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Codimension one foliations on solvable manifolds

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1. The objects of our interest are codimension one foliations on (possibly open) manifolds. If we consider C^{∞} foliations in general and put no particular conditions on them, then they can be extremely complicated even when the manifolds are simple. For example, S^{2k+1} ($k \geq 1$) admits foliations with nonzero secondary characteristic classes. In other words, the topology of the manifolds does not have a strong effect on the nature of the foliations they carry.

On the other hand, if we confine ourselves to foliations without closed leaves on closed 3-manifolds, then the topology of the manifolds does have a remarkable influence on the foliations. For example, closed 3-manifolds with finite fundamental groups do not admit such foliations ([N]); one can classify foliations on closed 3-manifolds with solvable fundamental groups ([GS], [P_1]); also, quite recently, complete understanding is obtained for foliations on the unit tangent bundle of a surface of genus >1 ([G], [Ma]).

For higher dimensional manifolds, the influence is not so clear. It is known that all closed manifolds of dimension ≥ 4 with Euler number 0 admit C^1 foliations without closed leaves ([Sch]). The problem about C^2 foliations remains unsolved. However if we focus our attention upon C^{ω} foliations, then there is a clue to the problem. I.e. they are known not to admit null transversals ([H_1], [H_2]). As a result, e.g., closed manifolds with finite fundamental groups do not admit C^{ω} foliations.

In this paper we consider foliations on manifolds of arbitrary dimensions and make two assumptions on them, i.e. that they do not have closed leaves and that they behave like C^{ω} foliations. Our purpose is to investigate the influence of the topology of the manifolds upon such foliations. In particular we shall show that foliations on manifolds with solvable fundamental groups admit 'transverse structures'.

To be precise, let $\mathscr{F}$ be a codimension one transversely oriented C^r foliation ($r \geq 0$) on a (possibly open) connected smooth manifold M. Throughout this paper we shall work under the following assumptions.

(I) $\Pi = \pi_1(M)$ *is solvable.*
(II) $\mathscr{F}$ *does not admit closed leaves.*
(III) *Every nontrivial leaf holonomy of* $\mathscr{F}$ *has an isolated fixed point.*

Denote by $\tilde{\mathscr{F}}$ the lift of $\mathscr{F}$ to the universal covering space $\tilde{M}$ of M. Our first result is the following.

THEOREM 1. *There exist a C^r submersion $D:\tilde{M}\to\mathbf{R}$ which carries each leaf of $\tilde{\mathscr{F}}$ to a point of $\mathbf{R}$ and a homomorphism $\phi:\Pi\to\mathrm{Diff}^r(\mathbf{R})$ such that $D(\gamma x)=\phi(\gamma)D(x)$ for any $\gamma\in\Pi$ and $x\in\tilde{M}$.*

In 1983 paper [P_2], J. Plante showed the following.

THEOREM. *If a solvable group Γ acts on the real line $\mathbf{R}$ in such a way that the fixed point set of any nontrivial element of Γ is isolated (possibly empty), then there exists a nontrivial locally finite measure μ on $\mathbf{R}$ and a homomorphism $a:\Gamma\to\mathbf{R}_{>0}$ such that for any $\gamma\in\Gamma$, $\gamma_*\mu=a(\gamma)\mu$. If further all the orbits are dense, then the Γ-action must be topologically conjugate to an affine action.*

Applying this to Theorem 1, we get the following corollary.

COROLLARY 2. *Suppose all the leaves of $\mathscr{F}$ are dense in M. Then $\mathscr{F}$ is topologically conjugate to a transversely affine foliation.*

The methods that we use here are to study the 1-connected (possibly non-Hausdorff) 1-manifold $\mathscr{X}=\tilde{M}/\tilde{\mathscr{F}}$, and the action of the fundamental group Π on $\mathscr{X}$.

The qualitative study of foliations often leads to the work on 1-manifolds. First of all, their fundamental properties are investigated in connection with foliations without holonomy on open manifolds ([HR]). Since then, works have been done especially on their relations with particular classes of foliations. For example, the completeness (as defined below) is discussed for foliations without holonomy on closed manifolds ([I]). There are extensive studies on foliations with singularities which are defined by closed 1-forms, especially in their connection with the fundamental group ([L_1], [L_2], [Si]). Here the algebraic key is [BNS]. Foliations with transversely affine or projective structures have also been investigated along this line ([M_1], [M_2], [M_3], [IMT]).

Another purpose of this paper is to give some criteria for the completeness of foliations. The foliation $\mathscr{F}$ is called *complete* if the leaf space $\mathscr{X}=\tilde{M}/\tilde{\mathscr{F}}$ is homeomorphic to $\mathbf{R}$. Notice that there are examples of noncomplete all-leaves-dense foliations on closed 3-manifolds. See [M_2].

THEOREM 3. *If Π is polycyclic, then $\mathscr{F}$ is complete.*

In [M_2], G. Meigniez already obtained Theorem 3 assuming $\mathscr{F}$ is transversely affine.

THEOREM 4. *If all the leaves of $\mathscr{F}$ have finitely generated fundamental groups, then $\mathscr{F}$ is complete.*

Before this paper is prepared, Thierry Barbot obtained the result that codimension one (un)stable foliations of Anosov flows on solvable manifolds are complete and topologically transversely affine. This is undoubtedly the most interesting case.

V. V. Solodov asserted ([S_2] Theorem 1) that under some mild condition, any foliation with a single Novikov component on a compact manifold must be complete provided the fundamental group of the manifold does not contain a free subgroup on two generators. But this is not correct. A counterexample due to Nobuo Tsuchiya will be given in Section 5. However this noncomplete foliation has a compact leaf. In fact it is almost without holonomy. We do not know the answer to the following problems.

PROBLEM. *Does there exist an noncomplete all-leaves-dense foliation on a compact solvable manifold?*

If we drop the condition 'compact', then the answer is positive. The following example is communicated by the referee.

Take a finitely presented solvable group Γ and a homomorphism $\chi : \Gamma \to \mathbf{R}$ of rank ≥ 2 such that $\chi \in \Sigma(\Gamma)$ in the notation of [BNS] but that $-\chi \notin \Sigma(\Gamma)$. For example, the metabelian group

$$\Gamma = \langle a, b \mid a[a, b]a^{-1} = b[a, b]b^{-1} = [a, b]^2 \rangle$$

and any rank 2 homomorphism χ such that $\chi(a), \chi(b) > 0$ will do ([BS]).

Let N be a compact manifold of dimension ≥ 3 with fundamental group Γ. The class $\chi \in H^1(N, \mathbf{R})$ is represented by a closed 1-form ω on N with Morse type singularities and dense leaves ([L_1]). Let M be the manifold obtained from N by deleting the singularities of ω. The foliation drawn by ω on M is of type V, to be defined later (Theorem 5.1, [BNS]).

Also communicated by the referee, we shall show in Section 5 that any group of orientation preserving homeomorphisms of $\mathbf{R}$ can be lifted to an action on a 1-connected non-Hausdorff 1-manifold.

The author is glad to express his thanks to the referee, whose valuable comments are indeed helpful for the improvement of this paper.

