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Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space

SHIN NAYATANI*

Dedicated to Professor Tadashi Nagano on his 60th birthday

1. Introduction

The index of a complete noncompact minimal surface in a Riemannian manifold is defined as the limit of indices of an increasing and exhausting sequence of compact domains in the surface. Fischer–Colbrie [6] and Gulliver–Lawson [7], [8] proved that a complete oriented minimal surface M in $\mathbf{R}^3$ has finite index if and only if it has finite total curvature. Fischer–Colbrie also observed that if M has finite total curvature its index coincides with the index of an operator associated to the extended Gauss map of M. The first quantitative study of this invariant was done by Tysk [13], who proved that the index of a complete oriented minimal surface in $\mathbf{R}^3$ is bounded from above by an explicit constant times the total curvature. Ejiri–Micallef [5] have also obtained an upper bound for the index. On the other hand, Choe [1] and the present author [11] have studied the lower bound for the index.

In this paper we shall study the index and the nullity of an operator L_G associated to an arbitrary holomorphic map $G : \Sigma \to S^2$, where Σ is a compact Riemann surface. We first consider a certain deformation $G_t : \Sigma \to S^2$, $0 < t < \infty$, with $G_1 = G$ and study the behavior of eigenvalues of L_{G_t} as t tends to zero or infinity (Theorem 1). We then give lower and upper bounds for the index and the nullity of L_{G_t} when t is sufficiently small or sufficiently large (Theorem 2). We point out here the works of Ejiri–Kotani [4] and Montiel–Ros [10], who have proved, among other things, that a function in the kernel of L_G is expressed as the support function of a complete branched minimal surface with planer ends whose extended Gauss map is G. Using this and our Theorem 2, we can give lower and upper

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bounds for the index and the nullity of L_G (Theorems 3, 4). Finally we compute the index and the nullity of L_{G_t} for all t when the meromorphic function associated to G is the derivative of the Weierstrass $\wp$-function for the unit square lattice (Theorem 5). In particular, we can determine the index of the Costa's surface to be five.

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2. Preliminaries

Let Σ be a compact Riemann surface and $G : \Sigma \to S^2$ a nonconstant holomorphic map, where S^2 is the unit sphere in $\mathbf{R}^3$ endowed with the complex structure induced by the stereographic projection from the north pole. We fix a conformal metric ds^2 on Σ and consider the operator $L = -\Delta - |dG|^2$, acting on functions on Σ, where Δ is the Laplace–Beltrami operator with respect to ds^2. We denote by Q the quadratic form associated to L. Thus for a function u on Σ

$$Q(u, u) = \int_\Sigma (|du|^2 - |dG|^2 u^2)\, dA,$$

where dA is the area element with respect to ds^2. We note that Q is independent of the particular choice of metric on Σ.

We now define $\mathrm{Ind}\,(G)$, the index of G, as the number of negative eigenvalues (counted with multiplicities) of L. It can also be defined as the dimension of a maximal supspace of $H^1(\Sigma)$ on which Q is negative definite. This latter definition justifies our notation. The kernel of L, $N(G) = \{u \in C^\infty(\Sigma) \mid Lu = 0\}$, is also an invariant of G. We define $\mathrm{Nul}\,(G)$, the nullity of G, as the dimension of $N(G)$. We note that $L(G) = \{a \cdot G \mid a \in \mathbf{R}^3\}$ is a three dimensional subspace of $N(G)$ and so $\mathrm{Nul}\,(G) \geq 3$.

We now consider on Σ the metric ds_G^2 induced by G from S^2. Thus $ds_G^2 = \frac{1}{2}|dG|^2\, ds^2$. This metric is singular precisely at the ramification points of G. For this choice of metric, the operator L becomes $L_G = -\Delta_G - 2$, where Δ_G is the Laplace–Beltrami operator with respect to ds_G^2. The eigenvalue problem for L_G can be solved via a standard variational approach. Hence if λ is an eigenvalue of L_G, its corresponding eigenspace is given by

$$V_\lambda(G) = \left\{ u \in H^1(\Sigma) \,\middle|\, Q(u, v) = \lambda \int_\Sigma uv\, dA_G \text{ for all } v \in H^1(\Sigma) \right\},$$

where $dA_G = \frac{1}{2}|dG|^2\, dA$ is the area element with respect to ds_G^2. By elliptic regularity,

$V_\lambda(G) \subset C^\infty(\Sigma)$. We point out the following variational characterization of the eigenvalues of L_G. For a function $u \neq 0$ on Σ we define

$$R_G(u) = Q(u, u) \Big/ \int_\Sigma u^2 \, dA_G.$$

Then the k-th eigenvalue (counted with multiplicities) $\lambda_k(G)$ is characterized by

$$\lambda_k(G) = \inf_{V_k} \sup \{R_G(u) \mid u \in V_k, u \neq 0\}, \tag{1}$$

where V_k runs through k-dimensional subspaces of $H^1(\Sigma)$. We note that $\mathrm{Ind}\,(G)$ coincides with the number of negative eigenvalues of L_G and $N(G)$ is nothing but $V_0(G)$.

We now let M be a complete oriented minimal surface in $\mathbf{R}^3$ with finite total curvature. By a theorem of Osserman [12], M is conformally equivalent to a compact Riemann surface with finitely many punctures and the Gauss map $G : M \to S^2$ extends to the compactified surface as a holomorphic map. As mentioned in the introduction, Fischer–Colbrie [6] showed that the index of M coincides with the index of the extended Gauss map.

3. Deformation of a holomorphic map and the index

Let Σ be a compact Riemann surface and $G : \Sigma \to S^2$ a nonconstant holomorphic map of degree d. We define a one-parameter group of conformal diffeomorphism $\mathscr{A}_t$, $t \in (0, \infty)$, of S^2 by

$$\Pi \circ \mathscr{A}_t \circ \Pi^{-1}(w) = tw, \qquad w \in \bar{\mathbf{C}},$$

where $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and $\Pi : S^2 \to \bar{\mathbf{C}}$ is the stereographic projection from the north pole. Let $G_t = \mathscr{A}_t \circ G$, $t \in (0, \infty)$. If g is the meromorphic function associated to G, that is, $g = \Pi \circ G$, then we have $\Pi \circ G_t = tg$. Since the correspondence $t \mapsto G_t$ is continuous with respect to the usual C^1-topology, it can be shown, using (1), that $\lambda_k(G_t)$ is continuous in t. A naturally arising question here is:

How does $\lambda_k(G_t)$ behave as t tends to zero or infinity?

An answer to this question is given by Theorem 1 below.

Let $P(G) = m_1 p_1 + \cdots + m_\nu p_\nu$ be the polar divisor of g, where p_i, $i = 1, \ldots, \nu$, are distinct. Note that $m_1 + \cdots + m_\nu = d$. For $i = 1, \ldots, \nu$ we define a holomorphic map $\tilde{G}_i : \bar{\mathbf{C}}_i \to S^2$ by $\Pi \circ \tilde{G}_i(z) = z^{m_i}$, where $\bar{\mathbf{C}}_i$ is a copy of $\bar{\mathbf{C}}$. Let $\tilde{\Sigma}$ be the disjoint union of $\bar{\mathbf{C}}_i$, $i = 1, \ldots, \nu$, and $\tilde{G} : \tilde{\Sigma} \to S^2$ the holomorphic map defined by $\tilde{G}(z) = \tilde{G}_i(z)$ if $z \in \bar{\mathbf{C}}_i$.

THEOREM 1. *Let* $G_t : \Sigma \to S^2$, $t \in (0, \infty)$, *and* $\tilde{G} : \tilde{\Sigma} \to S^2$ *be as above. Then for* $k = 1, 2, \dots$

$$\lim_{t \to 0} \lambda_k(G_t) = \lambda_k(\tilde{G}).$$

REMARK 1. Let $G^* : \Sigma \to S^2$ be the holomorphic map defined by $\Pi \circ G^* = 1/g$. It is then easy to verify that $G^* = PG$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since $P \in O(3)$, we have $ds^2_{G^*} = ds^2_G$. Hence the eigenvalues of L_{G^*} coincide with those of L_G. In particular,

$$\mathrm{Ind}\,(G^*) = \mathrm{Ind}\,(G) \quad \text{and} \quad \mathrm{Nul}\,(G^*) = \mathrm{Nul}\,(G). \tag{2}$$

This observation applied to G_t enables one to deduce from Theorem 1 the behavior of $\lambda_k(G_t)$ as t tends to infinity, which we omit.

