1}L_i) = \varepsilon \cdot \mathrm{Ker}\,\beta_{n-1}$, and prop. 2 and the corollary to prop. 3 yield the first exact sequence of prop. 5. Similarly, the injectivity of cup-product by ε yields the equality:

$$(\mathrm{Ker}\,\alpha_n) \cap [\varepsilon \cdot (H^{n-1}M \oplus (H^{n-1}F)^3)] = \varepsilon \cdot \mathrm{Ker}\,\alpha_{n-1},$$

which in turn yields the second exact sequence of prop. 5 *via* prop. 2 and the corollary to prop. 3. This concludes the proof of prop. 5 in case 1.

Remark. In particular, the groups $\mathscr{H}_3(r)$ and $\mathscr{H}^3(r)$ vanish under condition (a) or (b) of th. 1, thanks to the theorem of Rost and Merkurjev–Suslin mentioned in the introduction. In [MT], the vanishing of these groups for r odd is proven unconditionally; the same was proven by Rost [R]. The latter proofs use deep results of quadratic form theory and K-cohomology of function fields of quadrics. The proof given here is more elementary; in some sense it is 'algebraic' rather than 'transcendental'.

Proof of proposition 5 *in case* 2. Here we choose $\tilde{H}^i = M_i(—)$ [K2]. The following lemma does not appear in [K2]:

LEMMA 1. *Assume that the Milnor conjecture holds for* K *in degrees* $n-1$ *and* n. *Then there is an exact sequence*:

$$0 \to H^{n-2}(K, \mathbf{Q}/\mathbf{Z}(n))/2 \to H^{n-1}K \xrightarrow{\cdot\{-1\}} M_n(K) \xrightarrow{2} M_n(K) \xrightarrow{\rho} H^nK \to 0.$$

Proof. As in [K2], denote by $\check{Z}(n)$ the Galois complex $G_m^{\overset{L}{\otimes} n}[-n]$ (defined as an object of the derived category of Galois modules). The triangle $\check{Z}(n) \xrightarrow{2} \check{Z}(n) \to Z/2 \to \check{Z}(n)[1]$ yields a long exact sequence in hypercohomology, of which a part is the following:

$$0 \longrightarrow H^{n-1}(K, \check{Z}(n))/2 \longrightarrow \mathbb{H}^{n-1}K \xrightarrow{\partial} M_n(K) \xrightarrow{2} M_n(K) \xrightarrow{\rho_n} H^nK$$

The tautological products $\check{\mathbf{Z}}(i) \overset{\mathrm{L}}{\otimes} \check{\mathbf{Z}}(j) \to \check{\mathbf{Z}}(i+j)$ provide the theory $\mathrm{M}_*(\mathrm{K})$ with a graded product which commutes with the cup-product in mod 2 cohomology *via* the morphism ρ^* above; therefore, since the Milnor conjecture holds for K in degrees $n-1$ and n, ρ^{n-1} and ρ^n are surjective (observe that ρ^1 is the Kummer theory map). On the other hand, by [K2, prop. 2.2], there is a canonical isomorphism $\mathrm{H}^{n-2}(\mathrm{K}, \mathbf{Q}/\mathbf{Z}(n)) \xrightarrow{\cong} \mathrm{H}^{n-1}(\mathrm{K}, \check{\mathbf{Z}}(n))$. This proves lemma 1, except for the identification of ∂. Note that, by assumption, $\mathrm{H}^{n-1}\mathrm{K}$ identifies to $\mathrm{M}_{n-1}(\mathrm{K})/2$ and product by $\{-1\} \in \mathrm{K}^* = \mathrm{M}_1(\mathrm{K})$ from $\mathrm{M}_{n-1}(\mathrm{K})$ to $\mathrm{M}_n(\mathrm{K})$ factors through $\mathrm{M}_{n-1}(\mathrm{K})/2 = \mathrm{H}^{n-1}\mathrm{K}$; this gives sense to the claim "$\partial =$ product by $\{-1\}$". To prove this claim, we argue in 3 steps:

(1) *The claim is true for* $n = 1$. This is trivial.