This paper was prepared while the author was staying at Université Claude Bernard, Lyon I. The author wants to thank the institute and especially Gilbert Hector for his warm hospitality.

2. Like all the other parts of this paper, we assume that our manifold M and foliation $\mathscr{F}$ satisfy conditions (I) $\sim$ (III) of Section 1. First of all, let us recall the theorem of Haefliger. See [H_1], [H_2] or [CL]. When $r = 0$, we need additional arguments about general position. For this, see [S_1] and [HH].

THEOREM. *The foliation $\mathscr{F}$ cannot have a null transversal (i.e. a null homotopic closed curve which is transverse to $\mathscr{F}$).*

Let us consider the leaf space $\mathscr{X} = \tilde{M}/\tilde{\mathscr{F}}$ of the foliation lifted to the universal covering space.

LEMMA (2.1). *$\mathscr{X}$ is an oriented, connected and simply connected (possibly non-Hausdorff) 1-manifold.*

Proof. That $\mathscr{X}$ is a 1-manifold is an easy consequence of the theorem of Haefliger. The orientation of $\mathscr{X}$ is obtained by the transverse orientation of $\tilde{\mathscr{F}}$. The other statements follow from the corresponding properties of $\tilde{M}$. (Fundamental natures of $\mathscr{X}$ with full proofs will appear in [IMT].) □

Since the foliation $\tilde{\mathscr{F}}$ is the lift of the foliation $\mathscr{F}$ on M, the action of the fundamental group Π on $\tilde{M}$ yields an orientation preserving action of Π on $\mathscr{X}$. Let Γ be the quotient of Π by the subgroup which acts trivially on $\mathscr{X}$. Thus Γ acts on $\mathscr{X}$ effectively. In this paper we focus our attention on this Γ-action.

Properties of the foliation $\mathscr{F}$ can be observed through the action of Γ on $\mathscr{X}$. For example, a leaf of $\mathscr{F}$ corresponds to a Γ-orbit; A dense leaf corresponds to a dense orbit. One needs to be a bit cautious to see what kind of orbits correspond to closed leaves of $\mathscr{F}$. Let us call a subset S of $\mathscr{X}$ *closed discrete* if for any embedded compact interval J of $\mathscr{X}$, we have $\#(S \cap J) < \infty$. Then a closed leaf corresponds to a closed discrete orbit.

Notice that this notion is apparently a bit different from Hausdorff intuition. For example, imagine a non-Hausdorff 1-manifold $\mathscr{X}$ whose nonseparating points (to be defined below) are dense. Consider an embedded real line R in $\mathscr{X}$. Then the boundary ∂R contains all the points which form nonseparating pairs with some points of R. (See Figure 1.) Hence to a Hausdorff eye ∂R might look dense in R. But in our terminology, ∂R is 'closed discrete'.

Next let us interpret the condition (III) in terms of Γ-orbits. One has to notice in the first place that the fixed point set Fix (γ) of $\gamma \in \Gamma$ is not necessarily a closed set of $\mathscr{X}$. (Imagine e.g. the exchange of branches in Type V, Figure 2.) Therefore even under (III), Fix (γ) may not be a closed discrete set. Suppose $x_n \in$ Fix (γ) are distinct points and that $\lim_{n \to \infty} x_n = x_0$. It might happen that $x_0 \notin$ Fix (γ). (III) says

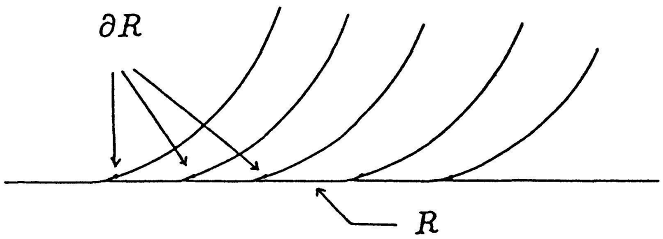

Figure 1

that if $x_0 \in \mathrm{Fix}\,(\gamma)$, then γ must be the identity near x_0. This is the same as saying that any component of $\mathrm{Fix}\,(\gamma)$ is either an open set or a single point which is isolated in $\mathrm{Fix}\,(\gamma)$. Here a point x of a subset S is called isolated in S if x has a neighbourhood U such that $U \cap S = \{x\}$.

Now let us summarize the conditions that we got for the induced Γ-action.

(I′) *Γ is a solvable group which acts effectively and orientation preservingly on an oriented* 1*-connected* 1*-manifold $\mathscr{X}$.*
(II′) *There are no closed discrete Γ-orbits.*
(III′) *For any $\gamma \in \Gamma$, any component of* $\mathrm{Fix}\,(\gamma)$ *is either an open set or an isolated point.*

Two distinct points x and y of $\mathscr{X}$ are said to form a nonseparating pair if for any open interval U (resp. V) containing x (resp. y), we have $U \cap V \neq \varnothing$. Define $U = U_- \cup \{x\} \cup U_+$ and $V = V_- \cup \{y\} \cup V_+$ in accordance with the orientation of $\mathscr{X}$. For a nonseparating pair x and y, choosing U and V sufficiently small, we have either of the following two cases. See [IMT] for detailed arguments.

(i) $U_- = V_-$ and $U_+ \cap V_+ = \varnothing$.
(ii) $U_- \cap V_- = \varnothing$ and $U_+ = V_+$.

DEFINITION (2.2). $\mathscr{X}$ is called of *type I* if it does not have nonseparating pairs, of *type W* if it admits nonseparating pairs of both (i) and (ii), and of *type V* otherwise.

See Figure 2. Saying $\mathscr{F}$ is complete is the same as saying $\mathscr{X}$ is of type I. In the rest of this section, we shall show the following.

PROPOSITION (2.3). *The* 1*-manifold $\mathscr{X}$ cannot be of type W.*

The key fact is the following lemma, which is a variant of so called Klein's criterion or the "Table Tennis Lemma", rather. The proof in the case of non-Hausdorff 1-manifolds will be found in [IMT].

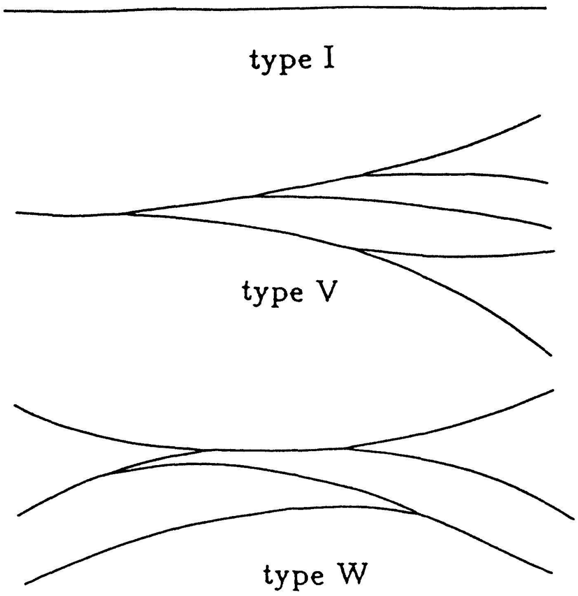

Figure 2

LEMMA (2.4). *Let P be a connected open subset of $\mathscr{X}$ with ∂P consisting of four distinct points a, b, c, d. Suppose $f, g \in \Gamma$ satisfy $f(a) = c$, $f(P) \cap P = \varnothing$, $g(b) = d$ and $g(P) \cap P = \varnothing$. Then f and g generate a free subgroup in Γ.* □

Notice the situation of Lemma (2.4) occurs only when $\mathscr{X}$ is of type W. See Figure 3. On the other hand if $\mathscr{X}$ is of type W and if all the leaves of $\mathscr{F}$ are dense (i.e., all the Γ-orbits are dense in $\mathscr{X}$), it is easy to find two elements f and g of Γ which satisfy the hypothesis of Lemma (2.4). Then we get that Γ cannot be solvable and this contradiction shows Proposition (2.3). However in order to show Proposition (2.3) in full generality, we need a bit more.