Proof of Theorem 1. Let $w_1, \dots, w_l$ be all the branching values of g other than 0 and ∞. Let $\Gamma = \bigcup_{j=1}^{l} \{sw_j \mid 0 \le s \le 1\}$. Then $\Omega = g^{-1}(\bar{\mathbf{C}} - \Gamma)$ has precisely ν components, each of which contains exactly one of the points p_i, $i = 1, \dots, \nu$. We denote by Ω_i the component which contains p_i. Let $\Gamma_i = \tilde{g}_i^{-1}(\Gamma)$, where $\tilde{g}_i = \Pi \circ \tilde{G}_i$. Then we have a biholomorphic map Ψ_i from Ω_i onto $\bar{\mathbf{C}}_i - \Gamma_i$ such that

$$\tilde{G}_i \circ \Psi_i = G. \tag{3}$$

Such Ψ_i is constructed by composing suitably the branches of m_i-valued analytic function $\tilde{g}_i^{-1}$ with g. By (3) Ψ_i gives an isometry between (Ω_i, ds^2_G) and $(\bar{\mathbf{C}}_i - \Gamma_i, ds^2_{\tilde{G}_i})$. We carry out the similar construction with g replaced by tg and obtain Γ_t, $\Gamma_{t,i}$ and $\Psi_{t,i} : (\Omega_i, ds^2_{G_t}) \to (\bar{\mathbf{C}}_i - \Gamma_{t,i}, ds^2_{\tilde{G}_i})$ corresponding to Γ, Γ_i and Ψ_i respectively. Note that $\Gamma_t = t\Gamma$ and $\Gamma_{t,i} = t^{1/m_i}\Gamma_i$. Let $\tilde{\Sigma}_t$ $(\subset \tilde{\Sigma})$ be the disjoint union of $\bar{\mathbf{C}}_i - \Gamma_{t,i}$, $i = 1, \dots, \nu$, and $\Psi_t : (\Omega, ds^2_G) \to (\tilde{\Sigma}_t, ds^2_{\tilde{G}})$ the isometry defined by $\Psi_t(p) = \Psi_{t,i}(p)$ if $p \in \Omega_i$. Thus we obtain the following diagram.

$$\begin{array}{ccc} \Omega & \xrightarrow{\Psi_t} & \tilde{\Sigma}_t \\ \cap & & \cap \\ (\Sigma, ds^2_{G_t}) & & (\tilde{\Sigma}, ds^2_{\tilde{G}}) \end{array}$$

Let $\lambda_k^D(t)$ (resp. $\lambda_k^N(t)$) denote the k-th eigenvalue of the Dirichlet (resp. Neumann) eigenvalue problem for L_{G_t} on Ω. They are characterized variationally as follows. For a function $u \neq 0$ on Ω we define

$$R_{G_t,\Omega}(u) = \int_\Omega (|du|^2 - 2u^2)\, dA_{G_t} \Big/ \int_\Omega u^2\, dA_{G_t}.$$

Then we have for $* = D, N$

$$\lambda_k^*(t) = \inf_{V_k} \sup \{R_{G_t,\Omega}(u) \mid u \in V_k, u \neq 0\}, \tag{4}$$

where V_k runs through k-dimensional subspaces of $H_0^1(\Omega)$ (resp. $H^1(\Omega)$) if $* = D$ (resp. N). It follows from (4) that $\lambda_k^*(t)$, $* = D, N$, are continuous in t.

Theorem 1 follows from (a) and (c) of the following lemma.

LEMMA 1. *With the above notations, we have for* $k = 1, 2, \ldots$
(a) $\lambda_k^N(t) \leq \lambda_k(G_t) \leq \lambda_k^D(t)$, $t \in (0, \infty)$;
(b) $\lambda_k^D(t)$ *(resp.* $\lambda_k^N(t)$*) is monotonically non-decreasing (resp. non-increasing) in* t;
(c) $\lim_{t\to 0} \lambda_k^*(t) = \lambda_k(\tilde{G})$ *for* $* = D, N$.

Proof of Lemma 1. Clearly $H_0^1(\Omega) \subset H^1(\Sigma) \subset H^1(\Omega)$. This fact together with (1) applied to G_t and (4) proves (a). We next prove (b). Since $(\Omega, ds^2_{G_t})$ is isometric to $(\tilde{\Sigma}_t, ds^2_{\tilde{G}})$, (4) can be rewritten as follows. For a function $u \neq 0$ on $\tilde{\Sigma}_t$ let

$$R_{\tilde{G},\tilde{\Sigma}_t}(u) = \int_{\tilde{\Sigma}_t} (|du|^2 - 2u^2)\, dA_{\tilde{G}} \Big/ \int_{\tilde{\Sigma}_t} u^2\, dA_{\tilde{G}}.$$

Then for $* = D, N$

$$\lambda_k^*(t) = \inf_{V_k} \sup \{R_{\tilde{G},\tilde{\Sigma}_t}(u) \mid u \in V_k, u \neq 0\}, \tag{5}$$

where V_k runs through k-dimensional subspaces of $H_0^1(\tilde{\Sigma}_t)$ (resp. $H^1(\tilde{\Sigma}_t)$) if $* = D$ (resp. N). Moreover, it is easy to see that for all $t \in (0, \infty)$

$$H_0^1(\tilde{\Sigma}_t) \subset H^1(\tilde{\Sigma}), \qquad H^1(\tilde{\Sigma}_t) \supset H^1(\tilde{\Sigma}), \tag{6}$$

and if $t < t'$ then

$$H_0^1(\tilde{\Sigma}_t) \supset H_0^1(\tilde{\Sigma}_{t'}), \qquad H^1(\tilde{\Sigma}_t) \subset H^1(\tilde{\Sigma}_{t'}). \tag{7}$$

(b) now follows from (5) and (7). It also follows using (1) applied to $\tilde{G}$, (5) and (6) that

$$\lambda_k^N(t) \leq \lambda_k(\tilde{G}) \leq \lambda_k^P(t), \qquad t \in (0, \infty). \tag{8}$$

To prove (c) we first establish the following facts:

$$H^1(\tilde{\Sigma}) = \overline{\bigcup_{0<t<\infty} H_0^1(\tilde{\Sigma}_t)}; \tag{9}$$

$$H^1(\tilde{\Sigma}) = \bigcap_{0<t<\infty} H^1(\tilde{\Sigma}_t). \tag{10}$$

In order to prove (9), it is sufficient to show

$$H^1(\bar{\mathbf{C}}_i) = \overline{\bigcup_{0<t<\infty} H_0^1(\bar{\mathbf{C}}_i - \Gamma_{t,i})}$$

for $i = 1, \ldots, \nu$. For each ε with $0 < \varepsilon < 1$, let $\varphi_\varepsilon : \bar{\mathbf{C}}_i \to [0, 1]$ be a Lipschitz cut off function defined as follows:

$$\varphi_\varepsilon(w) = \begin{cases} 0 & \text{if } 0 \leq |w| < \varepsilon^2, \\ (\log \varepsilon^2 - \log |w|)/\log \varepsilon & \text{if } \varepsilon^2 \leq |w| \leq \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to verify that there is a constant C, independent of ε, such that

$$\int_{\bar{\mathbf{C}}_i} |d\varphi_\varepsilon|^2 \, dA \leq C |\log \varepsilon|^{-1}, \tag{11}$$

where dA is the area element for an arbitrary metric on $\bar{\mathbf{C}}_i$. Let $u \in H^1(\bar{\mathbf{C}}_i)$. Then $\varphi_\varepsilon u \in H_0^1(\bar{\mathbf{C}}_i - \Gamma_{t,i})$ for all sufficiently small t. Moreover, it follows from (11) that $\varphi_\varepsilon u$ converges to u in $H^1(\bar{\mathbf{C}}_i)$ as ε tends to zero. Hence $u \in \overline{\bigcup_{0<t<\infty} H_0^1(\bar{\mathbf{C}}_i - \Gamma_{t,i})}$. This shows the inclusion $H^1(\bar{\mathbf{C}}_i) \subset \overline{\bigcup_{0<t<\infty} H_0^1(\bar{\mathbf{C}}_i - \Gamma_{t,i})}$. The reverse inclusion is obvious and thus (9) is proved. The proof of (10) is similar.

