Clones of spaces and maps in homotopy theory.

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This paper deals with certain infinite dimensional spaces which appear to be almost identical in the homotopy category. To explain precisely what is meant by "almost identical", I need a few definitions. Recall that two spaces, say X and Y, are said to have the same n-type if there exists a homotopy equivalence between $X^{(n)}$ and $Y^{(n)}$, their Postnikov approximations up through dimension n. These approximations can be obtained by attaching cells to the original spaces to kill off their homotopy groups in dimensions greater than n. Obviously, if X and Y are homotopy equivalent, then they have the same n-type for all n. However, the converse statement is false, indeed in [7] it is shown that counterexamples to the converse can occur when X is the classifying space of a compact Lie group.

Let $X_{(p)}$ denote the localization of X at a prime p in the homotopy theoretic sense of Bousfield–Kan, [1] or Sullivan, [15]. If X and Y are homotopy equivalent nilpotent spaces then so are $X_{(p)}$ and $Y_{(p)}$, for each prime p. Again the converse statement is false. In fact, a famous example of Rector [11] shows that when X is the infinite dimensional quarternionic projective space, there are, up to homotopy, uncountably many different Y's, each of finite type and each locally p-equivalent to X at each prime p.

In this paper we will regard two nilpotent spaces, X and Y, as almost identical if (i) they have the same n-type for all n and (ii) their localizations at each prime are homotopy equivalent. When this happens we will say that the X is a *clone* of Y. The obvious question is then – does it follow, when X is a clone of Y, that the two spaces are necessarily homotopy equivalent? I will show that the answer is no, even when the spaces are 1-connected with finite type. According to the definition just given, any space is a clone of itself, and so I will use the adjective *nontrivial* in describing these clones of a space which are not homotopy equivalent to it.

EXAMPLE 1. When $X = S^3 \times K(\mathbf{Z}, 3)$, there are, up to homotopy, uncountably many different clones of X. □

Proofs will be given later in this paper. Here is a simple construction of some clones of $S^3 \times K(\mathbf{Z}, 3)$. Partition the set of all primes into two subsets, say A and B. Let

Z denote the homotopy pull back of the diagram

$$\begin{array}{ccc} & & K(\mathbf{Z}_{(B)}, 3) \\ & & \downarrow \\ S^3_{(A)} & \longrightarrow & K(\mathbf{Q}, 3) \end{array}$$

wherein the maps are rational equivalences. Let Z' be a second pullback obtained by reversing the roles of A and B. Then define Y to be $Z \times Z'$. It will be shown that Y is a clone of X, and that different partitions give rise to different clones of X. I do not know, however, if every clone of X can be constructed in this manner.

One place where clones arise is in the study of the Mislin genus of an infinite dimensional space X. Recall that this genus, $\mathcal{G}(X)$, is defined to be the set of all homotopy types $[Y]$ where Y is a nilpotent space of finite type and where, for each prime p, $Y_{(p)} \simeq X_{(p)}$. If $X^{(n)}$ denotes the Postnikov approximation of X up through dimension n, then one has a map

$$\mathcal{G}(X) \to \lim_{\leftarrow} \mathcal{G}(X^{(n)})$$

that sends a homotopy type $[Y]$ to the sequence $([Y^{(1)}], [Y^{(2)}], [Y^{(3)}], \ldots)$. Very little is known about this function in general. For instance, is it always surjective? Notice that the preimage of $[X]$ under this function is just the set of clones of X. Thus Example 1 shows that this function need not be one-to-one.

EXAMPLE 2. Let $X = \mathbf{HP}^\infty$, the infinite dimensional quaternionic projective space. Then $X_{(S)}$ has nontrivial clones (uncountably many, in fact) if and only if S is an infinite subset of prime numbers whose complement is also infinite. Moreover, any two clones of $X_{(S)}$ become homotopy equivalent when localized at any finite set of primes. □

The verification of this example involves a surprising amount of arithmetic. In the proof, the existence of clones of $X_{(S)}$ is shown to depend on whether or not the index of a certain subgroup in $\mathbf{Z}^*_{(S)}$ is infinite. The subgroup in question consists of all positive units that have p-adic square roots for each prime $p \in S$. The question about the index, in turn, is shown to depend only on the cardinality of both S and its complement. The proof of this result, Theorem 2.3, is due in large part to Hugh Montgomery. I am very grateful to him for the clever proof he gave in response to my query and for allowing me to present it here.