(2) *Let* $x \in \mathrm{M}_n(\mathrm{K})$ *and* $y \in \mathrm{H}^0\mathrm{K}$. *Then* $\partial(y \cdot \rho^n(x)) = (\partial y) \cdot x$ (compare [CE, prop. XI.2.5]).

(3) Apply (2) with $y = 1$, taking account of (1) and using the surjectivity of ρ^n.

COROLLARY. *One has*

$$\rho({}_2\tilde{\mathrm{H}}^n(\mathrm{K})) = \varepsilon \cdot \mathrm{H}^{n-1}\mathrm{K} \subseteq \mathrm{H}^n\mathrm{K}$$

also for this choice of $\tilde{\mathrm{H}}^n$.

Assume now that $n = 2$. Then $\mathrm{H}^0(\mathrm{K}, \mathbf{Q}/\mathbf{Z}(2))/2 = \mathrm{Ker}\,(\mathrm{H}^1\mathrm{K} \xrightarrow{\cdot\{-1\}} {}_2\mathrm{M}_2(\mathrm{K}))$ has order ≤ 2. (In fact, if it is nontrivial it is generated by the class of a 2-primary root of unity ζ or $\zeta + \zeta^{-1}$, but we won't need this.)

We first prove that $\mathrm{Ker}\,(\theta) \subseteq \varepsilon \cdot \mathscr{H}_1(1)$. We need a general lemma.

LEMMA 2. *Let* $\mathscr{H}'_n(1)$ *be the homology of the complex*

$$\mathrm{H}^n\mathrm{M} \oplus (\mathrm{H}^n\mathrm{F})^2 \xrightarrow{\alpha'_n} \mathrm{H}^n\mathrm{L}_1 \oplus \mathrm{H}^n\mathrm{L}_2 \xrightarrow{\beta'_n} \mathrm{H}^n\mathrm{F}$$

where α'_n *and* β'_n *are respectively defined by*

$$\alpha'_n(u, v_1, v_2) = (\mathrm{N}_{\mathrm{M}/\mathrm{L}_1}u + (v_1)_{\mathrm{L}_1}, \mathrm{N}_{\mathrm{M}/\mathrm{L}_2}u + (v_2)_{\mathrm{L}_2}),$$

$$\beta'_n(\ell_1, \ell_2) = \mathrm{N}_{\mathrm{L}_1/\mathrm{F}}\ell_1 + \mathrm{N}_{\mathrm{L}_2/\mathrm{F}}\ell_2.$$

Then there is a natural isomorphism $\mathscr{H}_n(1) \xrightarrow{\cong} \mathscr{H}'_n(1)$.

Proof. It is trivial to check that the following diagram commutes:

$$\begin{array}{ccccc} H^nM\oplus(H^nF)^3 & \xrightarrow{\alpha_n} & \bigoplus_{i=1}^{3} H^nL_i & \xrightarrow{\beta_n} & X\otimes H^nF \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w''} \\ H^nM\oplus(H^nF)^2 & \xrightarrow{\alpha'_n} & H^nL_1\oplus H^nL_2 & \xrightarrow{\beta'_n} & H^nF \end{array}$$

where $w(u, v_1, v_2, v_3) = (u, v_1, v_2)$, $w'(\ell_1, \ell_2, \ell_3) = (\ell_1, \ell_2)$ and $w''((a_1)\otimes f_1 + (a_2)\otimes f_2) = f_1 + f_2$. Hence w' induces a map $\mathscr{H}_n(1)\to\mathscr{H}'_n(1)$. By the exact sequence associated to the quadratic extension L_3/F, $(\operatorname{Ker} w')\cap(\operatorname{Ker}\beta_n)\subseteq \operatorname{Im}\alpha_n$. Since w is surjective, it follows that $\mathscr{H}_n(1)\to\mathscr{H}'_n(1)$ is injective. Finally, we prove that $\operatorname{Ker}\beta'_n = w'(\operatorname{Ker}\beta_n)$, hence that $\mathscr{H}_n(1)\to\mathscr{H}'_n(1)$ is surjective. Let $(\ell_1, \ell_2)\in\operatorname{Ker}\beta'_n$. We have $N_{L_1/F}\ell_1 = N_{L_2/F}\ell_2 = x$ (say). Then $(a_1)\cdot x = (a_2)\cdot x = 0$. Therefore, $(a_1a_2)\cdot x = 0$. But then there exists $\ell_3\in H^nL_3$ such that $N_{L_3/F}\ell_3 = x$, and $(\ell_1, \ell_2, \ell_3)\in\operatorname{Ker}\beta_n$ is such that $w'(\ell_1, \ell_2, \ell_3) = (\ell_1, \ell_2)$. □

