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Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms

HOWARD MASUR AND JOHN SMILLIE

A quadratic differential on a Riemann surface M determines certain "topological" data: the genus of M; the orders of zeros and poles; and the orientability of the horizontal foliation. In this note we determine which collections of data can be realized by quadratic differentials with finite area. A pseudo-Anosov diffeomorphism of M also determines certain topological data: the genus of the M; the types of the singularities and the orientability of the stable foliation. As a corollary to our result on quadratic differentials we determine which topological data can be realized by pseudo-Anosov diffeomorphisms on oriented surfaces.

Let X be a closed Riemann surface of genus g with a system of holomorphic coordinate charts $\{U_\nu, h_\nu\}$. This means that $\{U_\nu\}$ is a covering of X by open sets; h_ν is a homeomorphism of U_ν to an open set in the complex plane and $h_\mu \circ h_\nu^{-1}$ is conformal whenever defined. Let q be a positive integer. A meromorphic q-differential ϕ on X is a set of meromorphic function elements ϕ_ν in the local parameters $z_\nu = h_\nu(p)$ for which the transformation law

$$\phi_\nu(z_\nu)\, dz_\nu^q = \phi_\mu(z_\mu)\, dz_\mu^q, \qquad dz_\mu = \frac{dz_\mu}{dz_\nu}\, dz_\nu,$$

holds whenever z_ν and z_μ are parameter values corresponding to the same point p of X. The differentials corresponding to $q = 2$ are classically called quadratic differentials and those corresponding to $q = 1$ are called abelian differentials. At any point p in X we may choose a parameter z so that p corresponds to $z = 0$. We say the differential ϕ has a zero (resp. pole) of order k at p if the meromorphic function element $\phi(z)$ has a zero (resp. pole) of order k at zero. The quadratic differential ϕ has zeroes and poles at a finite number of points $p_1 \dots p_n$. If the area is finite then all poles are simple. We will assume that this is the case. Define k_i to be the order of the zero at p_i, if p_i is a zero, and set $k_i = -1$, if p_i is a simple pole. Thus each quadratic differential ϕ determines certain data

$$(k_1, \dots k_n; \epsilon),$$

where $\epsilon = +1$ if ϕ is the square of an abelian differential, and $\epsilon = -1$ if it is not. In this situation we say that ϕ realizes the singularity data $k = (k_1, \ldots k_n; \epsilon)$. Theorem 1 describes precisely which sets of singularity data are realized by quadratic differentials. Note that the order in which the k_i's are listed is irrelevant for the realization problem.

There are two obvious necessary conditions on a set k of singularity data which must be satisfied in order for k to be realized. To describe the first condition we note that the data $k = (k_1 \ldots, k_n)$ determines the topological type of the surface. Define $\sigma(k)$ by the formula

$$\sigma(k) = \sum_{i=1}^{n} k_i.$$

It is a classical fact that if k can be realized on a surface of genus g then $\sigma(k) = 4(g-1)$. In particular we see that a necessary condition for the realization of the $n+1$-tuple k is that $\sigma(k) \equiv 0 \bmod 4$ and $\sigma(k) \geq -4$.

If a quadratic differential q has a zero or pole of odd order at a point p then near p the quadratic differential cannot be written as a square of an abelian differential. Thus if data $(k_1, \ldots k_n; \epsilon)$ can be realized by a quadratic differential then if any k_i is odd it must be the case that $\epsilon = -1$.

We will show that with four exceptions every $n+1$-tuple of integers, satisfying these two necessary conditions can be realized by a quadratic differential on a compact Riemann surface.

THEOREM 2. *Let $k = (k_1, \ldots, k_n; \epsilon)$ be an $(n+1)$-tuple where each $k_i \in \{-1, 1, 2, \ldots, \}$ and $\epsilon = \pm 1$. Then there is a quadratic differential ϕ on a closed surface realizing k if and only if*:

(a) $\sigma(k) \equiv 0 \bmod 4$ *and* $\sigma(k) \geq -4$;

(b) $\epsilon = -1$ *if any k_i is odd and*

(c) $(k_1, \ldots, k_n; \epsilon) \neq (4; -1), (1, 3; -1), (-1, 1; -1)$ *or* $(\ ; -1)$.

The set of all quadratic differentials with fixed data $(k_1, \ldots, k_n; \epsilon)$ forms a moduli space. These moduli spaces, called "strata", are studied in [V] and [MS]. Theorem 1 arose out of our interest in understanding these strata.

Geometric structures equivalent to quadratic differentials arise in contexts other than complex analysis. A quadratic differential on a surface M determines a pair of transverse foliations. The horizontal (resp. vertical) trajectories are arcs along which

$$q(z)\,dz^2 > 0 \qquad (\text{resp. } q(z)\,dz^2 < 0).$$

The horizontal (resp. vertical) measured foliation consists of the horizontal (resp. vertical) trajectories together with the transverse invariant measure

$$|\operatorname{Im} q^{1/2}\,dz| \qquad (\text{resp. } |\operatorname{Re} q^{1/2}\,dz|).$$

Conversely a pair of transverse measured foliations determine a quadratic differential. For more information see [HM] and [G].

Let f be a pseudo-Anosov homeomorphism of a surface of genus g with n punctures. According to Thurston [FLP] f determines stable and unstable transverse measured foliations. These foliations have common "p-pronged" singularities at a finite number of points x_i. Let p_i be the number of prongs of the singularity at x_i. Let ϵ be $+1$ or -1 depending on whether the stable foliation is or is not orientable.

THEOREM 2. *There is a pseudo-Anosov homeomorphism on a surface of genus g with n punctures realizing data $(p_1, \ldots, p_l; \epsilon)$ if and only if*:

(a) $\Sigma_{i=1}^{l} (p_i - 2) = 4(g - 1)$;

(b) $\epsilon = -1$ *if any p_i is odd*;

(c) $(p_1, \ldots, p_l; \epsilon) \neq (6; -1), (3, 5; -1), (1, 3; -1)$ *or* $(\ ; -1)$ *and*

(d) *The number of indices i for which $p_i = 1$ is less than or equal to n, the number of punctures.*

Proof of Theorem 1. A pair of transverse measured foliations with singularity data $(p_1, \ldots, p_l; \epsilon)$ determines a quadratic differential with singularity data $(k_1, \ldots, k_l; \epsilon)$ where $k_i = p_i - 2$. Moreover every quadratic differential can be constructed in this fashion from two transverse measured foliations. Thus the necessity of the conditions (a), (b) and (c) in Theorem 2 follows directly from Theorem 1. Condition (d) is a result of the convention that all 1 pronged singularities occur at punctures. The sufficiency of the conditions in Theorem 2 is a consequence of Theorem 1 and a deep theorem of Veech [V]. Veech's theorem implies (among other things) that if k can be realized by some quadratic differential ϕ then there is a quadratic differential ϕ' with the same data as ϕ and where ϕ' is constructed from a pseudo-Anosov diffeomorphism in the manner described above. Q.E.D.