Let G denote a 1-connected compact Lie group. The existence of nontrivial clones of BG, before localizing, is an open question. Only two special cases of it are

known ($G = \mathrm{SU}(2)$ and $\mathrm{SU}(3)$, wherein no nontrivial clones of BG exist). As Example 2 shows, the case of $\mathrm{BSU}(2)$, after localizing at any set of primes S, is well understood. This case, however is the exception. The following result is currently the most general one I know regarding clones of classifying spaces.

EXAMPLE 3. Let $X = BG_{(F)}$ where G is a compact connected Lie group of rank at least 2, and F is a finite set of two or more primes. Then the set of homotopy types of clones of X is a countably infinite set. □

Following Wilkerson, [16], let $\mathrm{SNT}(X)$ denote the set of homotopy types $[Y]$ with Postnikov approximations $Y^{(n)} \simeq X^{(n)}$ for all n. Notice that the set of homotopy types of clones of X is the intersection

$$\mathrm{Clones}(X) = \mathrm{SNT}(X) \cap \mathscr{G}(X).$$

This raises the question – Do nontrivial clones of X always exist when the sets $\mathrm{SNT}(X)$ and $\mathscr{G}(X)$ are both nontrivial? This seems to be a hard question. The case of $X = BG$, where the rank of G is at least 3, is a special case of it.

The study of clones reveals the strange manner in which the function $\mathrm{SNT}(\ )$ behaves with respect to localization. For, as Example 2 shows, there exists a space Y such that

$$\mathrm{SNT}(Y) \neq * \qquad \text{while } \mathrm{SNT}(Y_{(p)}) = * \quad \text{for every prime } p.$$

However, the example of $\mathrm{BSU}(3)$ shows that there exists a space X such that

$$\mathrm{SNT}(X) = * \qquad \text{while } \mathrm{SNT}(X_{(p)}) \neq * \quad \text{for every prime } p.$$

This last example was worked out in [7]. These examples seem to suggest that recovering $\mathrm{SNT}(X)$ from that of its localizations is going to be difficult if not impossible.

Clones of maps

We say that two maps, say $f, g : X \to Y$ are clones of each other if their localizations at each prime are homotopic and if their Postnikov approximations at each stage are homotopic.

It is worth noting that if X is a CW-complex with finite n-skeletons for each n, then the condition that f and g are homotopic at each prime p implies that their

restrictions to each skeleton of X are homotopic, by [5], Theorem 5.3. Hence, in this case, the Postnikov approximations of f and g would likewise be homotopy equivalent. However, if X does not have finite type, then the first condition does not, in general, imply the second one (ibid., Prop. 5.5).

One place where clones of maps often occur is among phantom maps. Recall that a phantom map is a based map from a CW-complex X to another space Y, whose restriction to each skeleton X_n is null-homotopic. Equivalently, a phantom map from X to Y is one whose projection onto each Postnikov approximation $Y^{(n)}$ is null homotopic. Let $\mathrm{Ph}(X, Y)$ denote the set of homotopy classes of phantom maps from X to Y. The next theorem contains the most general result I know regarding clones among phantom maps.

THEOREM 4. *Assume that X and Y are nilpotent CW-complexes of finite type. If X has the rational homotopy type of a suspension, or if Y has the rational homotopy type of a loop space, then the following statements are true*:

(i) $\mathrm{Ph}(X, Y)$ *has a natural, divisible, abelian group structure.*

(ii) *The map $Y \to \Pi Y_{(p)}$ whose pth component for each prime p, is the canonical map $Y \to Y_{(p)}$, induces an epimorphism*

$$\mathrm{Ph}(X, Y) \to \prod_p \mathrm{Ph}(X, Y_{(p)}).$$

(iii) *The map just displayed has a nonzero kernel whenever its domain,* $\mathrm{Ph}(X, Y)$ *is nontrivial. This kernel (consisting of clones of the constant map) is also a divisible group.* □

Remark. Part (i) generalizes results of Roitberg ([12], [13]) who reached the same conclusion assuming that X is a co-H-space or that Y is a H-space.

The result in part (ii) is not the most general one possible. The rational hypothesis on X or Y can be dropped here at the expense of losing the natural group structure on $\mathrm{Ph}(X, Y)$. More importantly, the induced map in part (ii) remains an epimorphism of sets. This follows from a $\varprojlim^1$ result of R. Steiner, ([14], Theorem 2.5).

