Discs in pseudoconvex domains.

Comment. Math. Helvetici 67 (1992) 129–145
0010-2571/92/010129-17$1.50 + 0.20/0

Discs in pseudoconvex domains

FRANC FORSTNERIČ AND JOSIP GLOBEVNIK

1. Introduction

Let $D \subset \mathbf{C}^N$ be a domain in the complex Euclidean space $\mathbf{C}^N$ $(N > 1)$, and let y be a point in D. There exist many closed complex one-dimensional subvarieties (curves) $V \subset D$ passing through y. For instance, it suffices to take the common zero set $f_1 = f_2 = \cdots = f_s = 0$ of suitably chosen holomorphic functions on D that vanish at y.

A special class of closed complex curves in D are the *proper analytic discs*, i.e., the images $F(\Delta)$ of proper holomorphic maps $F: \Delta \to D$ from the open unit disc $\Delta \subset \mathbf{C}$ into D. A natural question appears [6]: *Given a point $y \in D$, can we find a proper analytic disc in D passing through y?*

In this article we give a positive answer to this question for all bounded pseudoconvex domains in $\mathbf{C}^N$ with $\mathscr{C}^2$ boundary, and a counterexample for non-pseudoconvex domains with disconnected boundary. More precisely, we prove the following results:

THEOREM 1. *Let $D \subset\subset \mathbf{C}^N$ be a strongly pseudoconvex domain with boundary of class $\mathscr{C}^k$, with $N, k \geq 2$. Given a point $y \in D$ and a vector $X \in \mathbf{C}^N$, there is a mapping $F: \bar{\Delta} \to \bar{D}$ of class $\mathscr{C}^{k-0}(\bar{\Delta})$ that is holomorphic on the open disc Δ and satisfies $F(b\Delta) \subset bD$, $F(0) = y$, and $F'(0) = \lambda X$ for some $\lambda > 0$.*

Stated informally, the theorem asserts that through each point of a strongly pseudoconvex domain in any given direction there passes a proper analytic disc that is smooth up to the boundary. Here, as usual, $\mathscr{C}^{k-0} = \mathscr{C}^k$ if k is not an integer, and $\mathscr{C}^{k-0} = \bigcup_{0<\alpha<1} \mathscr{C}^{k-1,\alpha}$ if k is an integer.

We have a similar result for smoothly bounded weakly pseudoconvex domains, except that we are not able to get smoothness up to the boundary:

THEOREM 2. *Let $D \subset\subset \mathbf{C}^N$ $(N \geq 2)$ be a pseudoconvex domain with boundary of class $\mathscr{C}^2$. Given a point $y \in D$ and a vector $X \in \mathbf{C}^N$ there is a proper holomorphic map $F: \Delta \to \bar{D}$ satisfying $F(0) = y$ and $F'(0) = \lambda X$ for some $\lambda > 0$.*

If D admits a defining function that is plurisubharmonic near bD, then one can of course apply Theorem 1 to get a proper analytic disc in D through y that is of class $\mathscr{C}^{2-0}$ up to the boundary.

By technical modifications of our method one can construct proper analytic discs as above satisfying various additional properties. For instance, if $N \geq 3$, there exist proper holomorphic *embeddings* $F: \Delta \to D \subset \mathbf{C}^N$ satisfying $F(0) = y$ and $F'(0) = \lambda X$, and for $N = 2$ there exist holomorphic *immersions* with the same properties. Moreover, we shall see from the construction that one can prescribe, up to a positive scalar, any finite number of derivatives $F'(0), F''(0), \ldots$ of the map F at the origin. We leave out the details.

From Theorems 1 and 2 and from our Main Lemma in section two it follows immediately that there exist proper analytic discs in D containing a given finite subset of D:

COROLLARY 3. *Let $D \subset\subset \mathbf{C}^N$ $(N > 1)$ be a pseudoconvex domain with $\mathscr{C}^2$ boundary. For each finite set of points $y_1, y_2, \ldots, y_n \in D$ and vectors X_1, $X_2, \ldots, X_n \in \mathbf{C}^N$ there are a proper holomorphic map $F: \Delta \to D$ and points ζ_1, $\zeta_2, \ldots, \zeta_n \in \Delta$ such that for each j, $1 \leq j \leq n$, we have $F(\zeta_j) = y_j$ and $F'(\zeta_j) = \lambda_j X_j$ for some $\lambda_j > 0$. If D is strictly pseudoconvex with $\mathscr{C}^k$ boundary, there is an F as above that is of class $\mathscr{C}^{k-0}$ on $\bar{\Delta}$.*

If $D \subset\subset \mathbf{C}^N$ is a convex domain, then according to [6] there exist proper analytic discs in D passing through any given *discrete* subset of D. It is very likely that by combining the techniques of this paper with those in [6] one can prove the same result for all bounded pseudoconvex domains with $\mathscr{C}^2$ boundary.