There is a third equivalent way to describe the geometric structure induced by a quadratic differential. This geometric formulation will be useful in the constructive half of our proof of Theorem 1. Let M be a surface and let Σ be a finite subset of M. A Riemannian metric on $M - \Sigma$ is *flat* if it has Gaussian curvature zero. We say a Riemannian metric on $M - \Sigma$ has cone type singularities if, in a neighborhood of a point of Σ, the metric can be written as

$$ds^2 = dr^2 + (cr\, d\theta)^2,$$

where $c > 0$. In this case we say the metric has singularity with cone angle $2\pi c$. Parallel translation defines a homomorphism from $\pi_1(M - \Sigma) \to SO(2, \mathbf{R})$. The image of this map is contained in $\{\pm I\}$ if and only if M possesses a parallel line field. The image of this map is trivial if and only if M possesses a parallel vector field. A quadratic differential gives rise to a surface with a flat metric and cone type singularities that also posesses a parallel line field. Conversely every flat structure with cone type singularities and parallel line field produces a quadratic differential.

Our proof of Theorem 1 uses complex analysis to show the necessity of condition (c). It would also be interesting to understand geometrically why the structures listed in condition (c) cannot be realized. We will give a short discussion here.

The nonexistence of a structure of type $(\ ; -1)$ is a special (and particularly simple) case. A structure of type $(\ ; \epsilon)$ is a flat structure on the torus without singularities. It is a consequence of Bieberbach's classification of manifolds with flat metrics that every such structure is isometric to $\mathbf{R}^2$ modulo a lattice. Thus every such structure possesses a parallel vector field, not just a parallel line field. In particular $(\ ; -1)$ cannot be realized.

We now consider the other sets of singularity data listed in condition (c). Consider the problem of realizing flat structures with parallel line fields and prescribed sets of singularities on surfaces with boundary. It is a consequence of our proof of Theorem 1 that *any* data other than $(\ ; -1)$ can be realized on a surface with nonempty boundary so that the boundary components are parallel geodesics. One approach to realizing data on a surface M without boundary is to cut M along a closed curve, construct a flat structure with parallel boundary components and then glue the parallel boundary components together. This method fails in general because it is not always possible to realize the flat structure with boundary components of the same length. In fact the data sets $(4; -1)$, $(1, 3; -1)$, and $(-1, 1; -1)$ described in part (c) of Theorem 1 are precisely those which cannot be realized on surfaces with two parallel boundary components of equal length.

There is a related geometric realization problem. One can consider the problem of realizing prescribed sets of singularities by a flat structure with cone type singularities which does not necessarily possess a parallel vector field. It still makes sense to talk about cone angles in this case but the quantity ϵ is not defined. In this case a result of Troyonov, (see [T]), implies that the obvious necessary topological condition (condition (a) in Theorem 1) is in fact sufficient.

Proof of Theorem 1 (necessity of condition (c)). We will first prove that there is no quadratic differential ϕ on a closed surface realizing the topological data listed in (c). We are indebted to Irwin Kra for pointing out the proofs in the first two cases. If $(4; -1)$ or $(3, 1; -1)$ can be realized, they are realized on a surface of genus 2. Accordingly, let X be a Riemann surface of genus 2 which supports ϕ. Every Riemann surface of genus 2 is hyperelliptic. This means that it can be represented as a two sheeted branched covering of the Riemann sphere. Let z be the covering map and let $P_1, \ldots, P_6$ be the branch points in X.

Assume $z(P_i) \neq \infty$. There is a map w of X to the Riemann sphere which satisfies

$$w^2 = \prod_{j=1}^{6} (z - z(P_j)),$$

and there is an involution $\tau : X \to X$ satisfying $z \circ \tau = z$ and $w \circ \tau = -w$ and fixing precisely the points P_i. By III.7.5 Corollary 1 of Farkas–Kra the abelian differentials $\omega_1 = dz/w$ and $\omega_2 = z\,dz/w$ form a basis for the vector space of abelian differentials on X. Clearly $\tau^*(\omega_i) = -\omega_i$. By III.7.5 Corollary 2, the quadratic differentials ω_1^2, ω_2^2, and $\omega_1\omega_2$ form a basis for the vector space of quadratic differentials on X. Thus every quadratic differential ϕ satisfies $\tau^*(\phi) = \phi$. Suppose ϕ now is a quadratic differential with a single zero of order 4 at a point $p \in X$. We show ϕ must realize $(4; +1)$ and not $(4; -1)$. Since $\tau^*(\phi) = \phi$ we have $\tau(p) = p$. Therefore p is one of the branch points P_j. By Theorem III.7.2 of Farkas–Kra [FK] the points P_i are also the Weierstrass points of X. This means that for each P_i there is an abelian differential on X with a single zero of order 2 at P_i. Let ω_p be the abelian differential with a zero of order 2 at p. The quotient ϕ/ω_p^2 is therefore a meromorphic function on X without zeroes or poles and therefore is a constant function. Thus $\phi = c\omega_p^2$ and ϕ determines $(4; +1)$.

Next we show there is no quadratic differential ϕ on X with a zero of order 3 at a point $p_1 \in X$ and a zero of order 1 at a point $p_2 \in X$. If such a ϕ exists, we again have $\tau^*(\phi) = \phi$ and therefore $\tau(p_i) = p_i$ and so the p_i are Weierstrass points. Then $\phi/\omega_{p_1}^2$ is a meromorphic function with a simple pole at p_1 and a simple zero at p_2. Thus $\phi/\omega_{p_1}^2$ defines a degree 1 covering of X to the Riemann sphere, which is impossible.