Let $x\in\operatorname{Ker}(\mathscr{H}_2(1)\xrightarrow{\theta}\mathscr{H}_1(4))$. By prop. 2, pick $(\tilde\ell_1, \tilde\ell_2, \tilde\ell_3)\in {}_2\tilde Z_2(1)$ mapping to x. By lemma 1 and its corollary, write $\tilde\ell_i = \{-1, \ell'_i\}$, for some $\ell'_i\in H^1L_i$. By definition of $\tilde Z_2(1)$, we have:

$$\{-1, N_{L_1/F}\ell'_1\} = \{-1, N_{L_2/F}\ell'_2\} = \{-1, N_{L_3/F}\ell'_3\} \quad \text{in } M_2(F).$$

Since $\operatorname{Ker}(H^1F\xrightarrow{\cdot\{-1\}} {}_2M_2(F))$ has order ≤ 2, two among the three elements $N_{L_1/F}\ell'_1, N_{L_2/F}\ell'_2, N_{L_3/F}\ell'_3$ must be equal. Without loss of generality, we may assume that those are $N_{L_1/F}\ell'_1$ and $N_{L_2/F}\ell'_2$. Then $(\ell'_1, \ell'_2)\in\operatorname{Ker}\beta'_2$. Using lemma 2, this proves that after adding to x an element of $\varepsilon\cdot\mathscr{H}_2(1)$, we may assume $\tilde\ell_1 = \tilde\ell_2 = 0$. But then $x = 0$, by the exact sequence associated to the quadratic extension L_3/F.

Finally, we prove that $\operatorname{Ker}(\phi)\subseteq\varepsilon\cdot\mathscr{H}_1(0)$. Let $x\in\operatorname{Ker}(\mathscr{H}_2(0)\xrightarrow{\phi}\mathscr{H}_2(3))$. By prop. 2, pick $(\tilde u, \tilde v_1, \tilde v_2, \tilde v_3)\in {}_2\tilde Z_2(0)$ mapping to x. By lemma 1 and its corollary, write $\tilde u = \{-1, u'\}, \tilde v_i = \{-1, v'_i\}$ for some $u'\in H^1M, v'_i\in H^1F$. By definition of $\tilde Z_2(0)$, we have:

$$\{-1, N_{E/L_i}u'\} = \{-1, (v'_i)_{L_i}\}, \qquad i = 1, 2, 3.$$

We distinguish two cases:

Case 1: $H^0(F, \mathbf{Q}/\mathbf{Z}(2))$ *is infinite*. Then $\operatorname{Ker}(H^1F\xrightarrow{\cdot\{-1\}} {}_2M_2(F)) = 0$ and the same is true for the L_i and M. Therefore, $y = (u', v'_1, v'_2, v'_3)\in\mathscr{H}_1(0)$ is such that $x = \varepsilon\cdot y$.

Case 2: $H^0(F, \mathbf{Q}/\mathbf{Z}(2))$ *is finite*. Since the absolute Galois group of F acts on $\mathbf{Q}/\mathbf{Z}(2)$ through $(\mathbf{Z}_2^*)^2 = 1 + 8\mathbf{Z}_2$ which is procyclic, there must be an $i \in \{1, 2, 3\}$ such that $H^0(F, \mathbf{Q}/\mathbf{Z}(2)) = H^0(L_i, \mathbf{Q}/\mathbf{Z}(2))$. Then the natural map $\operatorname{Ker}(H^1F \xrightarrow{\cdot\{-1\}} {}_2M_2(F)) \to \operatorname{Ker}(H^1L_i \xrightarrow{\cdot\{-1\}} {}_2M_2(L_i))$ is bijective. Writing $N_{E/L_i}u' = z + (v_i')_{L_i}$, with $z \in \operatorname{Ker}(H^1L_i \xrightarrow{\cdot\{-1\}} {}_2M_2(L_i)) = \operatorname{Ker}(H^1F \xrightarrow{\cdot\{-1\}} {}_2M_2(F))$, we get $N_{E/F}u' = 0$ in H^1F. Therefore u' defines an element of $\mathscr{H}^1(4)$. Hence we get a canonical map:

$$\lambda : \operatorname{Ker}(\mathscr{H}_2(0) \xrightarrow{\phi} \mathscr{H}_2(3)) \longrightarrow \mathscr{H}^1(4),$$

such that the composite $\mathscr{H}_1(0) \xrightarrow{\cdot\varepsilon} \operatorname{Ker}(\mathscr{H}_2(0) \xrightarrow{\phi} \mathscr{H}_2(3)) \xrightarrow{\lambda} \mathscr{H}^1(4)$ is the isomorphism of prop. 1 (immediate verification).

Assume $x \in \operatorname{Ker}\lambda$. Then with the above notations, $u' = \delta^1(\ell_1, \ell_2, \ell_3)$ for suitable $\ell_i \in H^1L_i$. Then $y - \eta_1(0, 0, 0, \ell_1, \ell_2, \ell_3) = (0, v_1'', v_2'', v_3'')$ for suitable v_1'', v_2'', v_3''. In particular, x is the image of $(0, \{-1, v_1''\}, \{-1, v_2''\}, \{-1, v_3''\})$. But then $x = 0$, by the exact sequences attached to the three quadratic extensions L_1/F, L_2/F, L_3F. This shows that λ is injective, hence that $\operatorname{Ker}(\mathscr{H}_2(0) \xrightarrow{\phi} \mathscr{H}_2(3)) \subseteq \varepsilon \cdot \mathscr{H}_1(0)$. □

Remark. In [K1], part 2 of th. 1 was proved along the same lines, but using Milnor's K_2 instead of M_2. Instead of lemma 1 above, the main results of [Su] had to be used, namely the fact that ${}_2K_2(K) = \{-1, K^*\}$ for all fields K and the description of $\operatorname{Ker}(\cdot\{-1\} : K^*/K^{*2} \to K_2(K))$. This led to an argument reducing one to the case of finite fields and number fields. The present approach is both simpler, because there is no such reduction, and more elementary because it does not use the results of [Su]. This shows that one can use to an advantage the theory M_i, which can be thought of in some sense as the largest possible choice for a functor $\tilde{H}^i$ as in section 1, and the choice for which $\operatorname{Ker}(\cdot\{-1\} : H^{i-1}K \to \tilde{H}^i(K))$ is the smallest possible. Needless to say, this advantage has its limits, as I haven't been able to use M_3 to prove the vanishing of $\mathscr{H}_3(r)$ and $\mathscr{H}^3(r)$ unconditionally.

7. Miscellaneous remarks

1. Instead of looking at two series of 7-term complexes as in [MT] (or extended 8-term complexes as here), one can consider two infinite complexes:

$$\cdots \longrightarrow H^nM \xrightarrow{\delta_{n-1}} \bigoplus_{i=1}^{3} H^nL_i \xrightarrow{\beta_n} X \otimes H^nF \xrightarrow{\gamma_n} H^{n+1}F \xrightarrow{\text{res}} H^{n+1}M \longrightarrow \cdots$$

$$\cdots \longleftarrow H^{n+1}M \xleftarrow{\delta^{n+1}} \bigoplus_{i=1}^{3} H^{n+1}L_i \xleftarrow{\beta^n} G \otimes H^{n+1}F \xleftarrow{\gamma^n} H^nF \xleftarrow{\text{cor}} H^nM \longleftarrow \cdots$$

This is the viewpoint of [K1]. An advantage is that these complexes are a little more reminiscent of the long exact sequence associated to a quadratic extension. A disadvantage, however, is that they are *not* acyclic at $\bigoplus_{i=1}^{3} \mathrm{H}^n\mathrm{L}_i$ in general. Clearly, it is to deal with this problem that the complexes S_n and S^n somehow 'repeat' themselves. Here is a description of the homology at the bad spots:

PROPOSITION 6. *There are exact sequences*:

$$\operatorname{Ker} \alpha_n \longrightarrow (\mathrm{H}^n\mathrm{F})^3 \longrightarrow \operatorname{Ker} \beta_n/\operatorname{Im} \delta_{n-1} \longrightarrow \mathscr{H}_n(1) \longrightarrow 0;$$

$$0 \longrightarrow \mathscr{H}^n(1) \longrightarrow \operatorname{Ker} \delta^{n+1}/\operatorname{Im} \beta^n \longrightarrow (\mathrm{H}^{n+1}\mathrm{F})^3 \longrightarrow \operatorname{Coker} \alpha^n. \quad \square$$

COROLLARY. *Assume that* $\mathscr{H}_n(r) = \mathscr{H}^n(r) = 0$ *for all n. Then the two infinite complexes above are acyclic, except at* $\bigoplus_{i=1}^{3} \mathrm{H}^n\mathrm{L}_i$, *where their homology can be described by exact sequences*:

$$(\mathrm{H}^{n-1}\mathrm{F})^3 \oplus \bigoplus_{i=1}^{3} \mathrm{H}^n\mathrm{L}_i \xrightarrow{\eta'_n} (\mathrm{H}^n\mathrm{F})^3 \longrightarrow \operatorname{Ker} \beta_n/\operatorname{Im} \delta_{n-1} \longrightarrow 0;$$

$$0 \longrightarrow \operatorname{Ker} \delta^{n+1}/\operatorname{Im} \beta^n \longrightarrow (\mathrm{H}^{n+1}\mathrm{F})^3 \xrightarrow{\eta'^n} (\mathrm{H}^{n+2}\mathrm{F})^3 \oplus \bigoplus_{i=1}^{3} \mathrm{H}^{n+1}\mathrm{L}_i.$$

Here η'_n *and* η'^n *are defined similarly to* η_n *and* η^n *in section* 3. $\square$

2. Much of the theory developed in [MT] and here applies to more general situations than biquadratic extensions of fields. For example, instead of absolute Galois groups of fields, one may consider the case of two groups $\mathrm{H} \subseteq \mathrm{G}$ (profinite or discrete) with H normal in G and G/H isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$. One may also consider a biquadratic covering of topological spaces, or an étale biquadratic covering of schemes.

In all these cases, part 1 of [MT] and prop. 1, lemma 2, prop. 6 of the present paper hold verbatim. This is because they rely essentially on the long exact sequence for a quadratic extension (or a subgroup of index 2, or a quadratic covering), which exists quite generally. Similarly, the homomorphism of complexes defined in section 4 of this paper exists in all generality.

On the contrary, the vanishing of homology groups $\mathscr{H}_n(r)$ and $\mathscr{H}^n(r)$ is specific to fields and closely related to the Milnor conjecture. It is easy to check that the methods of this paper, hence theorem 1, extend to the case of semi-local rings for which the Milnor conjecture is known, under conditions similar to (a) and (b) in case 1 of theorem 1. Therefore, for an arbitrary scheme X, the complexes of *Zariski* *sheaves* associated to the complexes of étale cohomology groups analogous to S_n and S^n are acyclic when its local rings satisfy conditions of theorem 1 (*eg* $n \leq 2$, X smooth over a field).

On the other hand, the complexes S_n and S^n themselves need not be acyclic for a biquadratic covering $f: Y \to X$ of non-local schemes. The corresponding groups $\mathscr{H}_n(f, odd)$, $\mathscr{H}_n(f, even)$, $\mathscr{H}^n(f, odd)$, $\mathscr{H}^n(f, even)$ are invariants of this covering; it would be interesting to investigate them and compute them at least in special cases, for example in the case of a biquadratic covering of a smooth curve.

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CNRS-URA 212
Mathématiques - Université de Paris 7
2 Place Jussieu
75251 Paris Cedex 05 (France)
e-mail: Kahn@mathp7.jussieu.fr

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