DEFINITION (2.5). For a subset S of $\mathscr{X}$, define

$$\hat{S} = \bigcup \{J \mid J \text{ is an embedded closed interval joining two points of } S\}.$$

When $\hat{S} = \mathscr{X}$, we say *S fills up $\mathscr{X}$*.

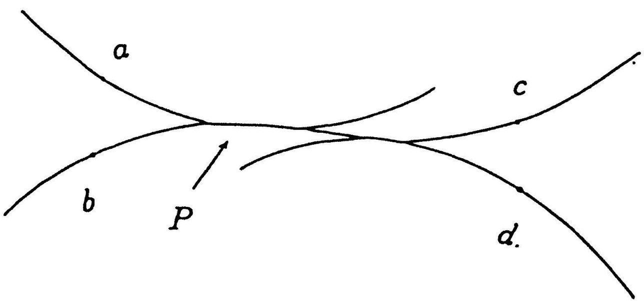

Figure 3

LEMMA (2.6). *For any subset S of $\mathscr{X}$, the boundary $\partial\hat{S}$ is a closed discrete set.*

Proof. For any interval I embedded in $\mathscr{X}$, clearly we have $I \cap \hat{S}$ is connected. Lemma (2.6) follows from this. $\square$

LEMMA (2.7). *Any orbit $\mathcal{O}$ of the Γ-action fills up $\mathscr{X}$.*

Proof. $\mathcal{O}$ is not closed discrete by (II′). Hence $\hat{\mathcal{O}}$ is not empty. By Lemma (2.6), $\partial\hat{\mathcal{O}}$ is a closed discrete set, invariant by the action of Γ. Hence by (II′) we have that $\partial\hat{\mathcal{O}} = \emptyset$. That is, $\mathcal{O}$ fills up $\mathscr{X}$. $\square$

Now by Lemma (2.7) it is easy to find two elements f and g which satisfy the condition of Lemma (2.4). The details are left to the reader. We have completed the proof of Proposition (2.3).

3. In this section we shall show Theorems 1 and 3. If $\mathscr{X}$ is of type I, then there is nothing to prove. So let us assume that $\mathscr{X}$ is of type V. (Recall Proposition (2.3).) Assume that the condition (i) of Section 2 occurs for each nonseparating pair. That is, there is only one end in the $-\infty$ direction. We use the following notations in $\mathscr{X}$.

DEFINITION (3.1).

(a) For x and $y \in \mathscr{X}$, denote $x \prec y$ if there is an orientation preserving embedding $f: [0, 1] \to \mathscr{X}$ such that $f(0) = x$ and $f(1) = y$.

(b) For $x \preceq y$, let

$$[x, y] = \{z \in \mathscr{X} \mid x \preceq z \preceq y\}$$

$$]-\infty, x[= \{z \mid z \prec x\}.$$

(c) Denote $x \approx y$ if $x = y$ or x and y form a nonseparating pair.

(d) For $\gamma \in \Gamma$, let

$$\mathrm{Fix}(\gamma) = \{x \in \mathscr{X} \mid \gamma x = x\},$$

$$\mathrm{Fix}^{\approx}(\gamma) = \{x \in \mathscr{X} \mid \gamma x \approx x\}.$$

We shall summarize properties which follow immediately from the definitions and the assumption that $\mathscr{X}$ is of type V.

LEMMA (3.2).

(a) *The relation $\preceq$ is a partial order.*

(b) *For any points x and $y \in \mathscr{X}$, there exists a point $z \in \mathscr{X}$ such that $z \prec x$ and $z \prec y$.*

(c) *If* $x \prec z$ *and* $y \prec z$, *then we have either* $x \preceq y$ *or* $y \preceq x$.

(d) *The relation* $\approx$ *is an equivalence relation.* □

Notice that $[x, y]$ is diffeomorphic to $[0, 1]$ if $x \prec y$. Also $]-\infty, x[$ is diffeomorphic to $]-\infty, 0[$. However for example for $x \in \mathscr{X}$, the set $\{z \in \mathscr{X} \mid x \prec z\}$ is not diffeomorphic to $]0, \infty[$. Clearly we have $\operatorname{Fix}(\gamma) \subset \operatorname{Fix}^{\approx}(\gamma)$.

LEMMA (3.3). *If an element* $\gamma \in \Gamma$ *satisfies* $\operatorname{Fix}^{\approx}(\gamma) = \varnothing$, *then there exists a unique* γ*-invariant properly*[1] *embedded copy of the real line*, $\operatorname{Axis}(\gamma)$, *called the axis of* γ.

Proof. Suppose $\operatorname{Fix}^{\approx}(\gamma) = \varnothing$. Given an arbitrary point $x \in \mathscr{X}$, choose a point z such that $z \prec x$ and $z \prec \gamma x$. Then we have $\gamma z \prec \gamma x$ and $z \prec \gamma x$. Therefore by Lemma (3.2) (c), either $\gamma z \prec z$, $\gamma z = z$ or $\gamma z \succ z$ holds. Since $\operatorname{Fix}(\gamma) = \varnothing$, $\gamma z \neq z$. Assume, to fix the idea, that $\gamma z \prec z$. Consider the set

$$\operatorname{Axis}(\gamma) = \bigcup \{\gamma^i([\gamma z, z]) \mid i \in \mathbf{Z}\}.$$

Since $\operatorname{Fix}^{\approx}(\gamma) = \varnothing$, $\{\gamma^i z\}$ must be closed discrete. Therefore $\operatorname{Axis}(\gamma)$ is a properly embedded copy of the real line, invariant by γ. It is easy to show that such a line is unique. □

Let

$$\Gamma > \Gamma_1 > \Gamma_2 > \cdots > \Gamma_n = \Omega > \{1\}$$

be the descending sequence of Γ. We are particularly interested in the action of the last normal subgroup Ω.

LEMMA (3.4). *For any element* $\omega \in \Omega \backslash \{1\}$, we have $\operatorname{Fix}^{\approx}(\omega) = \varnothing$.

Proof. Suppose the contrary and let

$$\Omega_F = \{\omega \in \Omega \backslash \{1\} \mid \operatorname{Fix}^{\approx}(\omega) = \varnothing\}.$$

Then $\omega \in \Omega_F$ has an axis. Since Ω is an abelian group, we have that $\operatorname{Axis}(\omega) = \operatorname{Axis}(\omega')$ for any $\omega, \omega' \in \Omega_F$. Denote this set by $\mathscr{B}$. Since Ω is a normal subgroup, we have that $\mathscr{B}$ is Γ-invariant. But then we get a nonempty Γ-invariant closed discrete set $\partial\mathscr{B}$, contradicting (II′). □

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[1] 'proper' means that the inverse image of a compact subset is compact.