Next suppose ϕ is a quadratic differential with a simple zero and a simple pole on a torus X. There exists a quadratic differential ψ on X which defines the flat structure without singularities (ψ defines $(\ ; +1)$). Then ϕ/ψ defines a degree 1 covering of X to the Riemann sphere, which is impossible. If ϕ defines $(\ ; -1)$, then ϕ/ψ is a constant, which also is impossible.

Sufficiency of the Conditions of Theorem in the case $g = 0$. The realization of $(n + 1)$-tuples, k, with $\sigma(k) = -4$ is particularly simple and we dispense with it here.

Data with $\sigma(k) = -4$ are realized on the sphere. Note that in this case some k_i is equal to -1 so by condition (b) $\epsilon = -1$. Suppose $k = (k_1, \dots k_n; -1)$, where $k_i = -1$ for $i > m$. Let $z_1, \dots z_n$ be distinct points on the Riemann sphere X. Then the quadratic differential ϕ defined by

$$\phi\, dz^2 = \frac{(z - z_1)^{k_1} \dots (z - z_m)^{k_m}}{(z - z_{m+1}) \dots (z - z_n)}\, dz^2$$

realizes k.

Our construction in the general case, $\sigma(k) \geq 0$, is by means of a "cut and paste" argument which relies on the geometric formulation of the quadratic differential structure. In the course of this proof it will be convenient to extend the definition of flat surfaces with cone points to include surfaces with boundaries and singularities on the boundaries. For our purposes it suffices to consider surfaces with the property that all boundary curves are unions of parallel geodesic segments. We require that for each boundary point p, there is a chart mapping a neighborhood of p to a neighborhood of the origin in the upper half plane so that with respect to this chart the metric has the form

$$ds^2 = dr^2 + (cr\, d\theta)^2.$$

The cone angle at p is then $c\pi$. In our examples c is always a positive integer.

Now suppose e_1 and e_2 are boundary edges of the same length on flat surfaces. Let p_1 and p_2 be endpoints of e_1 and e_2. If we glue e_1 to e_2 isometrically, identifying p_1 and p_2, then the cone angle at the identified point is the *sum* of the boundary cone angles at p_1 and p_2.

We will use the following terminology. Let M be a surface possibly with boundary. A *flat structure* on M is a flat Riemannian metric with cone type singularities and a specified horizontal line field. If M has a boundary then we assume in addition that the boundary curves are horizontal. If M does not have a boundary then a flat structure is equivalent to a quadratic differential. We will call a surface with a flat structure a *flat surface*.

Our strategy for constructing examples is to partition the singularity data into subsets which are as small as possible, realize these subsets on flat surfaces with parallel boundaries and then assemble these surfaces to give a flat structure on a connected surface without boundary. The first step is to break up the data k into minimal pieces.

NOTATION. If $k^1 = (k_1^1, \dots, k_{n_1}^1; \epsilon^1)$ and $k^2 = (k_1^2, \dots, k_{n_2}^2; \epsilon^2)$ and $\epsilon^1 = \epsilon^2 = \epsilon$ then we write $k = k^1 \oplus k^2$ for $(k_1^1, \dots, k_{n_1}^1, k_1^2, \dots, k_{n_2}^2; \epsilon)$.

Let $k = (k_1, \ldots, k_n; \epsilon)$ be an $(n+1)$-tuple. Define a collection k to be *admissible* if

(1) $\sigma(k) \equiv 0 \bmod 4$.

(2) $\sigma(k) \geq 0$.

We say k is minimal if it cannot be written as $k = k^1 \oplus k^2$ where k^1 and k^2 are admissible. Every admissible k can be written as $k = k^1 \oplus \cdots \oplus k^m$ where each k^j is minimal.

The following theorem shows that essentially every minimal collection can be realized on a flat surface with two parallel boundary components.

THEOREM 1′. *Let* $k = (k_1, \ldots, k_n; \epsilon)$ *be a minimal collection. Assume* $k \neq (\ ; -1)$.

(a) *If* $\epsilon = -1$, $k \neq (4; -1)$, $k \neq (3, 1; -1)$, *and* $k \neq (1, -1; -1)$, *then for any two positive numbers* l_1 *and* l_2, *there is a flat surface* M *realizing the data* k *with two horizontal boundary components of lengths* l_1 *and* l_2.

(b) *If* $k = (4; -1)$, $(3, 1; -1)$, *or* $(1, -1; -1)$, *then for any two positive numbers* $l_1 > l_2$ *there is a flat surface realizing the data* k *with two horizontal boundary components of lengths* l_1 *and* l_2.

(c) *If* $\epsilon = +1$, *then for any positive number* l *there is a flat surface realizing the data* k *with two horizontal boundary components of equal length* l.

DEFINITION. We call the k-tuples $(4; -1)$, $(3, 1; -1)$ and $(1, -1; -1)$ *restricted.*

We now show that Theorem 1′ implies Theorem 1.

Completion of the Proof of Theorem 1. Let $k = (k_1, \ldots, k_n, \epsilon)$ and suppose $\sigma \geq 0$ and k satisfies the hypotheses of Theorem 1. Write k as $k = k^1 \oplus \cdots \oplus k^m$ as a sum of m minimal collections $(k_1^j, \ldots, k_{n_j}^j, \epsilon^j)$. By convention $\epsilon^j = \epsilon$.

Suppose first that $m = 1$ so k itself is minimal. Hypothesis (c) of Theorem 1 implies that k is unrestricted. If $\epsilon = +1$ we apply conclusion (c) of Theorem 1′ to find a surface with two boundary components realizing k such that the boundary lengths are equal. If $\epsilon = -1$, conclusion (a) of Theorem 1′ says the boundary lengths are arbitrary and thus we may choose them to be equal. In either case we may glue the boundary components isometrically to realize k on a closed surface.

Now assume $m > 1$.

CASE A. $\epsilon = +1$.

Choose a flat structure X_i with two boundary components realizing each minimal collection so that the boundary lengths of all X_i are equal. This is possible by Theorem 1′ part (c). For each $1 \le j \le m-1$ glue a boundary component of X_j isometrically to a boundary component of X_{j+1} and glue a component of X_m to a component of X_1.

CASE B. $\epsilon = -1$ and at least one minimal collection is unrestricted.