Since nonzero divisible groups are never finite, one concludes in part (iii) that there are, up to homotopy, infinitely many different clones of the constant map in $\mathrm{Ph}(X, Y)$, whenever this group has more than one element in it. I do not know if the rational conditions on X or Y are really necessary here. In the proof, these conditions enable one to identify the set $\mathrm{Ph}(X, Y)$ with $\mathrm{Ext}(A, \mathbf{Z})$, where A is a certain torsion-free countable abelian group. The conclusions of parts (ii) and (iii)

are then a consequence of some homological algebra; namely that the obvious map

$$\mathrm{Ext}\,(A, \mathbf{Z}) \to \prod_p \mathrm{Ext}\,(A, \mathbf{Z}_{(p)})$$

it always surjective and that it always has a nontrivial kernel (unless, of course, its domain, Ext $(A, \mathbf{Z})$ is the trivial group). This particular result was one announced by Willi Meier in [9], but as far as I know, no proof of it was ever published. The slick proof of it in this paper is due to H. Pat Goeters; I am very grateful to him for allowing me to use it here.

Let me mention two examples which are relevant to Theorem 4. The first one is due to Harper and Roitberg. In [4], they study phantom maps whose domain is a finite Postnikov space and whose range is the iterated loops on a finite complex. In this case one can be more specific about clones of the constant map (or *special* phantom maps in their terminology). The following example is representative of their Theorem 2.2.

EXAMPLE 4.1. In Ph $(\mathbf{CP}^\infty, \Omega^n S^{n+3})$ the subgroup consisting of clones of the constant map is uncountably large and its index is also uncountably large. □

Compare this example to the next, in which every phantom map is a clone of the constant one.

EXAMPLE 4.2. Let

$$X = \mathrm{cofiber}\left\{\alpha_1 : \bigvee_{p \geq 3} S^{2p} \to S^3\right\}$$

where for each prime p, $\alpha_1 \mid S^{2p} = \alpha_1(p)$. Then Ph $(X, S^4) \neq 0$, while Ph $(X, S^4_{(p)}) = 0$ for all primes p. □

This phenomenon, where essential phantoms exist and yet all of them are clones of the constant map, also occurs in Ph $(\Omega^2 S^{2n+1}, S^{2n})$ for each $n \geq 2$. Both examples are verified in [3].

Here is another situation in which clones of maps arise quite naturally. Let Aut (X) denote the group of based homotopy classes of self equivalences of a space X and let $WI(X)$ denote the *weak identities* of X, that is, the subgroup of Aut (X) consisting of those classes which project to the identity class on each Postnikov approximation of X.

THEOREM 5. *Let X be a simply connected CW-complex of finite type. If the subgroup $WI(X)$ is nonzero, then it contains nontrivial clones of the identity map.* □

A simple example in which $WI(X) \neq 0$ is $X = \mathbf{CP}^\infty \times S^3$. In this case, $WI(X) \approx \mathrm{Ph}(\mathbf{CP}^\infty, S^3) \approx \mathbf{R}$, as rational vector spaces, [12]. It is still an open question if $WI(Y)$ could be nonzero when Y is the space of loops, free or based, on a finite complex.

This completes the discussion of the main results in this paper. Before giving the proofs, I wish to thank Hugh Montgomery and Pat Goeters for their contributions mentioned earlier. I also want to thank Jesper Møller for his help in the early stages of this project. In particular, it was he who first proved the existence of clones in Example 1 by methods different from those used here. He also deserves the credit for naming these things clones.

Proofs

Proof of Example 1. Let $Y = Z \times Z'$ as described in the introduction. It is clear from the construction that Y is in the genus of X where $X = S^3 \times K(\mathbf{Z}, 3)$. Thus the Postnikov approximation $Y^{(n)} \in \mathcal{G}(X^{(n)})$, for each natural number n. According to Zabrodsky, [17], there is a short exact sequence

$$\mathcal{E}_t(X^{(n)}) \xrightarrow{d} \mathbf{Z}_t^* / \pm 1 \longrightarrow \mathcal{G}(X^{(n)}) \longrightarrow *$$