Once (9) and (10) are established, the proof of (c) is standard. But we give it for completeness. In view of (b) and (8), $\lim_{t\to 0} \lambda_k^*(t)$, $* = D, N$, exist and $\lim_{t\to 0} \lambda_k^N(t) \leq \lambda_k(\tilde{G}) \leq \lim_{t\to 0} \lambda_k^P(t)$. Hence it suffices to prove

$$\lim_{t\to 0} \lambda_k^P(t) \leq \lambda_k(\tilde{G}), \tag{12}$$

and

$$\lim_{t\to 0} \lambda_k^N(t) \geq \lambda_k(\tilde{G}). \tag{13}$$

To prove (12) we take $u_1, \ldots, u_k \in H^1(\tilde{\Sigma})$ so that u_i is an eigenfunction of $L_{\tilde{G}}$ corresponding to $\lambda_i(\tilde{G})$ and

$$\int_{\tilde{\Sigma}} u_i u_j \, dA_{\tilde{G}} = \delta_{ij}, \qquad i, j = 1, \ldots, k.$$

Let V be the linear span of $u_1, \ldots, u_k$. By (9), for any $\varepsilon > 0$, there exist $v_1, \ldots, v_k \in \bigcup_{0 < t < \infty} H_0^1(\tilde{\Sigma}_t)$ such that

$$\|u_i - v_i\|_{H^1} \leq \varepsilon, \qquad i = 1, \ldots, k, \tag{14}$$

where $\| \cdot \|_{H^1}$ is the H^1-norm on $H^1(\tilde{\Sigma})$ defined in terms of an arbitrary metric on $\tilde{\Sigma}$. If ε is sufficiently small, $v_1, \ldots, v_k$ are linearly independent. Moreover, $v_i \in H_0^1(\tilde{\Sigma}_t)$, $i = 1, \ldots, k$, for all sufficiently small t. For such t, let V_t be the k-dimensional subspace of $H_0^1(\tilde{\Sigma}_t)$ spanned by $v_1, \ldots, v_k$. Then, by (5) and (14),

$$\begin{aligned}
\lambda_k^D(t) &\leq \sup \{R_{\tilde{G}, \tilde{\Sigma}_t}(v) \mid v \in V_t, v \neq 0\} \\
&\leq \sup\{R_{\tilde{G}}(u) \mid u \in V, u \neq 0\} + C(\varepsilon) \\
&= \lambda_k(\tilde{G}) + C(\varepsilon),
\end{aligned}$$

where $C(\varepsilon)$ is a constant depending only on ε such that $\lim_{\varepsilon \to 0} C(\varepsilon) = 0$. Thus (12) is proved.

To prove (13) we first note that, for each t, the H^1-norm on $H^1(\tilde{\Sigma}_t)$ defined in terms of the singular metric $ds_{\tilde{G}}^2$ is an admissible norm on $H^1(\tilde{\Sigma}_t)$. For each t we take $u_{1,t}, \ldots, u_{k,t} \in H^1(\tilde{\Sigma}_t)$ so that $u_{i,t}$ is a Neumann eigenfunction of $L_{\tilde{G}}$ on $\tilde{\Sigma}_t$ corresponding to $\lambda_i^N(t)$ and

$$\int_{\tilde{\Sigma}_t} u_{i,t} u_{j,t} \, dA_{\tilde{G}} = \delta_{ij}, \qquad i, j = 1, \ldots, k.$$

Let V_k be the linear span of $u_{1,t}, \ldots, u_{k,t}$. By (8) $\{u_{i,t}\}_{0 < t < \infty}$ is bounded with respect to the H^1-norm. Hence we can find, using the diagonal argument, a sequence $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} t_n = 0$ such that u_{i,t_n} converges to some $u_i \in \bigcap_{0 < t < \infty} H^1(\tilde{\Sigma}_t)$ L^2-strongly and weakly in $H^1(\tilde{\Sigma}_t)$ for all t. By (10), $u_i \in H^1(\tilde{\Sigma})$. The L^2-strong convergence implies that

$$\int_{\tilde{\Sigma}} u_i u_j \, dA_{\tilde{G}} = \delta_{ij}.$$

Let V be the k-dimensional subspace of $H^1(\tilde{\Sigma})$ spanned by $u_1, \ldots, u_k$. For

$u = \Sigma_{i=1}^{k} a_i u_i \in V$, let $u_n = \Sigma_{i=1}^{k} a_i u_{i,t_n}$. Then u_n converges to u L^2-strongly and weakly in $H^1(\tilde{\Sigma}_t)$ for all t. By the lower semicontinuity of the H^1-norm with respect to the weak convergence, we have

$$R_{\tilde{G}}(u) \leq \liminf_{n\to\infty} R_{\tilde{G},\tilde{\Sigma}_{t_n}}(u_n).$$

Hence by (1) applied to $\tilde{G}$

$$\begin{aligned}\lambda_k(\tilde{G}) &\leq \sup\{R_{\tilde{G}}(u) \mid u \in V, u \neq 0\}\\ &\leq \liminf_{n\to\infty} \sup\{R_{\tilde{G},\tilde{\Sigma}_{t_n}}(u) \mid u \in V_{t_n}, u \neq 0\}\\ &= \lim_{n\to\infty} \lambda_k^N(t_n),\end{aligned}$$

which establishes (13). Thus Lemma 1 is proved and the proof of Theorem 1 is complete.

We now recall that $\operatorname{Ind}(\tilde{G}_i) = 2m_i - 1$ and $\operatorname{Nul}(\tilde{G}_i) = 3$, $i = 1, \ldots, v$ (see [11]). Hence we have

$$\operatorname{Ind}(\tilde{G}) = 2d - v \quad \text{and} \quad \operatorname{Nul}(\tilde{G}) = 3v. \tag{15}$$

THEOREM 2. *Let $G : \Sigma \to S^2$ be a nonconstant holomorphic map of degree d and $G_t = \mathscr{A}_t \circ G$, $t \in (0, \infty)$. Let v be the number of distinct poles of $g = \Pi \circ G$. Then the following estimates hold for all sufficiently small t:*

$$\operatorname{Ind}(G_t) \geq 2d - v; \tag{16}$$

$$\operatorname{Ind}(G_t) + \operatorname{Nul}(G_t) \leq 2d + v + 1; \tag{17}$$

$$\operatorname{Nul}(G_t) \leq 2v + 1. \tag{18}$$

In particular, if $v = 1$, then we have

$$\operatorname{Ind}(G_t) = 2d - 1 \quad \text{and} \quad \operatorname{Nul}(G_t) = 3$$

for all sufficiently small t.

REMARK 2. It should be mentioned that the estimate (16) has been obtained in [10].

REMARK 3. Let ν' be the number of distinct zeros of g. Then, by (2), the estimates in Theorem 2 with ν replaced by ν' hold for all sufficiently large t.

Proof of Theorem 2. (16) is an immediate consequence of (15) and Theorem 1. (18) and the last assertion follow from (16) and (17). We now prove (17). We shall use the notations in the proof of Theorem 1. Recall that $u_t = (0, 0, 1)^t \cdot G_t$ is an element of $N(G_t)$. Let

$$V_t = \bigoplus_{\lambda < 0} V_\lambda(G_t) \oplus \left\{ u \in N(G_t) \,\middle|\, \int_\Sigma u u_t \, dA_{G_t} = 0 \right\}.$$

Let $u_{t,i} = u_t|_{\Omega_t}$ and denote by $V_\lambda^{(i)}(G_t)$ the eigenspace corresponding to the eigenvalue λ of the Neumann eigenvalue problem for L_{G_t} on Ω_i. Let $\tilde{u}_i = (0, 0, 1)^t \cdot \tilde{G}_i$. Then $u_{t,i} = \tilde{u}_i \circ \Psi_{t,i}$. Moreover, since $\tilde{u}_i(z) = (|z|^{2m_i} - 1)/(|z|^{2m_i} + 1)$, $z \in \bar{\mathbf{C}}_i$, $\tilde{u}_i$ is a radial function on $\bar{\mathbf{C}}_i$ and hence satisfies the Neumann condition on $\Gamma_{t,i}$ for all t. Therefore $u_{t,i} \in V_0^{(i)}(G_t)$. Let

$$W_{t,i} = \bigoplus_{\lambda < 0} V_\lambda^{(i)}(G_t) \oplus \left\{ u \in V_0^{(i)}(G_t) \,\middle|\, \int_{\Omega_t} u u_{t,i} \, dA_{G_t} = 0 \right\}.$$

If $u \in H^1(\Omega_i)$ is orthogonal to $W_{t,i}$ in $L^2(\Omega_i, dA_{G_t})$, then

$$\int_{\Omega_t} (|du|^2 - 2u^2) \, dA_{G_t} \geq 0$$

and the equality holds if and only if $u = a u_{t,i}$ for some $a \in \mathbf{R}$. By (15) and Lemma 1 (b) (c), there exists $t_0 > 0$ such that $\Sigma_{i=1}^{\nu} \dim W_{t,i} = 2d + \nu$ for all $t < t_0$. We now show that $\dim V_t \leq 2d + \nu$ for all $t < t_0$, which clearly implies (17). Suppose that $\dim V_t \geq 2d + \nu + 1$ for some $t < t_0$. Then we can find $u \in V_t - \{0\}$ such that $u|_{\Omega_t}$ is orthogonal to $W_{t,i}$ in $L^2(\Omega_i, dA_{G_t})$ for $i = 1, \ldots, \nu$. Thus

$$\begin{aligned} 0 &\geq \int_\Sigma (|du|^2 - 2u^2) \, dA_{G_t} \\ &= \sum_{i=1}^{\nu} \int_{\Omega_t} (|du|^2 - 2u^2) \, dA_{G_t} \geq 0. \end{aligned}$$

Hence we must have

$$\int_\Sigma (|du|^2 - 2u^2) \, dA_{G_t} = 0,$$

and

$$\int_{\Omega_i} (|du|^2 - 2u^2)\, dA_{G_t} = 0, \qquad i = 1, \ldots, \nu.$$

Therefore $u \in N(G_t)$ and $u|_{\Omega_i} = a_i u_{t,i}$ for some $a_i \in \mathbf{R}$, $i = 1, \ldots, \nu$. By the unique continuation principle, $u = au_t$ for some $a \in \mathbf{R}$, a contradiction. Thus (17) is proved and the proof of Theorem 2 is complete.