We may assume the unrestricted collection is realized on X_m. If $m = 2$ choose the arbitrary boundary length of X_2 to be equal to the boundary lengths of X_1 and glue isometrically. Suppose then that $m \ge 3$. Realize k^1 on X_1. Realize k^2 on X_2 so that one boundary length of X_2 is equal to a boundary length of X_1. Glue these boundaries together isometrically. Similarly we realize k^3 on X_3 and glue one boundary component of X_3 to the unglued component of X_2. Continue in this fashion eventually gluing a boundary component of X_{m-1} to the unglued boundary component of X_{m-2}. Since X_m has arbitrary boundary lengths we may choose them so that one boundary length is the same as the length of the unglued boundary of X_{m-1} and the other is the same as the length of the unglued boundary of X_1. We may glue X_m to the connected sum of the X_i along these boundaries to form the closed surface.

CASE C. $\epsilon = -1$ and all minimal collections are restricted.

Realize the minimal collections on flat surfaces X_i. For each X_i label the boundaries C_i^j, $j = 1, 2$ so that $|C_i^1| < |C_i^2|$ where the notation $|\gamma|$ denotes the length of γ. The hypothesis of theorem 1 excludes the case that the data consists of a single restricted collection so we may assume that $m \ge 2$. If $m = 2$ we may choose X_1 and X_2 so that $|C_1^j| = |C_2^j|$, $j = 1, 2$ and glue C_1^j to C_2^j. Suppose then $m \ge 3$. For $1 \le i \le m-2$ we may choose X_i so that $|C_i^2| = |C_{i+1}^1|$ and then form the connected sum of X_i and X_{i+1} by gluing C_i^2 isometrically to C_{i+1}^1. Now $|C_{m-1}^2| > |C_1^1|$. Now choose X_m so that $|C_m^2| = |C_{m-1}^2|$ and $|C_m^1| = |C_1^1|$. Glue C_m^2 isometrically to C_{m-1}^2 and C_m^1 isometrically to C_1^1. Q.E.D.

Before proving Theorem 1′ we will establish some properties of minimal collections.

PROPOSITION 1.1. *For each minimal collection* $(k_1, \dots k_n)$ *either*

(a) $n \le 4$ *or*

(b) *exactly one* $k_i > 0$.

Proof. Write m_- for the number of indices i for which $k_i = -1$.

CASE I. $m_- < 4$.

We will show (a) holds. For $m \le n$ let $\alpha_m = \Sigma_{i=1}^m k_i$. If $n > 4$, then for some $1 \le j_1 < j_2 \le 5$ we have $\alpha_{j_1} \equiv \alpha_{j_2} \bmod 4$ by the pigeon hole principle.

Thus $\alpha_{j_2} - \alpha_{j_1} = \Sigma_{i=j_1+1}^{j_2} k_i \equiv 0 \bmod 4$. This implies that both subcollections $k^1 = (k_{j_1+1}, \ldots k_{j_2})$ and $k^2 = (k_1, \ldots k_{j_1}, k_{j_2+1} \ldots, k_n)$ satisfy $\sigma(k^i) \equiv 0 \bmod 4$. The condition, $m_- < 4$, that there are fewer than 4 indices with $k_i = -1$, implies that $\sigma(k^i) \geq -3$ for both $i = 1, 2$. Therefore since $\sigma(k^i) \equiv 0 \bmod 4$ we have $\sigma(k^i) \geq 0$. We have subdivided $(k_1, \ldots, k_n)$ into two admissible collections contrary to the hypothesis.

CASE II. $m_- \geq 4$.

We will show that (b) applies. Suppose to the contrary that $k_1 > 0$ and $k_2 > 0$.

Consider first the situation when $m_- > k_1$ or $m_- > k_2$. Assume for the sake of definiteness that $m_- > k_1$. Consider the subcollection $k^1 = (k_1, -1, \ldots, -1)$ where the number of -1 entries in the subcollection is k_1. We have $\sigma(k^1) = 0$. Therefore k^1 and the complementary subcollection are admissible and we have subdivided $(k_1, \ldots, k_n)$, contrary to assumption.

Now consider the situation when $m_- \leq k_1$ and $m_- \leq k_2$. Since k is minimal, $k_i \neq 4$. Thus $k_1, k_2 \geq 5$. Form the subcollection $(k_1, -1, \ldots, -1)$ where the number of -1 entries m'_- satisfies $0 \leq m'_- \leq 3$ and $k_1 \equiv m'_- \bmod 4$. Then both this collection and the complementary collection are admissible and again we have contradicted the minimality of $(k_1, \ldots, k_n)$. Q.E.D.

The remainder of this paper will be devoted to the proof of Theorem 1′. We construct three different classes of examples. The lowest level are the building blocks. These are denoted below by $A(1)$, $A(m)$, B_1 and B_2 and are constructed by making identifications on the boundary of a metric cylinder. These building blocks will contain singularities on their boundaries.

We call the intermediate level pieces the *basic examples*. These are constructed by gluing together building blocks. The basic examples do not contain singularities on their boundaries and are in fact examples of surfaces with boundary which realize minimal data.

In the final step of the construction we use two inductive procedures to build all examples realizing minimal data from the basic examples.

Basic examples

Let C denote a metric cylinder, that is to say C is isometric to the product $S^1 \times I$. We choose the horizontal direction so that the boundaries are horizontal. C realizes the data $(\ ; +1)$.

Building Block $A(1)$. Let C be a metric cylinder and let C^0 be one of its boundary components. Divide C^0 into three segments and labelled a, b and b^{-1} so

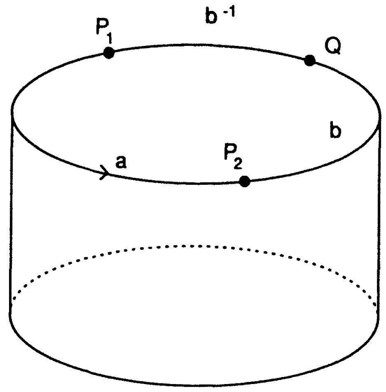

Figure 1

that $|b| = |b^{-1}|$. Let P_1, Q and P_2 be the vertices as shown in Figure 1. We construct a new surface, $A(1)$, by identifying b and b^{-1} by an orientation reversing isometry. Thus the endpoints of a, P_1 and P_2, are identified to become a single point P and a becomes a closed curve. The point P becomes a boundary singular point with cone angle 2π. The point Q becomes an interior singular point in $A(1)$ with cone angle π. In the language of quadratic differentials Q is a simple pole. In general we identify sides using the standard convention that a side labelled x and a side labelled x^{-1} are glued together by an orientation reversing isometry. The edge identifications force certain vertex identifications. We will identify two vertices only if their identification is forced in this way. $A(1)$ has two boundary components $A^0(1)$ and $A^1(1)$ corresponding to the two components of C; $|A^0(1)| = |a|$ and $|A^1(1)| = |a| + 2|b|$. Note that by choosing the diameter of C, $|a|$ and $|b|$ appropriately we can give the boundaries any assigned lengths with $|A^0(1)| < |A^1(1)|$.