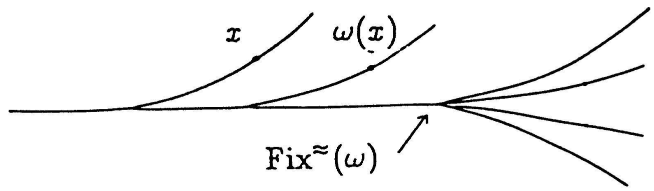

Figure 4

The original idea of the proof of Theorem 1 is to find out a good Ω-invariant measure and using it to define an equivariant submersion of $\mathscr{X}$ onto $\mathbf{R}$. However it turned out to be difficult and we have to be content with the worst one, i.e., a Dirac measure.

LEMMA (3.5). *There exists a nonempty Ω-invariant closed discrete subset $S \subset \mathscr{X}$.*

Proof. First of all assume for contradiction that for any element $\omega \in \Omega \backslash \{1\}$, we have $\mathrm{Fix}\,(\omega) = \varnothing$. By Lemma (3.4), we have $\mathrm{Fix}^{\approx}\,(\omega) = \varnothing$. Since $\mathrm{Fix}\,(\omega) = \varnothing$, one can show without much difficulty that $\mathrm{Fix}^{\approx}\,(\omega)$ is a single $\approx$ class. (See Figure 4.) Since Ω is abelian, $\mathrm{Fix}^{\approx}\,(\omega)$ must be kept invariant by any other element $\eta \in \Omega \backslash \{1\}$. That is, $\mathrm{Fix}^{\approx}\,(\omega) \subset \mathrm{Fix}^{\approx}\,(\eta)$. The converse inclusion also holds and hence $\mathrm{Fix}^{\approx} = \mathrm{Fix}^{\approx}\,(\omega)$ is independent of the choice of $\omega \in \Omega \backslash \{1\}$. Since Ω is a normal subgroup of Γ, $\mathrm{Fix}^{\approx}$ is Γ-invariant. But this is contrary to (II′), since $\mathrm{Fix}^{\approx}$ is closed discrete.

Thus we have shown that there exists an element $\omega \in \Omega \backslash \{1\}$ such that $\mathrm{Fix}\,(\omega) \neq \varnothing$.

If $\mathrm{Fix}\,(\omega)$ is closed discrete, then let the required set S be $\mathrm{Fix}\,(\omega)$. Notice that $\mathrm{Fix}\,(\omega)$ is Ω-invariant since Ω is an abelian group, completing the proof.

Suppose on the other hand that $\mathrm{Fix}\,(\omega)$ is not closed discrete. That is, there exist infinitely many points $x_n \in \mathrm{Fix}\,(\omega)$ such that $x_n \to x_0$. At this point we do not know whether $x_0 \in \mathrm{Fix}\,(\omega)$ or not. We only have that $x_0 \in \mathrm{Fix}^{\approx}\,(\omega)$.

For a while assume for contradiction that $\mathrm{Fix}\,(\omega)$ fills up $\mathscr{X}$. Then there exists $y \in \mathrm{Fix}\,(\omega)$ such that $x_0 \prec y$. This implies that $x_0 \in \mathrm{Fix}\,(\omega)$. Now by the condition (III) or by its equivalent (III′) we have that ω must be the identity near x_0. Next consider the interval $]-\infty, x_0[$. Notice that ω keeps $]-\infty, x_0[$ invariant. Therefore by the condition (III′) and the continuity of ω, we get that $]-\infty, x_0[\subset \mathrm{Fix}\,(\omega)$. Now take an arbitrary point $x \in \mathscr{X}$. Then there exists a point $z \in \mathscr{X}$ and a point $u \in \mathrm{Fix}\,(\omega)$ such that $z \prec x_0$ and $z \prec x \prec u$. Then again since ω is the identity near z and ω keeps $]-\infty, u[$ invariant, we have that $\omega(x) = x$. Therefore ω must act as the identity on $\mathscr{X}$. This is contrary to the assumption that Γ acts effectively.

Hence we have shown that $\widehat{\mathrm{Fix}\,(\omega)} \subsetneq \mathscr{X}$. Let $S = \partial\, \widehat{\mathrm{Fix}\,(\omega)}$. S is Ω-invariant and closed discrete as required. $\square$

Notice that for an Ω-invariant closed discrete subset S and for an element $\gamma \in \Gamma$, γS is again Ω-invariant and closed discrete. This is an immediate consequence of the normality of Ω. Let us set

$$\Gamma = \{\gamma_1 = 1, \gamma_2, \gamma_3, \gamma_4, \dots\}$$

and let

$$S_i = \bigcup_{j=1}^{i} \gamma_j S \qquad (1 \le i \le \infty).$$

LEMMA (3.6).
(a) S_i $(i < \infty)$ *is an increasing sequence of* Ω*-invariant closed discrete subsets.*
(b) S_∞ *fills up* $\mathscr{X}$.

Proof. (a) is immediate. For (b), since S_∞ is Γ-invariant and nonempty, we have S_∞ fills up $\mathscr{X}$ by Lemma (2.7). □

Let us define a mapping $p_i : S_i \to \mathbf{Z}$ for $i < \infty$. First fix the base point $x_0 \in S_i$. Given any $x \in S_i$, choose $z \in \mathscr{X}$ such that $z \prec x_0$ and $z \prec x$. Define

$$p_i(x) = \#([z, x] \cap S_i) - \#([z, x_0] \cap S_i).$$

Notice that this is well-defined, independent of the choice of z.

LEMMA (3.7). *Whenever* $z \prec x$ *and* $z \prec y$, *we have*

$$\#([z, y] \cap S_i) - \#([z, x] \cap S_i) = p_i(y) - p_i(x).$$

Proof. Notice that the L.H.S. is independent of the choice of z so far as it satisfies $z \prec x$ and $z \prec y$. Therefore we can choose z so that it also satisfies that $z \prec x_0$. Then Lemma (3.7) follows directly from the definition of p_i. □

COROLLARY (3.8). *For any* $\omega \in \Omega$ *and* $x, y \in S_i$, *we have*

$$p_i(\omega y) - p_i(\omega x) = p_i(y) - p_i(x).$$

Proof. Choose z so that $z \prec x$ and $z \prec y$. We have

$$\begin{aligned} p_i(\omega y) - p_i(\omega x) &= \#([\omega z, \omega y] \cap S_i) - \#([\omega z, \omega x] \cap S_i) \\ &= \#([z, y] \cap S_i) - \#([z, x] \cap S_i) = p_i(y) - p_i(x). \end{aligned}$$

The second equality follows from the Ω-invariance of S_i. □

COROLLARY (3.9). *There exists a homomorphism* $\phi_i : \Omega \to \mathbf{Z}$ *such that*

$p_i(\omega x) = p_i(x) + \phi_i(\omega)$.