which is defined as follows. In the middle term, $\mathbf{Z}_t^*$ denotes the group of units in the ring of integers modulo t. This number t depends upon $X^{(n)}$. The prime divisors of t include those primes p, for which there is p-torsion in the homotopy groups of $X^{(n)}$. Zabrodsky gives a description of the smallest possible exponents $v_p(t)$, in [17]. However, it should be noted that in this sequence, t can be taken to be sufficiently large in the multiplicative sense. The first term in the sequence, $\mathcal{E}_t(X^{(n)})$, denotes the monoid (under composition) of homotopy classes of those self-maps of $X^{(n)}$, which are local equivalences at each prime divisor of t. The function d then assigns to each such map the determinant of the linear transformation f^* on $H^3(X^{(n)}, \mathbf{Z})$. Zabrodsky shows that this image is a subgroup and that the quotient is isomorphic as an abelian group to $\mathcal{G}(X^{(n)})$. Since $X = S^3 \times K(\mathbf{Z}, 3)$ it is easy to find, for any integer d, a self map of X, and of $X^{(n)}$, which induces a linear transformation with determinant d on $H^3(\ , \mathbf{Z})$. Thus the first map in Zabrodsky's sequence is surjective and so $\mathcal{G}(X^{(n)}) = *$. Consequently, $Y^{(n)} \simeq X^{(n)}$ for each natural number n, and thus Y is a clone of X. However, notice that Y has the following simple property: There

is a basis, say $\{u, v\} \subseteq H^3(Y, \mathbf{Z}) \approx \mathbf{Z} \oplus \mathbf{Z}$, such that

$$\mathscr{P}^1 u_p \neq 0 \quad \text{and} \quad \mathscr{P}^1 v_p = 0, \qquad \text{for every } p \in A$$

while

$$\mathscr{P}^1 u_p = 0 \quad \text{and} \quad \mathscr{P}^1 v_p \neq 0, \qquad \text{for every } p \in B.$$

Here x_p denotes the image of an integral class x in $H^3(Y, \mathbf{Z}/p)$. Also when $p = 2$, one should replace $\mathscr{P}^1$ by Sq^2 in this property. If one takes a different partition $\{A', B'\}$ of the set of all primes, it is not difficult to verify that there is no basis of $H^3(Y, \mathbf{Z})$ with the corresponding property in terms of A' and B'. Since this property is clearly homotopy invariant, our claim follows that distinct partitions of the set of all primes give rise to distinct clones of X. □

Proof of Example 2. Recall that an H_0-space is one whose rationalization is an H-space. In particular, $\mathbf{HP}^\infty$ becomes an Eilenberg–MacLane space $K(\mathbf{Q}, 4)$ when rationalized and so the following result from [7] applies to it.

THEOREM 2.1. *Let X be a 1-connected, H_0-space with finite type over $\mathbf{Z}_P$ for some set of primes P. The following conditions are equivalent*:

(i) $\mathrm{SNT}(X) = *$.

(ii) *the usual map* $\mathrm{Aut}\, X \to \mathrm{Aut}\, X^{(n)}$, *has a finite cokernel for all* n.

(iii) *the map* $\mathrm{Aut}\, X \xrightarrow{f \mapsto f^*} \mathrm{Aut}\, H^{\leq n}(X; \mathbf{Z}_P)$ *has a finite cokernel for all integers* n.

□

Let B denote $\mathbf{HP}^\infty$. If we localize at a set of primes S, it is easy to see that for $n \geq 4$,

$$\mathrm{Aut}\, H^{\leq n}(B_{(S)}, \mathbf{Z}_{(S)}) \approx \mathbf{Z}^*_{(S)}$$

because the cohomology ring in question is a polynomial algebra on a single generator of degree 4, truncated at height $[n/4] + 1$. Each graded algebra automorphism of it is completely determined by what it does in degree 4. The following result then describes the image of the composite map,

$$\mathrm{Aut}\, B_{(S)} \to \mathbf{Z}^*_{(S)}.$$

This result is a consequence of the pioneering work of Sullivan, [15], and Rector, [11].

THEOREM 2.2. *Given $\lambda \in \mathbf{Z}^*_{(S)}$, there is a self-equivalence of $B_{(S)}$ that induces multiplication by λ in degree 4 if and only if λ has a square root in the p-adic numbers for each prime p in S.* □

Recall from number theory that when p is an odd prime and λ is a p-local unit, one has the Legendre symbol

$$(\lambda/p) = \begin{cases} 1 & \text{if } \lambda \text{ is a square mod } p \\ -1 & \text{if } \lambda \text{ is not a square mod } p \end{cases}$$

and that λ is the square of a p-adic integer if and only if $(\lambda/p) = 1$. For $p = 2$, set

$$(\lambda/2) = \begin{cases} 1 & \text{if } \lambda \text{ is a square mod } 8 \\ -1 & \text{if } \lambda \text{ is not a square mod } 8. \end{cases}$$

For a 2-local unit λ to be the square of a 2-adic integer it is necessary and sufficient that $(\lambda/2) = 1$. Thus letting $S = \{p\}$, it follows that the image of Aut $B_{(p)}$ in Aut $H^{\leq n}(B, \mathbf{Z}_{(p)})$ has index at most 2 or 4, depending on whether p is odd or even. In both cases it follows that SNT $(B_{(p)}) = *$ by Theorem 2.1.