Building Block $A(m), m > 1$. Let C be a metric cylinder and let C^0 be one of its boundary components. Divide C^0 into $2m+1$ segments and label them $a, b_1, b_1^{-1}, b_2, b_2^{-1} \ldots b_m b_m^{-1}$ (see Figure 2). Choose the lengths of segments so that

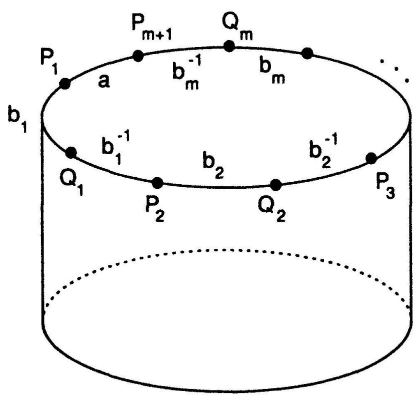

Figure 2

$|b_i| = |b_i^{-1}|$. Identify sides using the standard convention. The points $P_1 \ldots P_m$ are identified to a single point P. Let $A(m)$ be the resulting flat surface with boundary. $A(m)$ has two boundary components; one, $A^0(m)$, corresponds to the edge a and the other $A^1(1)$ corresponds to C^1. Then $|A^0(m)| = |a|$ and $|A^1(1)| = |a| + \Sigma\, 2|b_i|$. Note that by choosing the lengths of segments appropriately we can give the boundaries any assigned lengths with $|A^0(1)| < |A^1(1)|$. $A(m)$ has m singularities with cone angle π in its interior corresponding to $Q_1 \ldots Q_m$ and one singularity on the boundary $A^0(m)$ with boundary cone angle $(m+1)\pi$ at the point P.

We now construct the first set of basic examples.

Basic Examples I. $k = (m, -1, \ldots -1; -1)$ with $\sigma(k) = 0$.

If $m = 1$ then the data is restricted. Construct a surface M from C and $A(1)$ by gluing one component of C to $A^0(1)$. We have that the length of $A^1(1)$ is greater than the circumference of C, but otherwise the lengths are arbitrary. M has two singularities, one of cone angle π in the interior of $A(1)$ and one of cone angle $2\pi + \pi = 3\pi$ which arises from the identification of the boundary cone angle 2π singularity on $A^0(1)$ and a boundary cone angle π point on the boundary of C. Recall that boundary cone angles add under identification. The cone angle 3π singularity is a zero of order 1.

If $m > 1$ then we construct a surface M by gluing a copy of $A(1)$ to a copy of $A(m-1)$ so that the boundary singularities coincide. Using the fact that cone angles add we see that the singularity corresponding to the boundary singularity has cone angle $(m+2)\pi$. By choosing the boundary lengths of $A(1)$ and $A(m-1)$ appropriately we can realize any boundary lengths on M.

We now construct another building block, B_1, and use it to construct further basic examples.

Building Block B_1. Let C be a metric cylinder and let C^0 be one of its boundary components. Divide C^0 into 5 segments and label them $aba^{-1}b^{-1}c$ (see Figure 3).

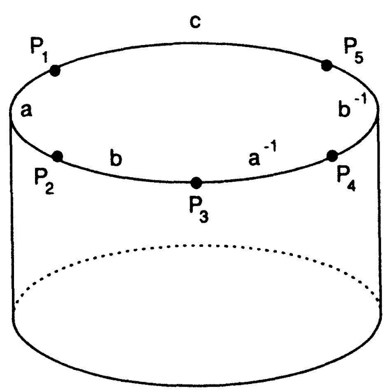

Figure 3

Choose the segments so that $|a| = |a^{-1}|$ and $|b| = |b^{-1}|$. Identify sides using the standard convention. Let B_1 be the resulting flat surface with boundary. B_1 has two boundary components; one, B_1^0, corresponds to the edge c and the other, B_1^1, corresponds to C^1. $|B_1^0| = |c|$ and $|B_1^1| = 2|a| + 2|b| + |c|$. Note that by choosing the lengths of segments appropriately we can give the boundaries any assigned lengths with $|B_1^0| < |B_1^1|$. All vertices are identified to a single point P. B_1 has one singularity at P with cone angle 5π on the boundary B_1^0.

We now construct the second group of basic examples.

Basic Examples II. $k = (4; -1)$, $(8; -1)$, $(5, -1; -1)$ and $(5, 3; -1)$.

We first construct the example M that realizes the restricted data $(4; -1)$. Construct a surface M from B_1 and C by gluing the boundary B_1^0 to one boundary of C. The singularity of cone angle 5π on B_1^0 gives a singularity of cone angle 6π in M. This is a zero of order 4. We can choose the boundary lengths of B_1 and the circumference of C to realize any boundary lengths $l_1 > l_2$.

We construct $(8; -1)$. Construct a surface M from two copies of B_1, calling them B_1 and B_1'. Glue boundaries B_1^0 and $B_1^{0\prime}$ so that the singularities coincide. The resulting singularity has cone angle 10π. The boundary lengths can be chosen arbitrarily.

We construct a surface realizing $(5, -1; -1)$. Construct a surface M from a copy of B_1 and a copy of $A(1)$. Glue the boundaries B_1^0 and $A^0(1)$ together so that the singularities coincide.

The resulting singularity has cone angle $5\pi + 2\pi = 7\pi$. There is one cone angle π singularity in the interior of M coming from $A(1)$. The boundary lengths can be chosen arbitrarily.

To construct $(5, 3; -1)$ we need one additional building block.

