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On the semicontinuity of curvatures

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Dedicated to Prof. Dr. Kurt Leichtweiß on the occasion of his Sixty-fifth Birthday

In this article, a convex hypersurface is the boundary of a convex body (compact, convex subset with non-empty interior) of Euclidean n-space, $\mathbf{R}^n$, Let $\mathscr{C}^n$ denote the space of convex hypersurfaces, endowed with the topology induced by the Hausdorff metric (see, for example, Busemann [1] or Leichtweiß [4]). Let $\mathscr{C}_2^n$ denote the subset of $\mathscr{C}^n$ consisting of the regular C^2-hypersurfaces with everywhere positive principal curvatures. For $Q \in \mathscr{C}_2^n$, and $x \in Q$, let $K(Q, x)$ denote the Gauss curvature of Q at the point x, and let

$$K^+(Q) = \max_{x \in Q} K(Q, x) \quad \text{and} \quad K^-(Q) = \min_{x \in Q} K(Q, x).$$

A special case of the main result of this article is that,

$$K^+ : \mathscr{C}_2^n \to (0, \infty) \quad \text{is lower semicontinuous,}$$

while

$$K^- : \mathscr{C}_2^n \to (0, \infty) \quad \text{is upper semicontinuous.}$$

Let $r_1(x), \ldots, r_{n-1}(x)$ denote the principal radii of curvature at the point $x \in Q$. The normalized j-th elementary symmetric function of the principal radii of curvature at x will be denoted by $s_j(x)$; i.e.,

$$\binom{n-1}{j} s_j(x) = \sum_{1 \le i_1 < \cdots < i_j \le n-1} r_{i_1}(x) \cdots r_{i_j}(x).$$

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For $Q \in \mathscr{C}_2^n$, define

$$C_j^+(Q) = \max_{x \in Q} s_j(x) \quad \text{and} \quad C_j^-(Q) = \min_{x \in Q} s_j(x).$$

Thus, $1/C_{n-1}^+(Q) = K^-(Q)$, and $1/C_{n-1}^-(Q) = K^+(Q)$. It will be shown that the functionals C_j^+ and C_j^- may be extended to all of $\mathscr{C}^n$ in such a manner that

$$C_j^+ : \mathscr{C}^n \to [0, \infty] \quad \text{is lower semicontinuous,}$$

and

$$C_j^- : \mathscr{C}^n \to [0, \infty] \quad \text{is upper semicontinuous.}$$

More general results will be established along the way. One of these results will establish a conjecture of Wm. J. Firey (see [2, p. 257]) regarding the upper semicontinuity of a certain functional for convex hypersurfaces.

Some preliminary notation and results will be helpful. Let $C^+(S^{n-1})$ denote the set of positive continuous functions on the unit sphere S^{n-1}. For $f \in C^+(S^{n-1})$, and $\alpha \in \mathbf{R}$, such that $\alpha \neq 0$, define

$$|f|_\alpha = \left\{ \frac{1}{\kappa_n} \int_{S^{n-1}} f(u)^\alpha \, dS(u) \right\}^{1/\alpha},$$

where the integration is with respect to spherical Lebesgue measure, S, on S^{n-1}, and $\kappa_n = S(S^{n-1})$ denotes the surface area of S^{n-1}. From the Hölder inequality it follows that,

$$|f|_{\alpha_1} \leq |f|_{\alpha_2} \qquad \text{whenever } \alpha_1 \leq \alpha_2. \tag{1}$$

For $f \in C^+(S^{n-1})$, and μ a Borel measure on S^{n-1}, let

$$\langle f, \mu \rangle = \frac{1}{\kappa_n} \int_{S^{n-1}} f(u) \, d\mu(u).$$

For $Q \in \mathscr{C}_2^n$ and $u \in S^{n-1}$, write $(Q, u) \in Q$ for the point of Q at which the outer unit normal is u. Let $r_1(Q, u), \ldots, r_{n-1}(Q, u)$ denote the principal radii of curvature at (Q, u), and let $s_j(Q, u)$ denote the normalized j-th elementary symmetric function of the principal radii of curvature at (Q, u). It will be convenient to view $s_j(Q, \cdot)$ as a function on S^{n-1}.