Virtually the same technique can be used to prove Theorems 1 and 2 for relatively compact pseudoconvex domains with $\mathscr{C}^2$ boundary in an N-dimensional Stein manifold. For strongly pseudoconvex domains one can use the embedding theorem of Fornæss and Henkin [9]. Again we shall not go into details of this.

We show by an example that pseudoconvexity cannot be entirely deleted from our hypothesis:

THEOREM 4. *For each $N \geq 2$ there exist a smoothly bounded domain $D \subset\subset \mathbf{C}^N$ and a point $x \in D$ such that there is no proper holomorphic map $F: \Delta \to D$ with $x \in F(\Delta)$.*

Here are some related open problems:

1. When D is a weakly pseudoconvex domain with smooth boundary that does not admit a plurisubharmonic defining function, can we find discs as in Theorem 2 that are smooth up to the boundary?

2. Does Theorem 2 still hold if we assume no boundary regularity of D?
3. Let $\mathcal{M}$ be a Stein manifold of dimension $n \geq 2$. Given a point $p \in \mathcal{M}$, does there exist a proper holomorphic map $F : \Delta \to \mathcal{M}$ with $F(0) = p$? If so, can one also prescribe the direction of $F'(0)$ as above? Can one find analytic discs in $\mathcal{M}$ that contain any given finite (or discrete) subset of $\mathcal{M}$?

Another related problem is the following. Suppose that for each $\zeta \in b\Delta$ we are given a strongly pseudoconvex domain $D_\zeta \subset\subset \mathbf{C}^N$ containing the origin. Suppose also that the boundaries bD_ζ depend continuously or even smoothly on $\zeta \in b\Delta$. The problem is to construct continuous maps $F : \bar{\Delta} \to \mathbf{C}^N$, holomorphic on Δ, such that $F(\zeta) \in bD_\zeta$ ($\zeta \in b\Delta$). Such maps are known to exist when all D_ζ are convex [1], [4], [7], [11], and in this case their graphs fill up the interior of the entire polynomially convex hull of the set $K = \bigcup_{\zeta \in b\Delta} \bar{D}_\zeta$. In the non-convex case the problem is well understood only for $N = 1$, see [3], [8], and [12]. Using the methods of this paper one can solve this problem under suitable additional assumptions on a defining function $P : b\Delta \times \mathbf{C}^N \to \mathbf{R}$ satisfying $D_\zeta = \{z \in \mathbf{C}^N : P(\zeta, z) < 0\}$.

The paper is organized as follows. In section two we state our Main Lemma, and based on this Lemma we prove Theorems 1 and 2 and Corollary 3. Section three contains technical results required in the proof of the Main Lemma in section four. In section five we construct the example claimed by Theorem 4.

This research was supported in part by grants from the Research Council of the Republic of Slovenia.

2. Proof of Theorems 1 and 2

We first show that it suffices to prove Theorems 1 and 2 and Corollary 3 for $N = 2$.

Let $D \subset\subset \mathbf{C}^N$ be a pseudoconvex domain with $\mathscr{C}^2$ boundary, $y \in D$ and $X \in \mathbf{C}^N \backslash \{0\}$. Choose a complex basis $X_1 = X, X_2, \ldots, X_N$ of $\mathbf{C}^N$. On the space $\mathbf{C}^2 \times \mathbf{C}^{2N}$ we use the coordinates $z = (z_1, z_2)$, $\lambda = (\lambda_1, \ldots, \lambda_N)$, $v = (v_1, \ldots, v_N)$. Let $\Phi : \mathbf{C}^2 \times \mathbf{C}^{2N} \to \mathbf{C}^N$ be the entire map defined by

$$\Phi(z, \lambda, v) = y + z_1 X_1 + z_2 X_2 + \sum_{j=1}^{N} (\lambda_j z_1^2 + v_j z_2^2) X_j.$$

Notice that $\Phi(0, \cdot, \cdot) \equiv y$ and $\partial\Phi/\partial z_j(0, \cdot, \cdot) = X_j$ for $j = 1, 2$.