Building Block B_2. Let C be a metric cylinder and let C^0 be one of its boundary components. Divide C^0 into 6 segments and label them $efe^{-1}d_1f^{-1}d_0$ (see Figure 4). Choose the segments so that $|e| = |e^{-1}|$ and $|f| = |f^{-1}|$. Identify sides using the standard convention. Let B_2 be the resulting flat surface with boundary. B_2 has two boundary components; one, B_2^0, corresponds to the edge d_1 and d_0 the other, B_2^1, corresponds to C^1. Then $|B_2^0| = |d_1| + |d_0|$ and $|B_2^1| = 2|e| + 2|f| + |d_1| + |d_0|$. Note that by choosing the lengths of segments appropriately we can give the boundaries any assigned lengths with $|B_2^0| < |B_2^1|$; B_2 has a boundary singularity at P with cone angle 2π and a boundary singularity at Q with cone angle 4π.

We construct the basic example $(5, 3; -1)$. Construct a flat surface M by gluing a copy of B_1 to a copy of B_2 by gluing B_1^0 to B_2^0. B_1^0 contains a singularity with cone angle 5π. B_2^0 contains singularities with cone angles 2π and 4π. Choose the identification of the boundaries so that the 5π singularity and the 2π singularity P are

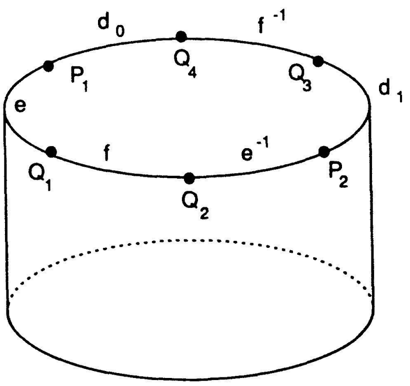

Figure 4

identified. The resulting singularity will have cone angle 7π. The 4π singularity Q will be identified with a nonsingular boundary point so the resulting singularity will have cone angle 5π. The boundary lengths can be chosen arbitrarily.

We now construct the last set of basic examples.

Basic Examples III. $(2,2;-1)$, $(3,2,-1;-1)$, $(3,3,-1,-1;-1)$, $(2,1,1;-1)$ and $(1,1,1,1;-1)$.

The paradigm for these constructions is the construction of $(2,2;-1)$. Take a metric cylinder C_1 and identify two diametrically opposite points on one boundary component of C_1. This gives two boundary circles c_1 and c_2 of equal length joined at a point P. Note this space is a 2-complex but not a manifold with boundary. Take a second metric cylinder C_2 with circumference $|c_1| = |c_2|$ and glue one boundary component isometrically to c_1, the other to c_2. Then P becomes a singularity with cone angle 4π which in the language of quadratic differentials is a zero of order 2. Now identify two not necessarily diametrically opposite points on the other boundary component of C_1 giving two boundary circles d_1 and d_2 of arbitrary length joined together at a point Q. Glue one boundary component of a third cylinder C_3 with circumference $|d_1|$ isometrically to d_1 and glue a boundary component of a fourth cylinder C_4 with circumference $|d_2|$ isometrically to d_2. The point Q is also a zero of order 2. This construction produces a surface with two boundary components corresponding to the unglued boundary components of C_3 and C_4 and they have arbitrary lengths $|d_1|$ and $|d_2|$ respectively. (See Figure 5.)

$(3,2,-1;-1)$: Replace the cylinder C_4 in the previous construction with a copy of $A(1)$. Glue the 2π singularity on the boundary of $A(1)$ to Q on C_1. This makes Q a cone angle 5π singularity and adds a singularity of cone angle π coming from the interior of $A(1)$.

$(3,3,-1,-1;-1)$: Replace the cylinder C_2 in the previous construction with a copy of $A(1)$. (This necessitates identifying points on the boundary of C_1 which are

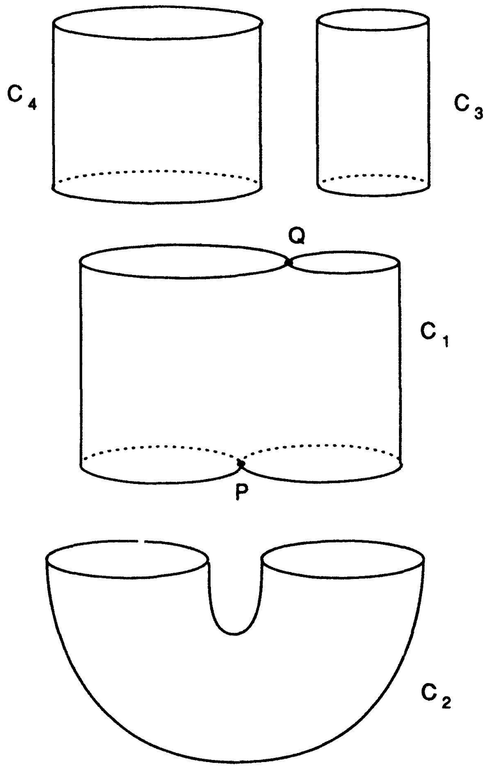

Figure 5

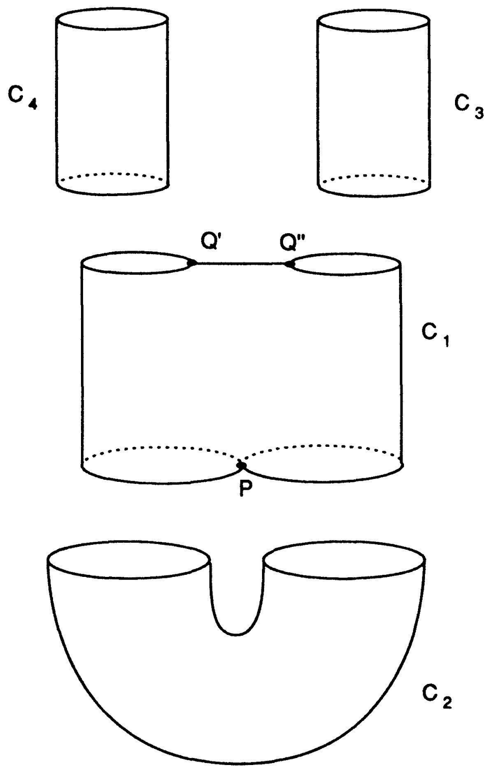

Figure 6

not diametrically opposite because the boundaries of $A(1)$ must have different lengths.) In this way P becomes a cone angle 5π singularity and a second cone angle π singularity is added.

$(2, 1, 1; -1)$: We identify disjoint intervals of the same length on the "top" component of C_1 by an orientation reversing isometry. (See Figure 6.) This gives two circles d_1 and d_2 based at distinct points Q' and Q'' and an interval joining them. The lengths $|d_1|$ and $|d_2|$ of the circles are arbitrary. The points Q' and Q'' are zeroes of order 1.