Now suppose $Y \in$ SNT $(B_{(S)})$, where S is any nonempty set of primes. Localizing at any prime $p \in S$, it follows that $Y_{(p)}$ is in SNT $(B_{(p)})$. It was just shown that SNT $(B_{(p)})$ has only one element and so it follows that $Y_{(p)} \simeq B_{(p)}$; thus $Y \in \mathcal{G}(B_{(S)})$. In other words, every member of SNT $(B_{(S)})$ is a clone of $B_{(S)}$. Now, the set SNT $(B_{(S)})$ is either the one element set or else it is uncountably large by Theorem 2 of [7]. To determine which alternative holds, one can use Theorems 2.1, 2.2, and the following result.

THEOREM 2.3. *Let S be a set of primes and let $\mathcal{U}$ denote the group of positive units in the ring of integers localized at S. Define*

$$\mathbf{G} = \{\lambda \in \mathcal{U} \mid (\lambda/p) = 1 \text{ for every } p \in S\}.$$

Then **G** *has finite index in $\mathcal{U}$ if and only if either S is a finite set or its complement is finite.* □

Note that $\mathcal{U}$ has index 2 in $\mathbf{Z}^*_{(S)}$. Thus, for the purposes of applying Theorem 2.1, it suffices to know whether the index of **G** in $\mathcal{U}$ is finite or not.

Let me begin the proof of 2.3 with the easiest case where S is a finite set of n primes. For each odd prime p in S, the function $\lambda \mapsto (\lambda/p)$ defines an epimorphism from $\mathcal{U}$ to $\mathbf{Z}/2$. If $2 \in S$, one can also map $\mathcal{U}$ onto the units in $\mathbf{Z}/8$. The subgroup

$\mathbf{G}$ is clearly the intersection (over S) of the kernels of the epimorphisms just considered. It follows that the index of $\mathbf{G}$ in $\mathcal{U}$ is at most 2^{n+1} in this case.

The next case to consider is where the complement of S consists of a finite set of primes, say $p_1, p_2, \ldots, p_n$. These primes freely generate the subgroup $\mathcal{U}$. Within $\mathcal{U}$ there is the subgroup $\mathcal{U}^2 = \{x^2 \mid x \in \mathcal{U}\}$; of index 2^n in $\mathcal{U}$. Clearly $\mathcal{U}^2 \subseteq \mathbf{G}$ and so, in this case, the index of $\mathbf{G}$ in $\mathcal{U}$ is at most 2^n.

We are left with the hardest case where both S and its complement are infinite. The following argument is due to Hugh Montgomery. Let $\mathbf{A}$ denote a doubly infinite matrix whose rows are indexed by the odd primes $p \in S$ and whose columns are indexed by the odd primes $q \notin S$. Each entry in $\mathbf{A}$ is an element of $\mathbf{Z}/2$ and is given as follows:

$$a_{p,q} = \begin{cases} 1 & \text{if } (q/p) = -1 \\ 0 & \text{if } (q/p) = 1. \end{cases}$$

CLAIM. *The subgroup* $\mathbf{G}$ *has finite index in* $\mathcal{U}$ *if and only if the matrix* $\mathbf{A}$ *has finite rank.* □

To verify this, consider the homomorphism, say

$$\mathcal{U} \xrightarrow{\varphi} \prod_{\text{odd } p \text{ in } S} \mathbf{Z}/2,$$

defined by the product of the Legendre symbols. The q-column in this matrix records the image of $\varphi(q)$ in additive notation. It is then easy to see that $\mathbf{A}$ has finite rank

$\Leftrightarrow \mathbf{A}$ has only finitely many linearly independent columns

$\Leftrightarrow$ the image of φ is finitely generated

$\Leftrightarrow$ the kernel of φ has finite index in $\mathcal{U}$.