4. Lower and upper bounds for the index

Let Σ be a compact Riemann surface and $G : \Sigma \to S^2$ a nonconstant holomorphic map. In this section we shall study lower and upper bounds for Ind (G) and Nul (G).

We first review briefly a result of Ejiri–Kotani [4] and Montiel–Ros [10]. They have proved that a nonlinear element of $N(G)$ (that is, an element of $N(G)$ which does not lie in $L(G)$) is expressed as the support function of a complete branched minimal surface with planar ends whose extended Gauss map is G. Using the Weierstrass representation, their result can be stated as follows (see [10]): Let g be the meromorphic function associated to G. Let $P(G)$ and $B(G) = e_1 p_1 + \cdots + e_\mu p_\mu$ be the polar and ramification divisors of g respectively, where e_i is the ramification index of g at p_i, that is, the multiplicity with which g takes its value at p_i. We define a divisor $D(G)$ on Σ by $D(G) = B(G) - 2P(G)$ and a vector space $H(G)$ by

$$H(G) = \left\{ \omega \in H^0(K(\Sigma) + D(G)) \;\middle|\; \operatorname{Res}_{p_i} \omega = 0, 1, \ldots, \mu, \right.$$
$$\left. \operatorname{Re} \int_\alpha (1 - g^2, i(1 + g^2), 2g)\omega = 0 \text{ for all } \alpha \in H_1(\Sigma, \mathbf{Z}) \right\},$$

where $K(\Sigma)$ is the canonical divisor of Σ. For $\omega \in H(G)$ let $X(\omega) : \Sigma - \{p_1, \ldots, p_\mu\} \to \mathbf{R}^3$ be the conformal harmonic map defined by

$$X(\omega)(p) = \operatorname{Re} \int^p (1 - g^2, i(1 + g^2), 2g)^t \omega.$$

Then $X(\omega) \cdot G$, the support function of $X(\omega)$, extends over to $p_1, \ldots, p_\mu$ smoothly and thus gives an element of $N(G)$. Conversely, every element of $N(G)$ is obtained in this way. In fact, the map $\imath : H(G) \to N(G)/L(G)$ defined by $\imath(\omega) = [X(\omega) \cdot G]$, the class containing $X(\omega) \cdot G$, is an isomorphism.

We define a complex vector space $\hat{H}(G)$ by

$$\hat{H}(G) = \{\omega \in H^0(K(\Sigma) + D(G)) \mid \mathrm{Res}_{p_i}\, \omega = 0, i = 1, \ldots, \mu\}.$$

If the genus of Σ is zero, then $H_1(\Sigma, \mathbf{Z}) = \{0\}$ and so $\hat{H}(G) = H(G)$. Let $G_t = \mathscr{A}_t \circ G$, $t \in (0, \infty)$. It is clear that $\hat{H}(G_t) = \hat{H}(G)$. Hence, if the genus of Σ is zero, $H(G_t) = H(G)$ and therefore

$$\mathrm{Nul}\,(G_t) = \mathrm{Nul}\,(G) \qquad \text{for all } t \in (0, \infty). \tag{19}$$

Using this and Theorem 2, we can prove the following

THEOREM 3. *Let $G : \bar{\mathbf{C}} \to S^2$ be a nonconstant holomorphic map of degree d and $\nu = \nu(G)$ the minimal number of distinct points in $G^{-1}(q)$ when q runs over S^2. Then we have*

$$\mathrm{Ind}\,(G) \geq 2d - \nu, \tag{20}$$

$$\mathrm{Ind}\,(G) + \mathrm{Nul}\,(G) \leq 2d + \nu + 1, \tag{21}$$

and

$$\mathrm{Nul}\,(G) \leq 2\nu + 1. \tag{22}$$

In particular, if $\nu(G) = 1$ then

$$\mathrm{Ind}\,(G) = 2d - 1 \quad \textit{and} \quad \mathrm{Nul}\,(G) = 3.$$

Proof. Since the composition of a rotation of S^2 and G does not affect the metric ds_G^2 and hence $\mathrm{Ind}\,(G)$ and $\mathrm{Nul}\,(G)$, we may assume without loss of generality that ν is the number of distinct poles of $g = \Pi \circ G$. Let $G_t = \mathscr{A}_t \circ G$, $t \in (0, \infty)$. Then by Theorem 2 we have

$$\mathrm{Ind}\,(G_t) \geq 2d - \nu \qquad \text{for all sufficiently small } t. \tag{23}$$

By (19) and the continuity of $\lambda_k(G_t)$,

$$\mathrm{Ind}\,(G_t) = \mathrm{Ind}\,(G) \qquad \text{for all } t. \tag{24}$$

(20) follows from (23) and (24). (21) follows in a similar way. (22) and the last assertion follow from (20) and (21).

REMARK 4. In [4] and [10], it has been proved that $\operatorname{Ind}(G) \leq 2d-1$ for any nonconstant holomorphic map $G: \bar{\mathbf{C}} \to S^2$ of degree d and the equality holds for a generic G.

EXAMPLE. Let m and n be positive integers with $m \geq 2$. Let $G: \bar{\mathbf{C}} \to S^2$ be the holomorphic map of degree $d = m + n$ defined by

$$\Pi \circ G(z) = z^m + \frac{1}{z^n}.$$

The divisor $D(G)$ is given by

$$D(G) = \begin{cases} 2z_1 + \cdots + 2z_d - m \cdot \infty - 2 \cdot 0 & \text{if } n = 1, \\ 2z_1 + \cdots + 2z_d - m \cdot \infty - n \cdot 0 & \text{if } n \geq 2, \end{cases}$$

where

$$z_j = \sqrt[d]{\frac{n}{m}}\, e^{(2\pi i/d)j}, \qquad j = 1, \ldots, d.$$

It is easy to see that the meromorphic differential

$$\omega = \frac{z^{n+1}}{(z - z_1) \cdots (z - z_d)} \left(\frac{z_1^{m-1}}{z - z_1} + \cdots + \frac{z_d^{m-1}}{z - z_d} \right) dz$$

is an element of $H^0(K(\bar{\mathbf{C}}) + D(G))$. Moreover, it can be shown, using the identity $z_1^k + \cdots + z_d^k = 0$, $k = 1, \ldots, m-2$, that $\operatorname{Res}_{z_j} \omega = 0$, $j = 1, \ldots, d$. Hence $\omega \in H(G)$. Since $H(G)$ is a complex vector space, we obtain

$$\operatorname{Nul}(G) = 3 + \dim_{\mathbf{R}} H(G) \geq 5.$$

On the other hand, we have $v(G) = 2$. Therefore we can conclude from (20) and (21) that

$$\operatorname{Ind}(G) = 2d - 2 \quad \text{and} \quad \operatorname{Nul}(G) = 5.$$

In the following theorem we give lower and upper bounds for $\operatorname{Ind}(G)$ and $\operatorname{Nul}(G)$ in terms of the degree of G and the genus of Σ.

THEOREM 4. *Let $G: \Sigma \to S^2$ be a holomorphic map of degree $d \geq 2$. Then*

$$\operatorname{Ind}(G) \geq d - 3\gamma + 1, \tag{25}$$

and

$$\mathrm{Ind}\,(G) + \mathrm{Nul}\,(G) \leq 3d + 3\gamma, \tag{26}$$

where γ is the genus of Σ.