$(1, 1, 1, 1; -1)$: We identify intervals on both boundary components of C_1. This produces four zeros of order one: P', P'', Q' and Q''. (See Figure 7.) As in the case of $(2, 2; -1)$ we require that the two boundary circles c_1 and c_2 on one component have equal length so that they can be glued to boundary components of the same cylinder C_2.

The next two propositions allow us to build all minimal examples from the basic ones and allow us to prove Theorem 1′.

DEFINITION. A *saddle connection* is a geodesic segment joining two singularities which has no singularities in its interior.

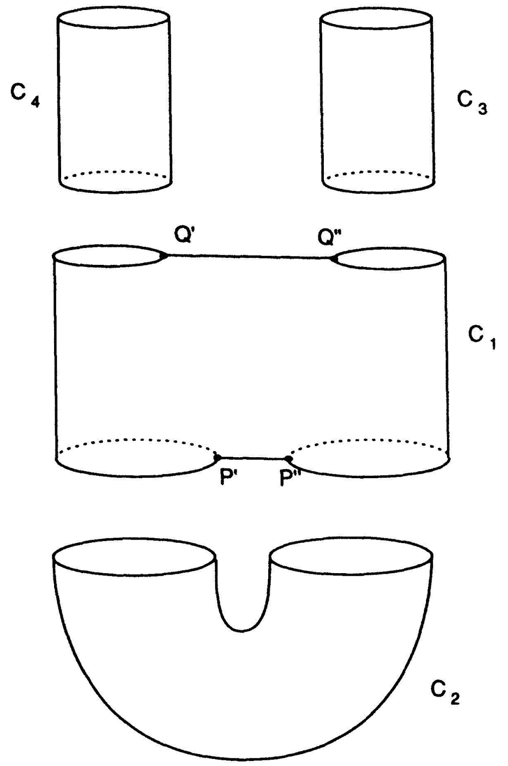

Figure 7

PROPOSITION 1.2. *Suppose M is a flat surface realizing $k = (k_1, k_2, \ldots, k_n; \epsilon)$. Suppose there is a horizontal saddle connection joining the zeroes of order k_1 and k_2. Then there is a new flat surface M' determining $k' = (k_1 + 2, k_2 + 2, \ldots, k_n; \epsilon)$.*

Proof. Let γ be the saddle connection joining p_1 and p_2, the zeroes of orders k_1 and k_2 and let d be its length. Tak a flat torus T, points $x_1, x_2 \in T$ and a horizontal geodesic segment γ' joining them of length d. Slit M and T along γ and γ', and let M' be the connected sum of M and T formed by gluing M and T together along the slits with p_1 glued to x_1 and p_2 glued to x_2. It is easy to see that 2π is added to each of the cone angles at p_1 and p_2. Thus 2 is added to the order of each zero.

Notice that if M has two boundary components which are horizontal geodesics without singularities then M' has the same property. Also note that M' has a horizontal saddle geodesic joining the two zeroes of order $k_1 + 2$ and $k_2 + 2$. If M has a horizontal saddle connection which is a loop based at a zero, then there is horizontal saddle connection which is a loop based at the corresponding zero of M'.

PROPOSITION 1.3. *Suppose M is a flat structure realizing $k = (k_1, \ldots, k_n; \epsilon)$. Suppose there is a closed horizontal saddle connection from the zero of order k_1 to itself. Then there is a flat structure M' determining $k' = (k_1 + 4, \ldots, k_n; \epsilon)$.*

Proof. Let p be the zero of order k_1. Let γ be the saddle connection based at p and suppose it has length d. Find a flat torus T, points x_1 and x_2 and a horizontal geodesic γ' with length $2d$ joining them. Slit T along γ' and then identify x_1 and x_2. The result is a flat torus T' with two boundary circles joined together at the point $x_1 = x_2$. Now slit M along γ. We form M' by gluing each side of the cut to one boundary circle of T'. We glue the point $x_1 = x_2$ to the point on the cut corresponding to p.

Note that M' has a closed horizontal saddle connection from the zero of order $k_1 + 4$ to itself. Suppose further that M has two horizontal boundary components, then M' has two horizontal boundary components with the same lengths as those of M and if there is closed horizontal saddle connection from a zero of M to itself, the corresponding zero of M' has such a saddle connection as well.

Proof of Theorem 1′.

CASE A. $\epsilon = +1$.

Since the k_i are even and k is minimal, it is easy to see that $n = 1$ or $n = 2$. The proof is by induction on $\sigma(k) = \Sigma_{i=1}^{2} k_i$. Assume $k_1 \geq k_2$. For the sake of applying Propositions 1.2 and 1.3 we make the following *induction hypotheses*:

(1) if $k_2 \neq 0$ there is a horizontal segment joining the two zeroes.

(2) if $k_1 > k_2$ there is a closed horizontal saddle connection from the zero of order k_1 to itself.

The only case when $\sigma(k) = 0$ is $(\ ; +1)$ which has already been constructed. Thus assume $\sigma(k) \geq 4$. To start the induction we construct the two examples for which $\sigma = 4$; namely $(2, 2; +1)$ and $(4; +1)$, and the example $(6, 2; +1)$. For the first, take a metric cylinder and choose a horizontal segment joining distinct points. Apply Proposition 1.2. Since the boundary lengths of the metric cylinder are equal, the same is true of $(2, 2; +1)$. To construct $(4; +1)$, choose a closed horizontal geodesic which is a waist curve of the metric cylinder. Apply Proposition 1.3. The induction hypotheses (1) and (2) are satisfied for both examples. The construction of $(6, 2; +1)$ is somewhat special since it does not follow the general pattern. We construct this example from $(4; +1)$ by joining the zero p of order 4 to a nonsingular point by a horizontal segment which is not a subsegment of the simple closed horizontal saddle connection from p to itself. Now apply Proposition 1.2. The remarks following the proof of Proposition 1.2 show that the induction hypotheses are satisfied for $(6, 2; +1)$ as well. Now suppose inductively we have constructed all examples k with $\sigma(k) = 4m$, $m \geq 1$, and k satisfies $\sigma(k) = 4m + 4$. If $k_1 \geq k_2 + 8$ let $k' = (k_1 - 4, k_2; +1)$. Then $k_1 - 4 > k_2$. The induction hypothesis (2) implies the existence of a horizontal saddle connection from the zero of order $k_1 - 4$ to itself. Proposition 1.3 allows us to construct k from k'. The remarks following Proposition 1.3 show that k satisfies the induction hypotheses (1) and (2). Suppose $k_2 \leq k_1 < k_2 + 8$. Since $k_1 + k_2 \geq 8$, $k_2 > 0$. The fact that $k_1 + k_2 \equiv 0 \bmod 4$ implies $k = (6, 2; +1)$, or $k_2 > 2$. The example $(6, 2; +1)$ has already been constructed so assume $k_2 > 2$. Let $k' = (k_1 - 2, k_2 - 2; +1)$. Since $k_2 - 2 > 0$, the induction hypothesis (1) says there is a saddle connection joining the two zeroes. We may apply Proposition 1.2 to k' to construct k. Again the remarks following Proposition 1.2 show that k satisfies (1) and (2).