If $2 \in S$, there is also a reduction mod 8 map,

$$\rho : \mathcal{U} \to (\mathbf{Z}/8)^*.$$

Since $\mathbf{G} = \ker \varphi \cap \ker \rho$, and $\ker \rho$ has finite index in $\mathcal{U}$, it follows that $\mathbf{G}$ has finite index in $\mathcal{U}$ if and only if $\ker \varphi$ does.

If 2 is not in S, it is then one of the units in $\mathcal{U}$. It is clear that $\mathbf{G}$ has finite index in $\mathcal{U}$ if and only if the image of

$$\mathcal{U} \xrightarrow{\varphi} \prod_{p \text{ in } S} \mathbf{Z}/2$$

is finitely generated. But for this purpose, it is enough to consider the sub-group of $\mathcal{U}$ generated by the odd primes not in S. The claim follows.

Assume now that the matrix $\mathbf{A}$ has finite rank r. I will show that this leads to a contradiction. Choose n larger than r. There is evidently a nontrivial linear relation among the first n rows of $\mathbf{A}$; that is, an equation

$$\sum c_p R_p = 0$$

where the sum is indexed by the first n odd primes in S and where the mod 2 coefficients c_p are not all zero. Let P denote the product of those odd primes in S for which $c_p \neq 0$. It follows, that for each odd prime $q \in S$,

$$\sum_{p \mid P} a_{p,q} = 0.$$

Expressed multiplicatively, in terms of the Legendre symbols, this says

$$\prod_{p \mid P} (q/p) = 1, \qquad \text{for each odd prime } q \in S.$$

Using the Jacobi symbol, this is simply $(q/P) = 1$ for all odd primes q not in S. By repeating this argument with rows and columns interchanged, we see that there is an integer Q (squarefree, and composed of a subset of the first n odd primes not in S) such that $(Q/p) = 1$ for all odd primes p in S.

Choose a residue class $a \pmod P$ for which $(a/P) = -1$. By quadratic reciprocity, as b runs through the odd positive integers, the value (Q/b) has period $4Q$. (This is not necessarily the least period.) Choose a value of $b \pmod{4Q}$ for which $(Q/b) = -1$. Since P and $4Q$ are relatively prime, it follows by the Chinese remainder theorem there is a $c \pmod{4PQ}$ for which $c = a \pmod P$ and $c \equiv b \pmod{4Q}$. Since c and $4PQ$ are relatively prime, it follows by Dirichlet's theorem that there is an odd prime l with $l \equiv c \pmod{4PQ}$. Thus $l \equiv a \pmod P$ and $l \equiv b \pmod{4Q}$, which is to say

$$(l/P) = -1 \quad \text{and} \quad (Q/l) = -1.$$

The first equation implies that l is in S, while the second implies that it is not. This contradiction completes the proof of Theorem 2.2.

The last statement in Example 2 is a consequence of the triviality of SNT $(X_{(S)})$ when S is a finite set of primes. This, in turn, follows from Theorems 2.1, 2.2, and 2.3. □

Proof of Example 3. Let $X = BG_{(F)}$ where G is a compact, connected Lie group and F is a finite set of two or more primes. Because F is finite, it follows from [17], Proposition 1.5, that $\mathscr{G}(X^{(n)}) = *$ for each integer n. Thus

$$Y \in \mathscr{G}(X) \Rightarrow Y \in \text{SNT}\,(X),$$

and consequently every member of the genus of X is a clone of X. However, the genus of X is an infinite set when rank $(G) \geq 2$, by Theorem 2.2 of [10]. That this set is countable can be seen by representing $\mathscr{G}(X)$ as a double coset space of the form $A\backslash G/B$ where G is the n-fold product of Aut (X_0), and n is the number of primes in F. □

Proof of Theorem 4. The proof of part (i) starts with the natural bijection of pointed sets,

$$\text{Ph}\,(X, Y) \approx \varprojlim{}^1 [X, \Omega Y^{(n)}].$$

See [1], Chapter IX, for this and background information on $\varprojlim^1$. Let $G_n = [X, \Omega Y^{(n)}]$. The finiteness conditions on X and Y imply that each G_n is a finitely generated nilpotent group. The rational conditions on X or Y imply that each rationalized group, $(G_n)_0 \approx [A, \Omega^2 B]$, and thus is abelian. Since the rationalization map,