Suppose $Q \in \mathscr{C}_2^n$. For $j = 1, \ldots, n-1$, and $\alpha \in \mathbf{R}$, such that $\alpha \neq 0$, define

$$\psi_j^\alpha(Q) = |s_j(Q, \cdot)|_\alpha.$$

When $\alpha = 1$ the superscript will be suppressed. The $\psi_j(Q)$ are just the classical integrals of mean curvature of Q. For $\alpha = -\infty$, 0, or ∞, define

$$\psi_j^\alpha(Q) = \lim_{r \to \alpha} \psi_j^r(Q).$$

Thus,

$$\psi_j^\infty(Q) = \max_{u \in S^{n-1}} s_j(Q, u) = C_j^+(Q),$$

while

$$\psi_j^{-\infty}(Q) = \min_{u \in S^{n-1}} s_j(Q, u) = C_j^-(Q).$$

For arbitrary convex hypersurfaces there are well-known extensions of the (indefinite integrals of the) elementary symmetric functions of the principal radii of curvature. Specifically, for each $Q \in \mathscr{C}^n$, and each j, there exists a Borel measure $S_j(Q, \cdot)$ on S^{n-1}, such that if $Q \in \mathscr{C}_2^n$, then $S_j(Q, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, and

$$\frac{dS_j(Q, \cdot)}{dS} = s_j(Q, \cdot), \tag{2}$$

where the derivative is a Radon–Nikodym derivative.

As will be seen, the functional ψ_j^α, on $\mathscr{C}_2^n$, can be extended to a functional Ψ_j^α, defined on all of $\mathscr{C}^n$. For $\alpha = 1$, define

$$\Psi_j(Q) = \Psi_j^1(Q) = \frac{1}{\kappa_n} \int_{S^{n-1}} dS_j(Q, u).$$

Thus, $\kappa_n \Psi_{n-1}(Q)$ is the surface area of Q, and the $\Psi_j(Q)$ are essentially the classical Quermassintegrals Q (see, for example, Leichtweiß [4]).

For $\alpha < 1$, and $\alpha \neq 0$, define $\Psi_j^\alpha(Q)$ by

$$\Psi_j^\alpha(Q) = \inf \{\langle g, S_j(Q, \cdot)\rangle / |g|_{\alpha/(\alpha-1)} : g \in C^+(S^{n-1})\}, \tag{3a}$$

and for $\alpha > 1$, define $\Psi_j^\alpha(Q)$ by

$$\Psi_j^\alpha(Q) = \sup\{\langle g, S_j(Q, \cdot)\rangle / |g|_{\alpha/(\alpha-1)} : g \in C^+(S^{n-1})\}. \tag{3b}$$

For $Q \in \mathscr{C}^n$, and all j,

$$\Psi_j^{\alpha_1}(Q) \leq \Psi_j^{\alpha_2}(Q), \qquad \text{whenever } \alpha_1 \leq \alpha_2. \tag{4}$$

To see this, note that if $\alpha_1 \leq 1 \leq \alpha_2$, then (4) follows from taking g to be a constant function in (3a) and (3b). If $\alpha_1 < \alpha_2 < 1$, or $1 < \alpha_1 < \alpha_2$, then (4) follows directly from (1) and definitions (3a) and (3b).