In order to prove this theorem we need the following

LEMMA 2. *Let $G_t = \mathcal{A}_t \circ G$, $t \in (0, \infty)$. There exists an integer n_0 (≥ 3) such that* $\mathrm{Nul}\,(G_t) \geq n_0$ *for all t and the equality holds except for a finite number of values of t. Moreover, if we let $n_- = \Sigma_{t \leq 1}(\mathrm{Nul}\,(G_t) - n_0)$, $n_+ = \Sigma_{t \geq 1}(\mathrm{Nul}\,(G_t) - n_0)$ and $n = \min\,(n_-, n_+)$, then*

$$n \leq 3\gamma, \tag{27}$$

and

$$n = 3\gamma \qquad \text{if and only if } n_- = n_+ = 3\gamma. \tag{28}$$

Proof of Theorem 4. Let v be the number of distinct poles of $g = \Pi \circ G$. We may assume without loss of generality that ∞ is a branching value of g and so $v \leq d - 1$. Let $G_t = \mathcal{A}_t \circ G$, $t \in (0, \infty)$. By Theorem 2 and Remark 3, if t_- is sufficiently small and t_+ is sufficiently large then $\mathrm{Ind}\,(G_{t_-}) \geq d + 1$ and $\mathrm{Ind}\,(G_{t_+}) \geq d$. It is easy to see from the continuity of $\lambda_k(G_t)$ that

$$\mathrm{Ind}\,(G) \geq \max\,(\mathrm{Ind}\,(G_{t_-}) - n_-, \mathrm{Ind}\,(G_{t_+}) - n_+).$$

By Lemma 2 the right-hand side can be estimated from below by $d - (3\gamma - 1)$ if $n \leq 3\gamma - 1$ and by $(d + 1) - 3\gamma$ if $n = 3\gamma$. Thus (25) is proved. The proof of (26) is similar.

Proof of Lemma 2. We first note that, since $\hat{H}(G_t) = \hat{H}(G)$,

$$H(G_t) = \left\{\omega \in \hat{H}(G) \,\middle|\, \mathrm{Re} \int_\alpha (1 - t^2 g^2, i(1 + t^2 g^2), 2tg)\omega = 0 \right.$$
$$\left. \text{for all } \alpha \in H_1(\Sigma, \mathbf{Z})\right\}.$$

Let $\{\omega_1, \ldots, \omega_n\}$ be a complex basis of $\hat{H}(G)$ and $\{\alpha_1, \ldots, \alpha_{2\gamma}\}$ a basis of

$H_1(\Sigma, \mathbf{Z})$. Let

$$s_{ij} = \int_{\alpha_i} \omega_j, \qquad t_{ij} = \int_{\alpha_i} g^2\omega_j \quad \text{and} \quad u_{ij} = \int_{\alpha_i} g\omega_j.$$

For a complex number z, we denote by $z^{(1)}$ and $z^{(2)}$ the real and imaginary parts of z respectively. Take $\omega \in \hat{H}(G)$. Then $\omega \in H(G_t)$ if and only if

$$\int_{\alpha_i} \omega = t^2 \overline{\int_{\alpha_i} g^2\omega} \quad \text{and} \quad \mathrm{Re} \int_{\alpha_i} g\omega = 0, \qquad i = 1, \ldots, 2\gamma. \tag{29}$$

If we write $\omega = \Sigma_{j=1}^n c_j\omega_j$, $c_j \in \mathbf{C}$, then

$$\int_{\alpha_i} \omega = \sum_{j=1}^n \{(c_j^{(1)}s_{ij}^{(1)} - c_j^{(2)}s_{ij}^{(2)}) + i(c_j^{(1)}s_{ij}^{(2)} + c_j^{(2)}s_{ij}^{(1)})\},$$

$$\int_{\alpha_i} g^2\omega = \sum_{j=1}^n \{(c_j^{(1)}t_{ij}^{(1)} - c_j^{(2)}t_{ij}^{(2)} + i(c_j^{(1)}t_{ij}^{(2)} + c_j^{(2)}t_{ij}^{(1)})\},$$

and

$$\mathrm{Re} \int_{\alpha_i} g\omega = \sum_{j=1}^n (c_j^{(1)}u_{ij}^{(1)} - c_j^{(2)}u_{ij}^{(2)}).$$

Hence the condition (29) is expressed as the system of linear homogeneous equations:

$$A(t^2)\mathbf{c} = \mathbf{0},$$

where $\mathbf{c} = (c_1^{(1)}, c_1^{(2)}, \ldots, c_n^{(1)}, c_n^{(2)})$, $\mathbf{0} = (0, 0, \ldots, 0, 0)$ and $A(x)$ is the $6\gamma \times 2n$ matrix given by

$$A(x) = \begin{pmatrix} A_{11}(x) & \cdots & A_{1n}(x) \\ \vdots & & \vdots \\ A_{2\gamma 1}(x) & \cdots & A_{2\gamma n}(x) \\ B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{2\gamma 1} & \cdots & B_{2\gamma n} \end{pmatrix},$$

$$A_{ij}(x) = \begin{bmatrix} s_{ij}^{(1)} - xt_{ij}^{(1)} & -(s_{ij}^{(2)} - xt_{ij}^{(2)}) \\ s_{ij}^{(2)} + xt_{ij}^{(2)} & s_{ij}^{(1)} + xt_{ij}^{(1)} \end{bmatrix},$$

$$B_{ij} = (u_{ij}^{(1)}, -u_{ij}^{(2)}), \qquad i = 1, \ldots, 2\gamma, j = 1, \ldots, n.$$

Note that

$$\begin{aligned} \operatorname{Nul}(G_t) &= 3 + \dim_{\mathbf{R}} H(G_t) \\ &= 3 + 2n - \operatorname{rank} A(t^2). \end{aligned} \tag{30}$$

For the moment we consider $A(x)$ as a matrix whose entries are polynomials with real coefficients. Let r be the rank of $A(x)$ and $\delta_k(x)$, $k = 1, \ldots, r$, the determinant divisors of $A(x)$. $\delta_k(x)$ is, by definition, the greatest common divisor of all the $k \times k$ minor determinants of $A(x)$ and is a polynomial of degree at most $\min(k, 4\gamma)$. An elementary fact in linear algebra says there exist nonsingular square matrices $P(x)$ and $Q(x)$ such that

$$P(x)A(x)Q(x) = \left[\begin{array}{ccc|c} e_i(x) & & 0 & \\ & \ddots & & 0 \\ 0 & & e_r(x) & \\ \hline & 0 & & 0 \end{array}\right],$$

where $e_{i-1}(x) \mid e_i(x)$, that is, $e_i(x)$ is divisible by $e_{i-1}(x)$, $i = 2, \ldots, r$. We note that $\delta_k(x) = e_i(x) \cdots e_k(x)$, $k = 1, \ldots, r$. Let $\tilde{A}(x)$ be the $4\gamma \times 2n$ matrix obtained by deleting the lower 2γ rows of $A(x)$:

$$\tilde{A}(x) = \begin{bmatrix} A_{11}(x) & \ldots & A_{1n}(x) \\ \vdots & & \ldots \\ A_{2\gamma 1}(x) & \ldots & A_{2\gamma n}(x) \end{bmatrix}.$$

Let $\tilde{r}$ and $\tilde{\delta}_k(x)$, $k = 1, \ldots, \tilde{r}$, be the rank and the determinant divisor of $\tilde{A}(x)$ respectively. We note that $r - 2\gamma \leq \tilde{r} \leq r$. Moreover, it is easy to see that $\tilde{\delta}_k(x)$ is an even function of x,

$$\delta_k(x) \mid \tilde{\delta}_k(x), \qquad k = 1, \ldots, \tilde{r}, \tag{31}$$

and

$$\tilde{\delta}_k(x) \mid \delta_{k+2\gamma}(x), \qquad k = 1, \ldots, r - 2\gamma. \tag{32}$$

From now on we consider $A(x)$, $x \in (0, \infty)$, as matrices whose entries are real numbers which vary with x. Clearly rank $A(x) \le r$ and rank $A(x) < r$ if and only if $\delta_r(x) = 0$. Therefore rank $A(x) = r$ except for a finite number of values of x. This together with (30) proves the first assertion of the lemma. To prove the second assertion, it suffices to show

$$n_- + n_+ \le 6\gamma. \tag{33}$$

Let $l(x) = r - \text{rank } A(x)$, $x \in (0, \infty)$. Note that $n_- = \Sigma_{x \le 1} l(x)$ and $n_+ = \Sigma_{x \ge 1} l(x)$. For a polynomial f we denote the degree of f by $d(f)$. If $l(\alpha) > 0$, then $e_{r - l(\alpha) + 1}(\alpha) = \cdots = e_r(\alpha) = 0$ and so $(x - \alpha)^{l(\alpha)} \mid \delta_r(x)$. Therefore

$$\begin{aligned} n_- + n_+ &= \sum_{0 < x < \infty} l(x) + l(1) \\ &\le d(\delta_r) + l(1). \end{aligned}$$