$$[X, \Omega Y^{(n)}] \to [X, \Omega Y^{(n)}]_{(0)}$$

has a finite kernel ([5], page 84), it follows that each G_n has a finite commutator subgroup, denoted G'_n. Consider the short exact sequence of towers,

$$\{G'_n\} \to \{G_n\} \to \{A_n\}$$

where A_n is G_n made abelian. Apply the 6-term $\varprojlim - \varprojlim^1$ sequence to this short exact sequence of towers and recall that $\varprojlim^1$ vanishes on towers of finite groups. It follows that the quotient map, $G_n \to A_n$, induces a bijection

$$\varprojlim{}^1 G_n \approx \varprojlim{}^1 A_n.$$

Clearly, this bijection is natural as well. Since the tower $\{A_n\}$ is abelian, there is a well known description of its $\lim^1$ term as the cokernel of a homomorphism into $\Pi_n A_n$. From this abelian group structure on the $\lim^1$ term and the natural bijections just described, one gets a natural abelian group structure on $\mathrm{Ph}(X, Y)$.

Since the groups G_n were finitely generated, so are their abelian quotients A_n. Let T_n denote the torsion subgroup of A_n. It is then a finite group which, for purposes of $\lim^1$ calculations, can be ignored. Indeed, apply the 6 term $\lim - \lim^1$ sequence to the short exact sequence of towers

$$\{T_n\} \to \{A_n\} \to \{F_n\}.$$

Since $\lim^1 T_n = 0$ for reasons noted earlier, the $\lim^1$ terms for $\{A_n\}$ and for the torsion free quotient $\{F_n\}$ are isomorphic. Hence, we may (and will) assume from now on that the groups A_n are torsion free.

It is easy to see that the group $\lim^1 A_n$ is divisible; one method amounts to applying $\lim^1$ to the short exact sequence of towers,

$$\{A_n\} \xrightarrow{d} \{A_n\} \longrightarrow \{A_n/d\}.$$

This could also be proved using Jensen's formula

$$\lim{}^1 A_n \approx \mathrm{Ext}(\lim \mathrm{Hom}(A_n, \mathbf{Z}), \mathbf{Z}). \tag{1}$$

We will have other uses for this isomorphism as well and we need to understand it better. Jensen obtains it in the following way in [6]. He first constructs a short exact sequence of towers

$$\{A_n\} \xrightarrow{i} \{B_n\} \xrightarrow{j} \{C_n\} \tag{2}$$

in which $B_n = \Sigma_{i \le n} A_i = A_n \oplus B_{n-1}$. The structure map $B_n \to B_{n-1}$ coincides with $A_n \to A_{n-1} \xrightarrow{i} B_{n-1}$ on the first factor and it restricts to the identity on the second factor. Since the map just described is an epimorphism, it follows that $\lim^1 B_n = 0$ and the 6-term $\lim - \lim^1$ sequence, applied to (2) reduces to the following

$$0 \longrightarrow \lim A_n \xrightarrow{i_*} \lim B_n \xrightarrow{j_*} \lim C_n \longrightarrow \lim{}^1 A_n \longrightarrow 0. \tag{3}$$

Apply $\mathrm{Hom}(\ , \mathbf{Z})$ to (2), denote it by $(\ )^*$, and then take direct limits. The result is a short exact sequence of groups,

$$\lim A_n^* \longrightarrow \lim B_n^* \longrightarrow \lim C_n^*. \tag{4}$$

It is easy to see that $\varinjlim B_n^* \approx \oplus A_n$, which is free. Consequently, $\operatorname{Ext}(\varinjlim B_n^*, \mathbf{Z}) = 0$. Moreover, if D denotes A, B, or C, one has a natural isomorphism

$$\operatorname{Hom}(\varinjlim D_n^*, \mathbf{Z}) \approx \varprojlim D_n$$

because the groups D_n are free abelian of finite rank. Therefore, the Hom–Ext sequence, applied to (4) reduces to

$$0 \longrightarrow \varprojlim A_n \xrightarrow{i_*} \varprojlim B_n \xrightarrow{j_*} \varprojlim C_n \longrightarrow \operatorname{Ext}(\varinjlim A_n^*, \mathbf{Z}) \longrightarrow 0. \tag{5}$$

Comparing (3) and (5), the isomorphism in (1) follows. Using this description it is not difficult to see that the following diagram commutes,

$$\begin{array}{ccc} \operatorname{Ext}(\varinjlim A_n^*, \mathbf{Z}) & \longrightarrow & \operatorname{Ext}(\varinjlim A_n^*, \mathbf{Z}_{(p)}) \\ \downarrow \approx & & \downarrow \approx \\ \varprojlim{}^1 A_n & \longrightarrow & \varprojlim{}^1 (A_n \otimes \mathbf{Z}_{(p)}) \end{array}$$

wherein the horizontal maps are induced by the inclusion $\mathbf{Z} \to \mathbf{Z}_{(p)}$. The rest of Theorem 4 is then a consequence of the following result. The proof here is due to H. Pat Goeters.