Proof. Corollary (3.8) says in particular that if $p_i(x) = p_i(y)$, then we have $p_i(\omega x) = p_i(\omega y)$. That is, the action of ω projects down to an endomorphism of the set $p_i(S_i) \subset \mathbf{Z}$. Again by (3.8), this mapping is the translation by a fixed integer, say $\phi_i(\omega)$. The required properties of ϕ_i can easily be established by (3.8). □

DEFINITION (3.10). Define a subgroup K of Ω by

$$K = \bigcup_{i \geq 1} \bigcap_{j \geq i} \mathrm{Ker}\,(\phi_j).$$

An element $\omega \in \Omega$ belongs to K if and only if $\phi_i(\omega) = 0$ for any sufficiently large i.

LEMMA (3.11). *We have* $K = \Omega$.

Proof. By definition, we have that $K \subset \Omega$. Let us show the converse. Choose an arbitrary element $\omega \in \Omega$. If $\omega = 1$, then there is nothing to prove. So let $\omega \neq 1$. Then by Lemma (3.4), $\mathrm{Fix}^{\approx}(\omega) = \varnothing$. Take a point $x \in \mathrm{Fix}^{\approx}(\omega)$ and consider $]-\infty, x[$. Notice that $]-\infty, x[$ is invariant by ω. Since S_∞ fills up $\mathscr{X}$ (Lemma (3.6)), for any sufficiently large i, we have $]-\infty, x[\cap S_i \neq \varnothing$. Take an arbitrary point $y \in]-\infty, x[\cap S_i$. Suppose for contradiction that $\omega(y) \neq y$. Then we have either $y \prec \omega y$ or $\omega y \prec y$. Assume $y \prec \omega y$. Then the sequence $\{\omega^k y \mid k \geq 0\}$ lies in the compact interval $[y, x]$, contrary to the closed discreteness of S_i. This absurdity shows that $\omega y = y$. Thus we have $\phi_i(\omega) = 0$ for any sufficiently large i and therefore $\omega \in K$. □

LEMMA (3.12). *Let* $\omega \in \Omega$ *and* $x \in \mathscr{X}$. *If either* $\omega x \preceq x$ *or* $x \preceq \omega x$ *holds, then we have* $\omega x = x$.

Proof. Assume the contrary. Without loss of generality, one may assume that $\omega x \prec x$. Then we have

$$\omega(]-\infty, x[) =]-\infty, \omega x[\subset]-\infty, x[.$$

For any point $y \in]-\infty, x[\cap S_i$, we have either $\omega y \preceq y$ or $y \preceq \omega y$. But then since $\omega \in K$ (Lemma (3.11)), we have $\omega y = y$. (The point y can be viewed to belong to S_j for arbitrarily large j.) That is, ω keeps points of $]-\infty, x[\cap S_\infty$ fixed. Consider an accumulation point z of S_∞. The orbit Γz fills up $\mathscr{X}$ (Lemma (2.7)). In particular we have $\Gamma z \cap]-\infty, x[\neq \emptyset$. That is, $S_\infty \cap]-\infty, x[$ is not closed discrete. Therefore ω must be the identity on $]-\infty, x[$. But this is a contradiction because $\omega x \in]-\infty, x[$. □

COROLLARY (3.13). *For any* $\omega \in \Omega$, *we have* Fix (ω) *is a nonempty open set. If* $x \in$ Fix (ω) *and if* $y \prec x$, *then we have* $y \in$ Fix (ω).

Proof. Let $\omega \in \Omega$. Choose an arbitrary point $x \in \mathscr{X}$ and a point $y \in \mathscr{X}$ such that $y \prec x$ and $y \prec \omega x$. Then it is easy to show that either $y \preceq \omega y$ or $\omega y \preceq y$ holds. Therefore by Lemma (3.12), we have $\omega(y) = y$, showing that Fix $(\omega) = \emptyset$.

Next let $x \in$ Fix (ω) and $y \prec x$. Then since $]-\infty, x[$ is kept invariant by ω, we have either $y \prec \omega y$ or $\omega y \prec y$. Therefore $y \in$ Fix (ω).

Finally it follows quickly from (III′) that Fix (ω) is an open set. □

COROLLARY (3.14). Ω *is not finitely generated.*

Proof. Suppose the contrary. Let $\omega_1, \omega_2, \dots, \omega_n$ be generators of Ω. Choose points $x_i \in$ Fix (ω_i) and a point $y \prec x_i$ $(1 \leq i \leq n)$. Then y is a common fixed point of ω_i. That is, if we put

$$\mathrm{Fix}\,(\Omega) \equiv \{x \in \mathscr{X} \mid \omega x = x, \ \forall \omega \in \Omega\},$$

we have

$$\mathrm{Fix}\,(\Omega) = \bigcap_i \mathrm{Fix}\,(\omega_i) \neq \emptyset.$$

Since Ω is a normal subgroup of Γ, we have Fix (Ω) is a Γ-invariant subset. It has the following property; If $x \in$ Fix (Ω) and if $y \prec x$, then $y \in$ Fix (Ω). Therefore we have that ∂ Fix (Ω) is a closed discrete Γ-invariant subset. Therefore by (II′), we have ∂ Fix $(\Omega) = \emptyset$ and Fix $(\Omega) = \mathscr{X}$. This contradicts that Γ acts effectively on $\mathscr{X}$. □

Recall that a solvable group is polycyclic if and only if all its subgroups are finitely generated ([W]). Therefore we have already shown that if $\mathscr{X}$ is of type V, then Γ and hence Π cannot be polycyclic. This finishes the proof of Theorem 3.

We shall continue the proof of Theorem 1. Let

$$\bar{\Gamma} = \Gamma/\Omega, \qquad \bar{\mathscr{X}} = \mathscr{X}/\Omega.$$

As a matter of fact, $\bar{\Gamma}$ acts on $\bar{\mathscr{X}}$.

PROPOSITION (3.15). $\bar{\mathscr{X}}$ *is a* 1-*connected* 1-*manifold of type I or V. The action of* $\bar{\Gamma}$ *satisfies the conditions* (II′) *and* (III′).

Proof. That $\bar{\mathscr{X}}$ is a 1-manifold follows from Lemma (3.12).

Next we shall show that there does not exist a nonseparating pair of type (ii) of Section 2. Suppose there exists a sequence $\{[x_n]\}$ of points of $\bar{\mathscr{X}}$ such that $[x_n] \downarrow [x]$ and that $[x_n] \downarrow [y]$. Let us show that $[x] = [y]$. Choose an open interval $I \subset \mathscr{X}$ containing the point $x \in \mathscr{X}$. One may assume that $x_n \in I$ $(n = 1, 2, \ldots)$. Likewise choose an open interval J containing the point $y \in \mathscr{X}$. One may assume that there exists a sequence $\{y_n\} \subset J$ such that $[y_n] = [x_n]$. As a matter of fact, we have $x_n \downarrow x$ and $y_n \downarrow y$. There exists an element $\omega_n \in \Omega$ such that $\omega_n x_n = y_n$. If we show that $\omega_1 x_n = y_n$ for any n, then this would imply that $\omega_1 x = y$, i.e., $[x] = [y]$. To show this consider the two points $\omega_1 x_n$ and $y_n = \omega_n x_n$. They satisfy $\omega_1 x_n \prec y_1$ and $\omega_n x_n \prec y_1$. Therefore we have either $\omega_n x_n \preceq \omega_1 x_n$ or $\omega_1 x_n \preceq \omega_n x_n$. Now $\omega_1 \omega_n^{-1} \in \Omega$ carries $y_n = \omega_n x_n$ to $\omega_1 x_n$. Therefore by Lemma (3.12), we have $y_n = \omega_1 x_n$.