Since $d(\delta_r) \le 4\gamma$, (33) holds if $l(1) \le 2\gamma$. We now suppose $l(1) = 2\gamma + s$, $s \ge 1$. Let $m = \Sigma_{x \ne 1} l(x)$. Using (32) and the fact that $\tilde{\delta}_k(x)$ is an even function, we can deduce

$$\begin{aligned} d(\tilde{\delta}_{r-2\gamma}) &\le 2(d(\delta_r) - (l(1) + m)) \\ &\le 2(2\gamma - s - m), \end{aligned}$$

and therefore

$$d(\tilde{\delta}_{r-2\gamma-j}) \le \max(2(2\gamma - s - m - j), 0), \qquad 0 \le j \le r - 2\gamma - 1.$$

By (31),

$$d(\delta_{r-2\gamma-j}) \le \max(2(2\gamma - s - m - j), 0). \tag{34}$$

If we choose $j = s - 1$, then $r - 2\gamma - j = r - l(1) + 1$, so that (34) becomes

$$d(\delta_{r-l(1)+1}) \le \max(2(2\gamma - 2s + 1 - m), 0).$$

Since $d(\delta_{r-l(1)+1}) > 0$, we must have $m \leq 2\gamma - 2s$. Therefore

$$\begin{aligned} n_- + n_+ &= \sum_{x \neq 1} l(x) + 2l(1) \\ &\leq (2\gamma - 2s) + 2(2\gamma + s) = 6\gamma, \end{aligned}$$

getting (33). This completes the proof of Lemma 2.

5. Example – Index of the Costa's surface

Let L be the square lattice in $\mathbf{C}$ generated by 1 and i and P the Weierstrass $\wp$-function for L. Let $G : \Sigma = \mathbf{C}/L \to S^2$ be the holomorphic map of degree three defined by

$$\Pi \circ G([z]) = P'(z),$$

where P' is the derivative of P and $[z]$ is the point in Σ corresponding to $z \in \mathbf{C}$. In this section we shall compute the index and the nullity of $G_t = \mathscr{A}_t \circ G$ for all $t \in (0, \infty)$.

In our computation essential is the fact that P and P' are highly symmetric. Consider the conformal maps λ, κ and τ of the complex plane defined by

$$\lambda(w_2 + z) = w_2 + iz, \qquad \kappa(w_2 + z) = w_2 + \bar{z},$$
$$\tau(w_2 + z) = \lambda^2(\kappa(w_2 + z)) = w_2 - \bar{z},$$

where $w_2 = (1 + i)/2$. λ is the rotation by $\pi/2$ about w_2 and κ (resp. τ) is the reflection through the horizontal (resp. vertical) line through w_2. It is easy to see that λ, κ and τ induce conformal diffeomorphisms of Σ.

LEMMA 3. *Let λ, κ and τ be as above. Then we have*

$$P \circ \lambda = -P, \qquad P \circ \kappa = \bar{P}, \qquad P \circ \tau = \bar{P}, \tag{35}$$

and

$$P' \circ \lambda = iP', \qquad P' \circ \kappa = \overline{P'}, \qquad P' \circ \tau = -\overline{P'}. \tag{36}$$

Proof. (35) is observed in [9]. Taking derivative of (35), we get

$$\lambda^*(P'\,dz) = -P'\,dz, \qquad \kappa^*(P'\,dz) = \overline{P'}\,d\bar{z}, \qquad \tau^*(P'\,dz) = \overline{P'}\,d\bar{z}.$$

Since $\lambda^*\,dz = i\,dz$, $\kappa^*\,dz = d\bar{z}$ and $\tau^*\,dz = -d\bar{z}$, (36) follows.

REMARK 5. In terms of the holomorphic map G, (36) may be rewritten as

$$G \circ \lambda = LG, \qquad G \circ \kappa = KG, \qquad G \circ \tau = TG, \tag{37}$$

where

$$L = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that (37) holds with G replaced by G_t, $t \in (0, \infty)$.

We now examine the ramification locus of P' and the values of P and P' on it. Let $w_1 = 1/2$, $w_3 = i/2$ and $e_j = P(w_j)$, $j = 1, 2, 3$. Then P satisfies the differential equation

$$(P')^2 = 4(P - e_1)(P - e_2)(P - e_3).$$

In our case e_j's are real, $e_1 = -e_3 > 0$ and $e_2 = 0$. Therefore

$$(P')^2 = 4P(P - e_1)(P + e_1). \tag{38}$$

We first note that P' has poles of order three, hence ramifies with the ramification index three, at the lattice points, where P also has poles of order two. The other ramification points of P' are exactly the zeros of P'' and all with ramification index two. Actually P'' has four simple zeros in $F = \{x + iy \mid 0 \le x, y \le 1\}$, which we denote by z_j, $j = 1, \ldots, 4$. They are located as in Figure 1, so that

$$z_j = \lambda^{j-1} z_1, \qquad j = 2, 3, 4. \tag{39}$$

By (38), it follows that

$$P'' = 6P^2 - 2e_1^2.$$

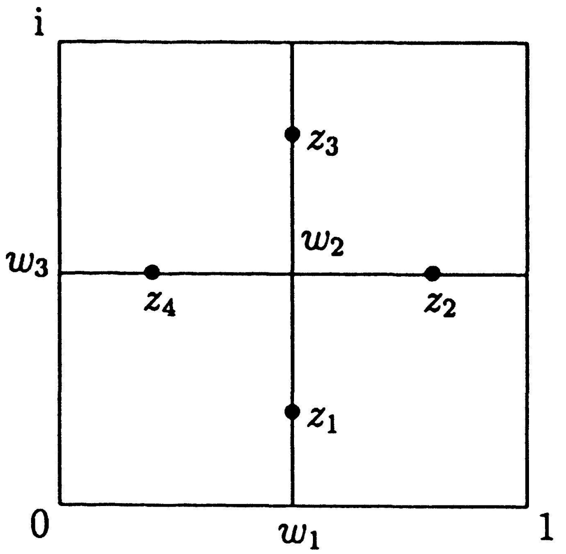

Figure 1. The line segments $\overline{w_1 z_j}$, $j = 1, \ldots, 4$, have the same length.

Letting $A = e_1/\sqrt{3}$, this becomes

$$P'' = 6(P - A)(P + A) \tag{40}$$

and thus $P(z_j) = \pm A$. By (38) we also have $P'(z_j) = \pm B$, $\pm iB$, where $B = (8e_1^3/3\sqrt{3})^{1/2}$. We point out here the fact that P is positive real and P' is "positive" pure imaginary in the interior of the vertical line segment $\overline{w_1 w_2}$ (see [3]). Hence we can conclude

$$P(z_1) = A \quad \text{and} \quad P'(z_1) = iB.$$

$P(z_j)$ and $P'(z_j)$, $j = 2, 3, 4$, can be determined using (39), (35) and (36).

In the following lemma we collect some formulas which are also needed in the sequel.

LEMMA 4. (a) *Let* $\alpha, \beta : [0, 1] \to \mathbf{C}$ *be the paths*

$$\alpha(s) = \frac{i}{3} + s, \qquad \beta(s) = \frac{1}{3} + si.$$

Then

$$\int_\alpha P\,dz = -\pi, \qquad \int_\beta P\,dz = \pi i.$$

(b)

$$\frac{1}{P-e_1}=\frac{1}{2e_1^2}(P(z-w_1)-e_1), \qquad \frac{1}{P-e_3}=\frac{1}{2e_1^2}(P(z-w_3)-e_3).$$

For the proof see [2].

Since P' has poles of order three at the lattice points, we have, from Theorem 2, that

$$\operatorname{Ind}(G_t)=5 \quad \text{and} \quad \operatorname{Nul}(G_t)=3 \qquad \text{for all sufficiently small } t. \tag{41}$$

By Lemma 2, $\operatorname{Nul}(G_t)=3$ except for finitely many values of t. We shall now compute $\operatorname{Nul}(G_t)$ for all t and describe the spaces $H(G_t)$ explicitly. We first note that, by (30), the space $\hat{H}(G)$ has complex dimension at most three.