We now divide the general case of minimal $k = (k_1, \ldots, k_n; -1)$ with $\sigma(k) \geq 0$ into two subcases based on the dichotomy of Proposition 1.1. We can assume $k_1 > 0$.

CASE B. $\epsilon = -1$ and k_1 is the only positive entry.

The proof is by induction on $\sigma(k)$. For the sake of applying Proposition 1.3 we make the following *Induction Hypothesis*: There is a closed horizontal loop based at the single zero of order k_1.

We construct the examples $(4m; -1)$ by induction on m. We have constructed the examples for $m = 1, 2$. These examples indeed satisfy the induction hypothesis. Suppose we have constructed $(4m; -1)$. We may apply Proposition 1.3 to construct $(4m + 4; -1)$. By the remark following Proposition 1.3 $(4m + 4; -1)$ satisfies the induction hypothesis. Notice the unrestricted example $(8; -1)$ cannot be constructed from the restricted example $(4; -1)$ using Proposition 1.3 because we cannot then get arbitrary boundary lengths. That is why it was constructed separately.

Now we construct the examples $k = (k_1, -1, \ldots, -1; -1)$ by induction on k_1. We have constructed all examples with $\sigma(k) = 0$ as well as $(5, -1; -1)$. For each the induction hypothesis holds. Suppose inductively we have constructed all examples for which $k_1 \leq m$ where $m \geq 5$. Suppose $k = (m + 1, \ldots)$. Let $k' = (m - 3, -1, \ldots)$ and $m - 3 \geq 2$. Then $k_1 - 3 \geq 2$ so k' is unrestricted. We construct the unrestricted k from k' by applying Proposition 1.3. Notice again $(5, -1; -1)$ cannot be constructed from $(1, -1; -1)$ since the latter is restricted and we cannot get arbitrary boundary lengths. That is why we constructed $(5, -1; -1)$ separately.

CASE C. $\epsilon = -1$ and there is more than one positive k_i.

The proof is by induction on σ. By Proposition 1.1, $n \leq 4$. We can assume k_1 is the largest positive entry. We make the following

Induction Hypothesis: If k is not the restricted example $(3, 1; -1)$ then for each k_i satisfying $k_i \geq \max(1, k_1 - 4)$ there is a closed horizontal saddle connection based at the zero of order k_i.

There are no minimal examples with $\sigma = 0$. If $\sigma(k) = 4$, $k_1 \le 3$. The possibilities are the basic examples constructed already in III: $(2, 2; -1)$, $(3, 2, -1; -1)$, $(3, 3, -1; -1)$, $(2, 1, 1; -1)$, $(1, 1, 1, 1; -1)$ and the restricted example $(3, 1; -1)$. It is easy to check that the induction hypothesis is satisfied by the basic examples. The example $(3, 1; -1)$ is constructed from $(1, -1; -1)$ by applying Proposition 1.2 to the horizontal saddle connection joining the zero and the pole. Suppose inductively we have constructed all examples k' with $\sigma(k') = 4m$ where $m \ge 1$ and $k = (k_1, k_2, \ldots, k_n; -1)$ satisfies $\sigma(k) = 4m + 4 \ge 8$. We can assume $k \ne (5, 3; -1)$ since it was already constructed.

Assume first that $k_1 \le 3$. Since $n \le 4$ and $\sigma(k) \ge 8$, the possibilities are

$$(3, 3, 2; -1),\ (3, 3, 3, -1; -1),\ \text{and}\ (3, 3, 3, 3; -1).$$

We construct $(3, 3, 2; -1)$ and $(3, 3, 3, -1; -1)$ from $(1, 1, 2; -1)$ and $(3, 1, 1; -1)$ respectively by applying Proposition 1.2 to the horizontal segment joining the two zeroes of order 1. Since for both $(1, 1, 2; -1)$ and $(3, 1, 1; -1)$ there is a horizontal loop from each zero to itself, the same is true for the examples built from them. Thus the induction hypothesis is still satisfied. To build $(3, 3, 3, 3; -1)$ we first construct $(3, 3, 1, 1; -1)$ from $(1, 1, 1, 1; -1)$ by applying Proposition 1.2 to a segment joining two zeroes of order 1. Then $(3, 3, 3, 3; -1)$ is constructed from $(3, 3, 1, 1; -1)$ by again applying Proposition 1.2 to a segment joining the remaining two zeroes with order 1.

Now assume $k_1 > 3$. Then $k_1 \ge 5$ by minimality. The case of $(7, 1; -1)$ is special since it is an exception to the induction hypothesis. We recall the construction of the flat surface realizing $(5, -1; -1)$ which has a horizontal saddle connection joining the zero and the pole. We apply Proposition 1.2 to construct $(7, 1; -1)$. For any other case let $k' = (k_1 - 4, k_2, \ldots, k_n; -1)$. Since k is assumed not to be $(5, 3; -1)$, $k' \ne (1, 3; -1)$ so k' is unrestricted. Moreover $k_1 - 4 \ge \max(1, k_i - 4)$, for all i, and so by the induction hypothesis there is a closed horizontal saddle connection based at the zero of order $k_1 - 4$. Applying Proposition 1.3 to k', constructs k. Again the induction hypothesis holds for k. This completes the proof of Theorem 1′.

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Department of Mathematics
University of Illinois at Chicago
Chicago, Illinois 60680

and

Department of Mathematics
Cornell University
Ithaca, NY 19850

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