Thus we have shown that there does not exist a nonseparating pair of type (ii). This implies that $\bar{\mathscr{X}}$ is 1-connected and of type either I or V.

The other properties are easy to establish and are left to the reader. □

End of proof of Theorem 1. If in Proposition (3.15), $\bar{\mathscr{X}}$ is of type I, then we are done. If not, consider the action of $\bar{\Gamma}$ on $\bar{\mathscr{X}}$. We can apply the same argument that we developed in this section. Notice that the step of $\bar{\Gamma}$ is less than that of Γ. Now an abelian group cannot act on a 1-manifold of type V in such a way that no orbits are closed discrete. (Recall Lamma (3.5).) Therefore we will obtain a 1-manifold of type I at some stage. This shows Theorem 1.

4. We shall show Theorem 4. Let us assume that the foliation $\mathscr{F}$ is not complete, that is, the 1-manifold $\mathscr{X}$ is of type V. Our goal is to show that there exists a leaf of $\mathscr{F}$ whose fundamental group is not finitely generated.

By Theorem 1, we have obtained a submersion $D : \tilde{M} \to \mathbf{R}$ and a homomorphism $\phi : \Pi \to \mathrm{Diff}^r(\mathbf{R})$. Clearly D yields a submersion, also denoted by D, from $\mathscr{X}$ to $\mathbf{R}$. Define the quotient group Γ of Π as in Section 2. Clearly ϕ induces a homomorphism, also denoted by ϕ, from Γ to $\mathrm{Diff}^r(\mathbf{R})$. In summary, we have the following.

PROPOSITION (4.1). *There exists a submersion* $D : \mathscr{X} \to \mathbf{R}$ *and a homomorphism* $\phi : \Gamma \to \mathrm{Diff}^r(\mathbf{R})$ *such that* $D(\gamma x) = \phi(\gamma)D(x)$ *for any* $\gamma \in \Gamma$ *and* $x \in \mathscr{X}$. □

Let $\Lambda = \phi(\Gamma)$. Proposition (3.15) implies that the action of Λ on $\mathbf{R}$ also satisfies the conditions (I′) $\sim$ (III′). That is, we have:

(I′) *A solvable group* Λ *acts on* $\mathbf{R}$ *preserving the orientation*;
(II′) *There are no discrete* Λ*-orbits*;
(III′) *For any* $\lambda \in \Lambda$, $\mathrm{Fix}(\lambda)$ *is discrete.*

By virtue of the theorem of Plante, we have the following.

PROPOSITION (4.2). *There exists a locally finite nontrivial measure μ on* $\mathbf{R}$ *and a homomorphism* $a : \Lambda \to \mathbf{R}_{>0}$ *such that* $\lambda_* \mu = a(\lambda)\mu$ *for any* $\lambda \in \Lambda$. □

Fix a base point, say 0, of $\mathbf{R}$. Define a mapping $E : \mathbf{R} \to \mathbf{R}$ by

$$E(x) = \begin{cases} \mu([0, x[) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu([x, 0[) & \text{if } x < 0. \end{cases} \tag{*}$$

Also define a mapping $b : \Lambda \to \mathbf{R}$ by $b(\lambda) = E(\lambda(0))$. Define a mapping ψ from Λ to the group of the orientation preserving affine transformations, $\mathrm{Aff}^+(\mathbf{R})$, by

$$\psi(\lambda)(x) = a(\lambda)x + b(\lambda).$$

It is a routine work to establish the following lemma. The proof is left to the reader.

LEMMA (4.3). *E is a monotone increasing map.* $\psi : \Lambda \to \mathrm{Aff}^+(\mathbf{R})$ *is a homomorphism. For any* $\lambda \in \Lambda$ *and* $x \in \mathbf{R}$, *we have* $E(\lambda x) = \psi(\lambda)E(x)$. □

Next let us show the continuity of E and its consequences. Let us prove first of all that $\psi(\Lambda)$ is neither trivial nor free cyclic. First of all, since μ is nontrivial, we have that $\mathrm{Im}(E)$ contains at least two points, say $\alpha < \beta$.

Assume for contradiction that $\psi(\Lambda)$ is trivial. Let $c = \sup E^{-1}(\alpha)$. Clearly c must be kept fixed by the Λ-action, contrary to (II$''$).

Next assume that $\psi(\Lambda)$ is free cyclic, generated by a translation, say by 1. For each integer n, let $c_n = \sup\{x \in \mathbf{R} \mid E(x) \le n\}$. Clearly we have $\mu((c_{n-1}, c_n]) = 1$. Since μ is locally finite, $\{c_n\}$ is a discrete Λ-orbit, again contradicting (II$''$).

Finally consider the case where $\psi(\Lambda)$ is free cyclic, generated by a homothety. One can divide into subcases according to $\mathrm{Im}(E)$. An argument similar to the above works to show that this case is also impossible. The details are left to the reader.

Now we have shown that $\psi(\Lambda)$ is neither trivial nor free cyclic. By the nature of the group $\mathrm{Aff}^+(\mathbf{R})$, we are led to either of the following two cases.

Case 1. *$\psi(\Lambda)$ contains arbitrarily small translations.*
Case 2. *$\psi(\Lambda)$ is an abelian group consisting of homotheties.*

Let us show that Case 2 can be reduced to Case 1. For this, we need to replace E, ψ and μ by new maps E', ψ' and a new measure μ'. (Simply we will take the logarithm.)

Assume Case 2. Let the common fixed point of $\psi(\Lambda)$ be c. Then Im (E) is contained either in $\mathbf{R}_{>c}$ or $\mathbf{R}_{<c}$. For otherwise one could find a discrete Λ-orbit by considering either $\inf E^{-1}(\mathbf{R}_{>c})$ or $\sup E^{-1}(\mathbf{R}_{<c})$. Suppose, to fix the idea, that Im $(E) \subset \mathbf{R}_{>c}$. Define E' by $E'(x) = \log(E(x) - c)$ and ψ' by $\psi'(\lambda)(y) = y + \log a(\lambda)$, where $a(\lambda)$ is the slope of $\psi(\lambda)$. It is easy to check that E' and ψ' satisfy the conditions of (4.3) and Case 1. Using (*), one can define a new measure μ' from E'.

Now rename E', ψ' and μ' by E, ψ and μ. In this way, we may assume without loss of generality that Case 1 holds.

Then it follows from the condition of Case 1 that μ is nonatomic and therefore E is continuous.

An element $\gamma \in \Gamma$ is called a *translation* of $\mathscr{X}$ if $\psi(\phi(\gamma))$ is a nontrivial translation. The translation number of $\psi(\phi(\gamma))$ is called the *translation number* of γ. A translation γ has an axis, since $\mathrm{Fix}^{\approx}(\gamma) = \emptyset$.