Let p_0 be the point in Σ corresponding to the lattice points and p_j, $j=1,\ldots,4$, the points in Σ corresponding to z_j. Since $D(G)=\Sigma_{j=1}^4 2p_j-3p_0$, an element of the space $H^0(K(\Sigma)+D(G))$ is written as $f\,dz$, where f is a meromorphic function on Σ with poles of order ≤ 2 at p_j, $j=1,\ldots,4$, and with zeros of order ≥ 3 at p_0. It is easy to check that

$$\omega_1=\frac{P\,dz}{(P^2-A^2)^2}, \qquad \omega_2=\frac{(P^2+A^2)\,dz}{(P^2-A^2)^2}, \qquad \omega_3=\frac{PP'\,dz}{(P^2-A^2)^2}$$

are **C**-linearly independent elements of $H^0(K(\Sigma)+D(G))$. Moreover, it can be shown that the residues of ω_i at p_j, $j=1,\ldots,4$, all vanish. We shall carry out the computation for ω_1. Since $P(z_1)=A$, $P'(z_1)=iB$ and $P''(z_1)=0$, we have

$$P=A+iB(z-z_1)+O(z-z_1)^3,$$

near z_1. Using this we compute

$$\frac{P}{(P^2-A^2)^2}=\frac{1}{-4AB^2(z-z_1)^2}+O(1)$$

and thus $\operatorname{Res}_{p_1}\omega_1=0$. By (35) we have $\lambda^*\omega_1=-i\omega_1$. Hence

$$\begin{aligned}\operatorname{Res}_{p_j}\omega_1&=\operatorname{Res}_{\lambda^{j-1}(p_1)}\omega_1=\operatorname{Res}_{p_1}(\lambda^{j-1})^*\omega_1\\&=(-i)^{j-1}\operatorname{Res}_{p_1}\omega_1=0, \qquad j=2,3,4.\end{aligned}$$

Computations for ω_2, ω_3 are more or less similar. Thus we have shown that $\dim_{\mathbf{C}} \hat{H}(G) = 3$ and ω_1, ω_2 and ω_3 form a basis of $\hat{H}(G)$.

Let ω and η be meromorphic differentials on Σ. We denote $\omega \sim \eta$ if there exists a meromorphic function f on Σ such that $\omega = \eta + df$.

Using (40) we compute

$$d\left(\frac{P^2}{P'(P^2 - A^2)}\right) = \left(-\frac{2A^2 P}{(P^2 - A^2)^2} - \frac{6P^2}{P'^2}\right) dz.$$

Hence

$$\begin{aligned}\omega_1 &\sim -\frac{9}{e_1^2}\frac{P^2}{P'^2}\,dz = -\frac{9}{8e_1^2}\left(\frac{1}{P - e_1} + \frac{1}{P + e_1}\right) dz \quad \text{(by (38))} \\ &= -\frac{9}{16e_1^4}(P(z - w_1) + P(z - w_3))\,dz \quad \text{(by Lemma 4(b))}.\end{aligned}$$

Thus, by Lemma 4(a),

$$\int_\alpha \omega_1 = \frac{9\pi}{8e_1^4}, \qquad \int_\beta \omega_1 = -\frac{9\pi i}{8e_1^4}, \tag{42}$$

where α and β are the paths as in Lemma 4(a). Similar computations yield

$$P'\omega_1,\ P'\omega_2,\ \omega_3 \text{ and } P'^2\omega_3 \sim 0, \tag{43}$$

and

$$\begin{aligned}&\int_\alpha P'^2\omega_1 = \int_\alpha P'\omega_3 = 3, \qquad \int_\beta P'^2\omega_1 = \int_\beta P'\omega_3 = 3i, \\ &\int_\gamma \omega_2 = \frac{3}{4e_1^2}, \qquad \int_\beta \omega_2 = \frac{3i}{4e_1^2}, \\ &\int_\alpha P'^2\omega_2 = -6\pi, \qquad \int_\beta P'^2\omega_2 = 6\pi i.\end{aligned} \tag{44}$$

Let $\omega = \Sigma_{j=1}^3 c_j\omega_j$, $c_j \in \mathbf{C}$. By the definition of $H(G_t)$, $\omega \in H(G_t)$ if and only if

$$\int_\delta \omega = t^2 \overline{\int_\delta P'^2\omega} \quad \text{and} \quad \operatorname{Re}\int_\delta P'\omega = 0 \qquad \text{for } \delta \in \{\alpha, \beta\}.$$

By (42), (43) and (44), these conditions are equivalent to $c_3 = 0$ together with the system of equations:

$$\begin{aligned} \frac{9\pi}{8e_1^4} c_1 + \frac{3}{4e_1^2} c_2 &= t^2\overline{(3c_1 - 6\pi c_2)}, \\ -\frac{9\pi i}{8e_1^4} c_1 + \frac{3i}{4e_1^2} c_2 &= t^2\overline{(3ic_1 + 6\pi ic_2)}. \end{aligned} \tag{45}$$

Considered as equations with unknown c_1 and c_2, (45) has a nontrivial solution only when $t^2 = 3\pi/8e_1^4$, $1/8\pi e_1^2$. When $t^2 = 3\pi/8e_1^4$ (resp. $1/8\pi e_1^2$), $(c_1, c_2) = (1, 0)$ (resp. $(0, i)$) is the unique nontrivial solution of (45) up to a real multiple. We state what we have proved as

LEMMA 5. *Let* $t_1 = (3\pi/8e_1^4)^{1/2}$ *and* $t_2 = (1/8\pi e_1^2)^{1/2}$. *Then*

$$\mathrm{Nul}\,(G_t) = \begin{cases} 4 & \text{if } t = t_1, t_2, \\ 3 & \text{otherwise.} \end{cases}$$

The vector spaces $H(G_{t_1})$ *and* $H(G_{t_2})$ *have real dimension one and are spanned by* ω_1 *and* $i\omega_2$ *respectively, where*

$$\omega_1 = \frac{P\, dz}{(P^2 - A^2)^2} \quad \text{and} \quad \omega_2 = \frac{(P^2 + A^2)\, dz}{(P^2 - A^2)^2}.$$

REMARK 6. Since $e_1 = 6.875\cdots$, it follows that $t_1 = 0.02296\cdots$ and $t_2 = 0.02901\cdots$. Thus we have $t_1 < t_2$.

For simplicity we set $V_\lambda(t) = V_\lambda(G_t)$. Remark 5 implies, since $K, T \in O(3)$, that κ and τ are isometries with respect to the metric $ds^2_{G_t}$ for all t. Hence they act $L^2(\Sigma, dA_{G_t})$-orthogonally on $V_\lambda(t)$. Since κ and τ are both involutive and commutative, $V_\lambda(t)$ splits $L^2(\Sigma, dA_{G_t})$-orthogonally as

$$V_\lambda(t) = V_\lambda(t)^{00} \oplus V_\lambda(t)^{01} \oplus V_\lambda(t)^{10} \oplus V_\lambda(t)^{11},$$

where $V_\lambda(t)^{ij} = \{u \in V_\lambda(t) \mid u \circ \kappa = (-1)^i u, u \circ \tau = (-1)^j u\}$, $i, j = 0, 1$.

Let $\Omega = \{x + iy \mid 0 < x, y < \frac{1}{2}\}$ and $L_j = \overline{w_{j-1} w_j}$, $j = 1, \ldots, 4$, where $w_0 = w_4 = 0$. The boundary of Ω, $\partial\Omega$, is the union of L_j, $j = 1, \ldots, 4$. We identify Ω with the corresponding domain in Σ and consider the eigenvalue problem

$$L_{G_t} u = \lambda u \quad \text{in } \Omega \tag{46}$$

with various boundary conditions:

(NN) $\partial u/\partial \nu = 0$ on $\partial\Omega$;

(ND) $\partial u/\partial \nu = 0$ on $L_1 \cup L_3$, $u = 0$ on $L_2 \cup L_4$;

(DN) $u = 0$ on $L_1 \cup L_3$, $\partial u/\partial \nu = 0$ on $L_2 \cup L_4$;

(DD) $u = 0$ on $\partial\Omega$,

where ν is the unit outward normal to $\partial\Omega$. Let $\lambda_k^{NN}(t)$ (resp. $\lambda_k^{ND}(t)$, $\lambda_k^{DN}(t)$, $\lambda_k^{DD}(t)$) denote the k-th eigenvalue of this problem with the boundary condition (NN) (resp. (ND), (DN), (DD)). We note that these are continuous in t. Let $u \in V_\lambda(t)^{00}$. Then it is easy to see that $u|_\Omega$ satisfies (46) together with the boundary condition (NN). Conversely, if a function v satisfies (46) and (NN), then, by extending v so that the resulting function is invariant by κ and τ, we get a function belonging to $V_\lambda(t)^{00}$. Thus we have a natural bijective correspondence between $V_\lambda(t)^{00}$ and the space of solutions of (46) satisfying the boundary condition (NN). We can establish the similar correspondence between $V_\lambda(t)^{01}$ (resp. $V_\lambda(t)^{10}$, $V_\lambda(t)^{11}$) and the space of solutions of (46) satisfying the boundary condition (ND) (resp. (DN), (DD)).