For $\alpha = -\infty$, or ∞, define Ψ_j^α on $\mathscr{C}^n$ by:

$$\Psi_j^\alpha(Q) = \lim_{r \to \alpha} \Psi_j^r(Q),$$

and define

$$\Psi_j^0(Q) = \lim_{r \to 0+} \Psi_j^r(Q).$$

Thus (4) shows that

$$\Psi_j^\infty(Q) = \sup_{1 < r < \infty} \Psi_j^r(Q), \tag{5a}$$

$$\Psi_j^{-\infty}(Q) = \inf_{-\infty < r < 0} \Psi_j^r(Q), \tag{5b}$$

and

$$\Psi_j^0(Q) = \inf_{0 < r < 1} \Psi_j^r(Q). \tag{5c}$$

PROPOSITION. *For* $j = 1, \ldots, n-1$, *and* $-\infty \leq \alpha \leq \infty$,

$$\Psi_j^\alpha(Q) = \psi_j^\alpha(Q), \qquad \textit{whenever } Q \in \mathscr{C}_2^n. \tag{6}$$

To see this first note that from (2) it follows that for $Q \in \mathscr{C}_2^n$ and $g \in C^+(S^{n-1})$,

$$\langle g, S_j(Q, \cdot)\rangle = \frac{1}{\kappa_n} \int_{S^{n-1}} g(u) s_j(Q, u)\, dS(u). \tag{7}$$

The case $\alpha = 1$ of (6) follows by taking $g = 1$ in (7). Suppose that $-\infty < \alpha < 1$, and $\alpha \neq 0$. From (7), and the Hölder inequality [3, p. 140], it follows that

$$\psi_j^\alpha(Q) = |s_j(Q, \cdot)|_\alpha \leq \langle g, S_j(Q, \cdot)\rangle / |g|_{\alpha/(\alpha-1)},$$

with equality if and only if the function $g/s_j(Q, \cdot)^{\alpha-1}$ is a constant function on S^{n-1}. This proves (6) when $-\infty < \alpha < 1$, and $\alpha \neq 0$. The case $1 < \alpha < \infty$ of (6) is established in exactly the same way. The cases of (6) where $\alpha = -\infty$, 0, or ∞ now follow since in these cases both $\Psi_j^\alpha(Q)$ and $\psi_j^\alpha(Q)$ were defined as limits of $\Psi_j^r(Q) = \psi_j^r(Q)$, for real $r \neq 0$.

THEOREM. *For all* $j = 1, \ldots, n-1$, *the functional*

$$\Psi_j^\alpha : \mathscr{C}^n \to [0, \infty], \quad \textit{is upper semicontinuous when } -\infty \leq \alpha \leq 1,$$

while the fuctional

$$\Psi_j^\alpha : \mathscr{C}^n \to [0, \infty], \quad \textit{is lower semicontinuous when } 1 \leq \alpha \leq \infty.$$

Proof. From the weak continuity of the measures S_j (see for example, Schneider [8]), it follows that for fixed $g \in C^+(S^{n-1})$, the function $\Gamma_g : \mathscr{C}^n \to (0, \infty)$, defined for $Q \in \mathscr{C}^n$ by

$$\Gamma_g(Q) = \langle g, S_j(Q, \cdot)\rangle / |g|_{\alpha/(\alpha-1)},$$

is continuous. For $1 < \alpha < \infty$, the functional Ψ_j^α is lower semicontinuous, on $\mathscr{C}^n$, since it is just the supremum of the continuous (on $\mathscr{C}^n$) functionals Γ_g. The case $\alpha = \infty$ now follows since by (5a), the functional Ψ_j^∞ is just a supremum (over all $r > 1$) of the lower semicontinuous functionals Ψ_j^r. The cases where $-\infty \leq \alpha \leq 1$ follow in exactly the same manner.

For $\alpha = n/(n+1)$, the upper semicontinuity of $\psi_{n-1}^\alpha : \mathscr{C}_2^n \to (0, \infty)$ was established in [7]. The case of the Theorem where $n = 3$, $\alpha = -1$, and $j = n - 1$, was conjectured by Wm. J. Firey at the 1974 Oberwolfach meeting on convex bodies (see [2, p. 257]). The case where $\alpha = n/(n+1)$ and $j = n - 1$ is due to Leichtweiß [5] (see also Leichtweiß [6]).

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