THEOREM 4.3. *Let A be a countable, torsion free abelian group. The diagonal embedding $\mathbf{Z} \to \Pi_p \mathbf{Z}_{(p)}$ induces a map*

$$\operatorname{Ext}(A, \mathbf{Z}) \longrightarrow \prod_p \operatorname{Ext}(A, \mathbf{Z}_{(p)})$$

which is always surjective and which has a nonzero kernel whenever $\operatorname{Ext}(A, \mathbf{Z}) \neq 0$. □

Proof. Let P denote the product $\Pi_p \mathbf{Z}_{(p)}$ and consider the short exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\delta} P \longrightarrow C \longrightarrow 0.$$

Here δ is the diagonal embedding and C is its cokernel. Since δ induces isomorphisms under $\operatorname{Tor}(\ , \mathbf{Z}/p)$ and $(\ ) \otimes \mathbf{Z}/p$ for each prime p, it follows that C is torsion free and divisible. Therefore the third term in the following portion of the $\operatorname{Hom}(A, \ )$–$\operatorname{Ext}(A, \ )$ sequence

$$\longrightarrow \operatorname{Ext}(A, \mathbf{Z}) \xrightarrow{\delta_*} \operatorname{Ext}(A, P) \longrightarrow \operatorname{Ext}(A, C),$$

is zero and so δ_* is surjective as claimed. Since A is countable and torsion free, it follows from Stein's theorem ([2], page 94) that

$$A \approx B \oplus F$$

where F is free and $\mathrm{Hom}(B, \mathbf{Z}) = 0$. Since $\mathrm{Ext}(F, \mathbf{Z}) = \mathrm{Ext}(F, P) = 0$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(A, \mathbf{Z}) & \xrightarrow{\delta_*} & \mathrm{Ext}(A, P) \\ \downarrow \approx & & \downarrow \approx \\ \mathrm{Ext}(B, \mathbf{Z}) & \xrightarrow{\delta'_*} & \mathrm{Ext}(B, P) \end{array}$$

and thus $\ker \delta = 0$ iff $\ker \delta' = 0$. Suppose that $\ker \delta' = 0$. Then, in the following portion of the $\mathrm{Hom}(B, \ ) - \mathrm{Ext}(B, \ )$ sequence,

$$\mathrm{Hom}(B, \mathbf{Z}) \longrightarrow \mathrm{Hom}(B, P) \longrightarrow \mathrm{Hom}(B, C) \longrightarrow \mathrm{Ext}(B, \mathbf{Z}),$$

the first group is zero and the image of the last map is zero. This forces the map in the middle to be an isomorphism. However, since C is divisible, so is $\mathrm{Hom}(B, C)$. The isomorphisms

$$\mathrm{Hom}(B, C) \approx \mathrm{Hom}(B, P) \approx \prod_p \mathrm{Hom}(B, \mathbf{Z}_{(p)})$$

then imply the last group is divisible as well. However, it is easy to see that this can happen only if $B = 0$. □

Proof of Theorem 5. Let aut X denote the space of self equivalences of X; hence $\mathrm{Aut}\, X = \pi_0(\mathrm{aut}\, X)$. A Postnikov decomposition of X then induces a short exact sequence of groups,

$$\varprojlim{}^1 \pi_1 \mathrm{aut}\, X^{(n)} \longrightarrow \mathrm{Aut}\, X \longrightarrow \varprojlim \mathrm{Aut}\, X^{(n)}$$

according to [1], page 254. Thus $WI(X) \approx \varprojlim{}^1 \pi_1 \mathrm{aut}\, X^{(n)}$. Since each $X^{(n)}$ has finite type, the tower $\{\pi_1 \mathrm{aut}\, X^{(n)}\}$ is one of finitely generated abelian groups. The properties of its $\varprojlim{}^1$ term are those described in the proof of Theorem 4 and Theorem 4.3. □

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