LEMMA 6. *$\lambda_k^{NN}(t)$ (resp. $\lambda_k^{DD}(t)$) is monotonically non-increasing (resp. non-decreasing) in t.*

Proof. We first point out the following fact (see [3]): P' maps the horizontal line segment $\overline{w_0 w_1}$ onto the nonpositive real axis and the vertical line segment $\overline{w_1 w_2}$ onto $\{iv \mid 0 \leq v \leq B\}$. This fact together with (36) implies that $P'(\partial\Omega)$, which is nothing but the stereographic projection of $G(\partial\Omega)$, is as in Figure 2. Observe that $G(\partial\Omega)$ divides S^2 into two components. Since G is an open map, Ω is simply connected, and $S^2 - G(\partial\Omega)$ contains no branching values of G, we may conclude that G maps Ω biholomorphically onto either of these components. The fact that the area of Ω with respect to the metric ds_G^2 is 3π determines $G(\Omega)$ to be the larger one (see Figure 2). Thus G maps (Ω, ds_G^2) isometrically onto an open three-quarter of S^2 from which two geodesic segments, emanating from the south pole and running toward the north pole, are deleted. Clearly the same statement holds with G replaced by G_t, and as t increases, the length of the deleted geodesic segments also increases. By the argument similar to that in the proof of Lemma 1 (b), the assertions follow.

We now examine the symmetry of elements of $N(G_t) = V_0(t)$. We have the distinguished subspace $L(G_t)$ spanned by

$$(G_t)_1 = (1, 0, 0)' \cdot G_t, \qquad (G_t)_2 = (0, 1, 0)' \cdot G_t, \qquad (G_t)_3 = (0, 0, 1)' \cdot G_t.$$

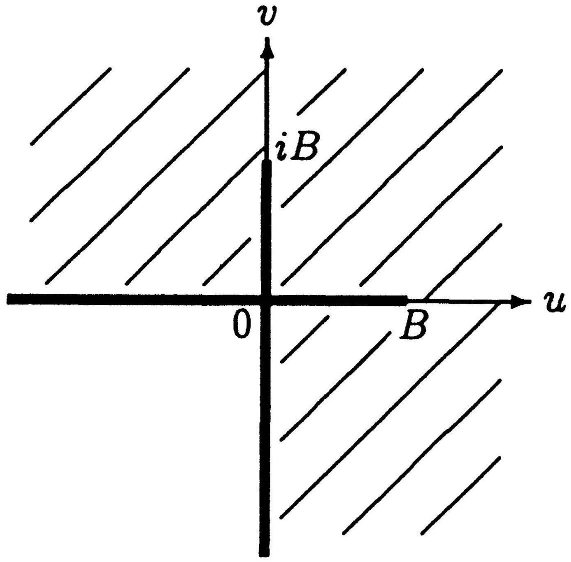

Figure 2. $P'(\partial\Omega)$ is shown as the thick lines and $P'(\Omega)$ as the shaded region.

It is easy to verify, using Remark 5,

$$(G_t)_1 \in V_0(t)^{01}, \qquad (G_t)_2 \in V_0(t)^{10}, \qquad (G_t)_3 \in V_0(t)^{00}. \tag{47}$$

Moreover, Lemma 5 implies that if $t = t_1, t_2$ $N(G_t)$ contains a nonlinear element, unique up to a real multiple and addition of an element of $L(G_t)$. Tracing the description of $N(G_t)$ in §4, we can write them down as follows: Let $X_i : \Sigma - \{p_0, p_1, \ldots, p_4\} \to \mathbf{R}^3$, $i = 1, 2$, be the conformal harmonic maps defined by

$$X_1(p) = \operatorname{Re} \int_{p_0}^{p} (1 - t_1^2 P'^2, i(1 + t_1^2 P'^2), 2t_1 P')^t \omega_1,$$

$$X_2(p) = \operatorname{Re} \int_{p_0}^{p} (1 - t_2^2 P'^2, i(1 + t_2^2 P'^2), 2t_2 P')^t i\omega_2,$$

where ω_i, $i = 1, 2$, are as in Lemma 5. Then, for each $i \in \{1, 2\}$, $u_i = X_i \cdot G_{t_i}$ extends over to $p_0, p_1, \ldots, p_4$ smoothly and gives an element of $N(G_{t_i}) - L(G_{t_i})$.

LEMMA 7. *Let u_1 and u_2 be as above. Then*

$$u_1 \in V_0(t_1)^{00} \quad \text{and} \quad u_2 \in V_0(t_2)^{11}.$$

Proof. It suffices to show

$$\begin{aligned} &u_1 \circ \kappa = u_1, \qquad u_1 \circ \tau = u_1; \\ &u_2 \circ \kappa = -u_2, \qquad u_2 \circ \tau = -u_2. \end{aligned} \tag{48}$$

Let $\Phi = (1 - t_1^2 P'^2, i(1 + t_1^2 P'^2), 2t_1 P')'\omega_1$. It is easy to verify, using (35) and (36), that $\kappa^*\Phi = K\bar{\Phi}$. Therefore

$$\begin{aligned} X_1 \circ \kappa(p) &= \mathrm{Re} \int_{p_0}^{\kappa(p)} \Phi = \mathrm{Re} \int_{\kappa^{-1}(p_0)}^{p} \kappa^*\Phi \\ &= \mathrm{Re}\left(K \int_{p_0}^{p} \bar{\Phi}\right) = KX_1(p). \end{aligned}$$

Since $G_{t_1} \circ \kappa = KG_{t_1}$ and $K \in O(3)$ (see Remark 5), we obtain

$$\begin{aligned} u_1 \circ \kappa &= (X_1 \circ \kappa) \cdot (G_{t_1} \circ \kappa) = KX_1 \cdot KG_{t_1} \\ &= X_1 \cdot G_{t_1} = u_1. \end{aligned}$$

The similar computations show the other equalities in (48).

THEOREM 5. *Let $G : \Sigma \to S^2$ be as in the beginning of this section. Let $G_t = \mathscr{A}_t \circ G$, $t \in (0, \infty)$. Then*

$$\mathrm{Ind}\,(G_t) = \begin{cases} 5 & \text{if } t \le t_1,\ t_2 \le t, \\ 6 & \text{if } t_1 < t < t_2, \end{cases}$$

and

$$\mathrm{Nul}\,(G_t) = \begin{cases} 4 & \text{if } t = t_1, t_2, \\ 3 & \text{otherwise}, \end{cases}$$

where $t_1 = (3\pi/8e_1^4)^{1/2}$ and $t_2 = (1/8\pi e_1^2)^{1/2}$.

Proof. We have only to compute $\mathrm{Ind}\,(G_t)$. By (47) and Lemma 7, we have

$$\begin{aligned} &\dim V_0(t)^{00} = \begin{cases} 2 & \text{if } t = t_1, \\ 1 & \text{otherwise}, \end{cases} \\ &\dim V_0(t)^{11} = \begin{cases} 1 & \text{if } t = t_2, \\ 0 & \text{otherwise}, \end{cases} \\ &\dim V_0(t)^{01} = \dim V_0(t)^{10} = 1 \qquad \text{for all } t. \end{aligned} \tag{49}$$

We fix $t_0 \in (0, t_1)$ and let $n^{ij} = \dim \bigoplus_{\lambda<0} V_\lambda(t_0)^{ij}$ for $i, j = 0, 1$. By (49), Lemma 6 and the observation preceding it, we obtain

$$\dim \bigoplus_{\lambda<0} V_\lambda(t)^{00} = \begin{cases} n^{00} & \text{if } t \le t_1, \\ n^{00}+1 & \text{if } t > t_1, \end{cases}$$

$$\dim \bigoplus_{\lambda<0} V_\lambda(t)^{11} = \begin{cases} n^{11} & \text{if } t < t_2, \\ n^{11}-1 & \text{if } t \ge t_2, \end{cases}$$

$$\dim \bigoplus_{\lambda<0} V_\lambda(t)^{01} = n^{01}, \qquad \dim \bigoplus_{\lambda<0} V_\lambda(t)^{10} = n^{10} \qquad \text{for all } t.$$

Since $\mathrm{Ind}\,(G_t) = \Sigma_{i,j} \dim \bigoplus_{\lambda<0} V_\lambda(t)^{ij}$ and thus $\Sigma_{i,j} n^{ij} = 5$ by (41), we get the desired result.

COROLLARY 1. *The index of the Costa's surface* [2] *is five.*

Proof. The Costa's surface, which we call M, is a complete minimal surface of total curvature -12π, hence of finite total curvature. The compactified surface $\bar{M}$ and the extended Gauss map $\bar{G} : \bar{M} \to S^2$ are given by

$$\bar{M} = \mathbf{C}/L \quad \text{and} \quad \Pi \circ \bar{G}([z]) = \frac{1}{t_2 P'(z)}$$

respectively. Hence by (2) and Theorem 5 we have

$$\mathrm{Ind}\,(M) = \mathrm{Ind}\,(\bar{G}) = \mathrm{Ind}\,(G_{t_2}) = 5.$$

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