Now let us summarize the properties of the action of Γ on $\mathscr{X}$. The proof is easy and omitted. ((3) below follows from (2) and (III").)

PROPOSITION (4.4).
(1) *Γ contains a translation of arbitrarily small translation number.*
(2) *μ is nonatomic and E is continuous.*
(3) *ψ is a monomorphism.*
(4) *The union of axes of all the translations on Γ coincides with $\mathscr{X}$.* □

Let $\mu_{\mathscr{X}} = D^*\mu$ and let $\mathscr{M} = \mathrm{Supp}(\mu_{\mathscr{X}})$. We shall show that $\mathscr{M}$ is a unique minimal set for the Γ-action. First we need a lemma.

LEMMA (4.5). *Let $x \in \mathscr{M}$. Suppose there exist points x_i and x_i' such that $x_i \preceq x \preceq x_i'$ and that $\mu_{\mathscr{X}}([x_i, x_i']) \to 0$. Then we have either $x_i \to x$ or $x_i' \to x$.*

Proof. Let $\{I_n\}$ be a fundamental system of neighbourhoods of x, each homomorphic to an open interval. Let us define

$$I_n^- = \{y \in I_n \mid y \prec x\},$$
$$I_n^+ = \{y \in I_n \mid x \prec y\}.$$

Since $x \in \mathrm{Supp}(\mu_{\mathscr{X}})$, one has that $\mu_{\mathscr{X}}(I_n^-) > 0$ for any n or $\mu_{\mathscr{X}}(I_n^+) > 0$ for any n. Assume the former. Then given n, we have $\mu_{\mathscr{X}}([x_i, x_i']) < \mu_{\mathscr{X}}(I_n^-)$ for any sufficiently large i. That is, $x_i \in I_n$. Therefore we have $x_i \to x$. □

LEMMA (4.6). *Any Γ-orbit in $\mathscr{X}$ contains $\mathscr{M}$ in its closure. Especially, $\mathscr{M}$ is a unique minimal set for the Γ-action. The set $\mathscr{M}$ either coincides with $\mathscr{X}$ or is locally homeomorphic to a Cantor set.*

Proof. By Proposition (4.4), there exists a translation in Γ of arbitrarily small translation number. Take any points $x \in \mathcal{M}$ and $y \in \mathcal{X}$. Let us show the following; For any small $\epsilon > 0$, there exist points $y', y'' \in \Gamma y$ such that $y' \preceq x \preceq y''$ and that $\mu_{\mathcal{X}}([y', y'']) < \epsilon$. By the previous lemma, this suffices for the proof of the first part.

First choose three translations γ_1, γ_2 and γ_ϵ as follows. (By Proposition (4.4) (4), this is possible.)

- $x \in \mathrm{Axis}(\gamma_1)$ and $y \in \mathrm{Axis}(\gamma_2)$.
- The translation number of γ_ϵ is positive and smaller than ϵ.

For some integers n and m, we have $\gamma_1^n(x)$, $\gamma_2^m(y) \in \mathrm{Axis}(\gamma_\epsilon)$. Then there exists k such that $\gamma_\epsilon^k(\gamma_2^m(y)) \preceq \gamma_1^n(x) \preceq \gamma_\epsilon^{k+1}(\gamma_2^m(y))$. Now one can choose $y' = \gamma_1^{-n}(\gamma_\epsilon^k(\gamma_2^m(y)))$ and $y'' = \gamma_1^{-n}(\gamma_\epsilon^{k+1}(\gamma_2^m(y)))$.

For the remaining part, notice that since $\mu_{\mathcal{X}}$ is nonatomic, there are no isolated points in $\mathcal{M}$. Also since $\mathcal{M}$ is minimal, $\partial\mathcal{M}$ is either empty or coincides with $\mathcal{M}$. If $\partial\mathcal{M} = \emptyset$, then clearly we have $\mathcal{M} = \mathcal{X}$. If $\partial\mathcal{M} = \mathcal{M}$, then $\mathcal{M}$ is locally a Cantor set. □

Now we have finished the preparations for the proof of Theorem 4. Let $N = \mathrm{Ker}(\phi) = \mathrm{Ker}(\psi\phi) \subset \Gamma$.

PROPOSITION (4.7). *N is nontrivial.*

Proof. By the argument of Corollary (3.13), one knows that $\Omega \subset N$. □

The following two lemmas are easy to establish. The proofs are left to the reader. For $x \in \mathcal{X}$, denote by N_x the isotropy subgroup of N at x.

LEMMA (4.8). *For $v \in N$, the set* $\mathrm{Fix}(v)$ *is a nonempty open subset of $\mathcal{X}$. If $x \in \mathrm{Fix}(v)$ and $y \prec x$, then we have $y \in \mathrm{Fix}(v)$.* □

LEMMA (4.9). *If $y \prec x$, then we have $N_y \supset N_x$.* □

Now we shall show the following key lemma.

LEMMA (4.10). *There exists a point $x \in \mathcal{X}$ such that the isotropy subgroup N_x is not finitely generated.*

Proof. Assume for contradiction that for any point $x \in \mathcal{X}$, N_x is finitely generated. Consider the global fixed point set $\mathrm{Fix}(N_x)$ of N_x. Since N_x is finitely generated, $\mathrm{Fix}(N_x)$ is a nonempty open set and if $y \in \mathrm{Fix}(N_x)$ and $z \prec y$, then we have $z \in \mathrm{Fix}\,(N_x)$. Let

$$V_x = \{z \in \mathrm{Fix}\,(N_x) \mid x \preceq z\}.$$

V_x is a nonempty connected set and if $z \in V_x$, we have $N_z = N_x$ by Lemma (4.9). Let

$$\mathscr{D} = \{x \in \mathscr{X} \mid N_{x_n} \neq N_x,\ \exists x_n \uparrow x\}.$$

Notice that since N is a normal subgroup, we have $N_{\gamma x} = \gamma N_x \gamma^{-1}$ for any $\gamma \in \Gamma$. In particular we have $\mathscr{D}$ is Γ-invariant. Choose $x \in \mathscr{D}$. Then by Lemma (4.6), we have $\gamma_n(x) \downarrow y$ for some $\gamma_n \in \Gamma$ and for some $y \in \mathscr{M}$. But this is absurd, since we have $N_z = N_y$ for any $z \in V_y$. This contradiction shows that $\mathscr{D}$ is empty.

Now it follows easily that $N_x = N_y$ for any x and $y \in \mathscr{X}$. This shows, via Lemma (4.8), that $N = N_x$ for any $x \in \mathscr{X}$. That is, N acts on $\mathscr{X}$ trivially. A contradiction. □

LEMMA (4.11). *There exists a point $y \in \mathscr{X}$ such that the isotropy subgroup Γ_y is not finitely generated.*

Proof. Let

$$\mathscr{Y} = \{y \in \mathscr{X} \mid N_y \text{ is not finitely generated}\}.$$

By the previous lemma, we have $\mathscr{Y} \neq \varnothing$. It is easy to show that $\mathscr{Y}$ is Γ-invariant. By Lemma (4.6), we obtain that $\mathrm{Cl}\,(\mathscr{Y})$ is an uncountable set. Consider a copy R of the real line properly embedded in $\mathscr{X}$. We shall use the coordinates of $\mathbf{R}$ in R. By the argument of Lemma (4.10), we know that if N_x is finitely generated for some $x \in R$, then there exists an $\epsilon > 0$ such that if $x < y < x + \epsilon$, then N_y is finitely generated. Therefore any component of $R \backslash \mathscr{Y}$ is of the form either $[a, b[$ or $]a, b[$. This shows that the set $(\mathrm{Cl}\,(\mathscr{Y}) \backslash \mathscr{Y}) \cap R$ is countable. That is, $\mathrm{Cl}\,(\mathscr{Y}) \backslash \mathscr{Y}$ is countable. Therefore $\mathscr{Y}$ is an uncountable set.

Consider the set

$$\mathscr{S} = \{x \in \mathscr{X} \mid N_x \neq \Gamma_x\}.$$

Γ is countable and for each $\gamma \in \Gamma \backslash N$, $\mathrm{Fix}\,(\gamma)$ is countable. Therefore $\mathscr{S}$ is countable. That is, there exists a point $y \in \mathscr{Y} \cap (\mathscr{X} \backslash \mathscr{S})$. □

Proof of Theorem 4. Let L be the leaf of $\mathscr{F}$ which corresponds to the point y of Lemma (4.11). Thus Γ_y and hence Π_y is not finitely generated. Let $\tilde{L}$ be the leaf of $\tilde{\mathscr{F}}$ which covers L. We have the following exact sequence.

$$1 \to \pi_1(\tilde{L}) \to \pi_1(L) \to \Pi_y \to 1.$$

Hence we have obtained that $\pi_1(L)$ is not finitely generated. This conclusion is derived from the hypothesis that $\mathscr{F}$ is not complete. Therefore if all the leaves of $\mathscr{F}$ have finitely generated fundamental groups, then $\mathscr{F}$ must be complete. □

5. First we shall construct an example, due to Nobuo Tsuchiya, of a non-complete real analytic foliation with a single Novikov component, on a closed 4-manifold whose fundamental group is solvable. Consider the 4-manifold $D^3 \times S^1$, Let $\mathscr{G}$ be the bundle foliation corresponding to the projection to the second factor. Let $f: S^1 \to \operatorname{Int}(D^3 \times S^1)$ be an embedding transverse to $\mathscr{G}$ and which winds twice along the S^1-direction. Let $N(f)$ be its tubular neighbourhood and let $P = D^3 \times S^1 \backslash \operatorname{Int}(N(f))$. Turbulize the foliation $\mathscr{G}|_P$ along the boundary ∂P as in Figure 5. Paste the two boundary components by some diffeomorphism to obtain a closed oriented 4-manifold M. Then we get a transversely oriented foliation $\mathscr{F}$ on M. We can do all this to obtain a transversely real analytic foliation. It is not difficult to show that the corresponding 1-manifold is of type V and looks like Figure 6. Also we have

$$\pi_1(M) = \langle H, T \mid H^{-1}T^2H = T \rangle.$$

That is, M has a solvable fundamental group.

Next we shall show that any group of orientation preserving homeomorphisms of $\mathbf{R}$ can be lifted up to an action on a 1-manifold of type V.

THEOREM 5. *For any group $\Gamma_{\mathbf{R}}$ of orientation preserving homeomorphisms of* $\mathbf{R}$, *there exists a* 1-*manifold $\mathscr{X}$ of type V, a group Γ which acts on $\mathscr{X}$, a submersion*

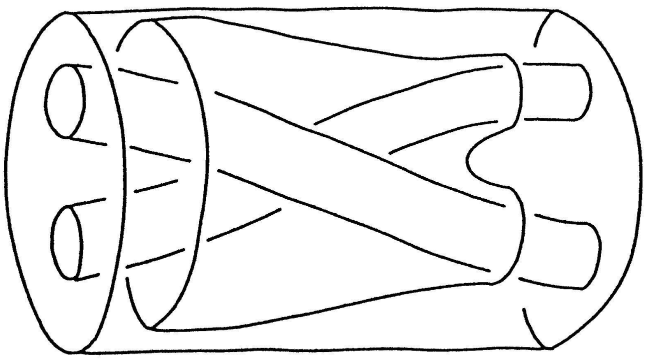

Figure 5

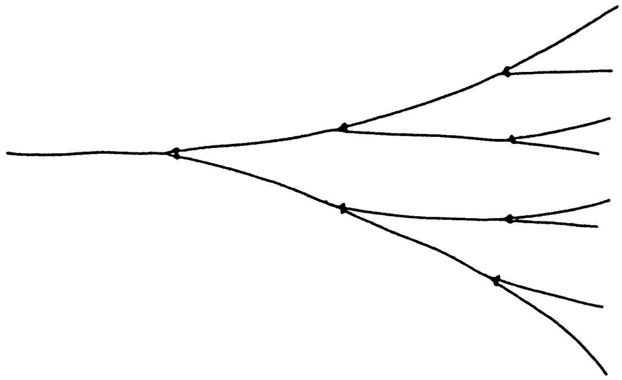

Figure 6

$D : \mathscr{X} \to \mathbf{R}$ and a homomorphism $\phi : \Gamma \to \mathbf{R}$ such that $D(\gamma x) = \phi(\gamma)D(x)$ ($\gamma \in \Gamma$, $x \in \mathscr{X}$).

Furthermore if $\Gamma_{\mathbf{R}}$ is finitely generated (resp. solvable, minimal on $\mathbf{R}$), then Γ is also finitely generated (resp. solvable, minimal on $\mathscr{X}$).

Proof. Fix a $\Gamma_{\mathbf{R}}$-orbit $\mathcal{O} \subset \mathbf{R}$. Define a $\mathbf{Z}/2\mathbf{Z}$ vector space Σ by

$$\Sigma = \{\sigma : \mathcal{O} \to \mathbf{Z}/2\mathbf{Z} \mid \sigma(x) = 0 \text{ except finite } x \in \mathcal{O}\}.$$

Define the left action of $\Gamma_{\mathbf{R}}$ on Σ by $(\gamma_*\sigma)(x) = \sigma(\gamma^{-1}x)$. Form the semidirect product $\Gamma = \Sigma \tilde{\times} \Gamma_{\mathbf{R}}$. The homomorphism $\phi : \Gamma \to \Gamma_{\mathbf{R}}$ is defined to be the canonical projection.

On the other hand, define an equivalence relation on $\Sigma \times \mathbf{R}$ by

$$(\sigma, x) \sim (\tau, y) \Leftrightarrow x = y, \quad \sigma = \tau \text{ on }]-\infty, x[.$$

The quotient space $\mathscr{X}$ is obviously a 1-manifold of type V. Γ acts on $\mathscr{X}$ by

$$(\sigma, \gamma)(\tau, x) = (\sigma + \gamma_*\tau, \gamma(x)).$$

Define $D : \mathscr{X} \to \mathbf{R}$ by $D(\sigma, x) = x$. All the required properties are easy to establish. □

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