If $\Phi(z, \lambda, v) = p \in bD$, then at least one of the variables z_1, z_2 is nonzero, hence Φ is a submersion here. It follows that Φ is transverse to bD. By the transversality theorem we conclude that for almost all values $\lambda_0, v_0 \in \mathbf{C}^N$ the map

$\Phi(\cdot, \lambda_0, v_0) : \mathbf{C}^2 \to \mathbf{C}^N$ is transverse to bD. Choosing λ_0 and v_0 sufficiently small we insure that the connected component Ω_0 of the preimage $\{z \in \mathbf{C}^2 : \Phi(z, \lambda_0, v_0) \in D\}$ containing the origin is a bounded pseudoconvex domain in $\mathbf{C}^2$ with $\mathscr{C}^2$ boundary. If D is strongly pseudoconvex then so is Ω_0.

Suppose that Theorems 1 and 2 and Corollary 3 hold in dimension two. If $F_0 : \Delta \to \Omega_0$ is a proper holomorphic map satisfying $F_0(0) = 0$ and $F_0'(0) = (\lambda, 0)$, then

$$F(\zeta) = \Phi(F_0(\zeta), \lambda_0, v_0) \qquad (\zeta \in \Delta)$$

is a proper holomorphic map of Δ into D satisfying Theorem 1 resp. 2. Similarly one proves Corollary 3.

From now on we shall only consider the case $N = 2$. Let $D \subset\subset \mathbf{C}^2$ be a pseudoconvex domain with $\mathscr{C}^2$ boundary. According to [2] there is a $\mathscr{C}^2$ defining function τ for D such that $D = \{\tau < 0\}$, $\nabla\tau \neq 0$ near bD, and there is a negative strictly plurisubharmonic function ρ on D such that near bD we have $\rho = -(-\tau)^\epsilon$. In particular, there is a $T < 0$ such that the gradient $(\nabla\rho)(z)$ is nonvanishing for $T < \rho(z) < 0$, and the domains

$$D(t) = \{z \in D : \rho(z) < t\} \qquad (T < t < 0)$$

are strongly pseudoconvex with $\mathscr{C}^2$ boundary. When D itself is strongly pseudoconvex we can of course choose a defining function ρ for D satisfying these properties.

For each $t_1, t_2, T < t_1 < t_2 < 0$, let

$$\mathscr{V}(t_1, t_2) = \{z \in D : t_1 < \rho(z) < t_2\}.$$

MAIN LEMMA. *Let $T < t_1 < t_2 < 0$. There is a $\mu_0 > 0$, depending only on t_1, t_2, with the following property: Let $0 < r < 1$ and let $F : \bar{\Delta}\backslash r\Delta \to \mathscr{V}(t_1, t_2)$ be a continuous map satisfying $\rho(F(\zeta)) > c$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$. Suppose that μ is a positive continuous function on $b\Delta$ such that $\mu(\zeta) \leq \mu_0$ $(\zeta \in b\Delta)$, let $\epsilon > 0$, and let $0 < R < 1$. Then there exists a continuous map $G : \Delta \to \mathbf{C}^2$ that is holomorphic on Δ and satisfies*

(i) $F(\zeta) + G(\zeta) \in D$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$,
(ii) $\rho(F(\zeta) + G(\zeta)) > c$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$,
(iii) $|\rho(F(\zeta) + G(\zeta)) - \rho(F(\zeta)) - \mu(\zeta)| < \epsilon$ $(\zeta \in b\Delta)$,
(iv) $|G(\zeta)| < \epsilon$ $(|\zeta| \leq R)$, *and*
(v) $G(0) = 0$, $G'(0) = 0$.

Remark. The Lemma also holds if we choose finitely many points $\zeta_1, \zeta_2, \ldots, \zeta_n \subset \Delta$ and replace (v) by the following stronger condition:
(v′) $G(\zeta_i) = 0$ and $G'(\zeta_j) = 0$ for $1 \le j \le n$.

We defer the proof of the Main Lemma to section four below.

Proof of Theorem 1. It suffices to prove the following: If $T < t_0 < 0$, $y \in D(t_0)$, and $X \in \mathbf{C}^2$, there is a map $F : \bar{\Delta} \to \overline{D(t_0)}$ of class $\mathscr{C}^{2-0}$ that is holomorphic on Δ, $F(0) = y$, and $F'(0) = \lambda X$ for some $\lambda > 0$. (If ρ is of class $\mathscr{C}^k$, the same proof will give $F \in \mathscr{C}^{k-0}(\bar{\Delta})$.)

Choose t_1 and t_2 such that $T < t_1 < t_0 < t_2 < 0$, and let μ_0 be as in the Main Lemma, chosen small enough such that $t_0 - \mu_0 \ge t_1$. Denote by $\mathbf{B}^2$ the open unit ball in $\mathbf{C}^2$. There is an $\epsilon > 0$ such that

$$\overline{D(t_0)} + 2\epsilon\mathbf{B}^2 \subset D(t_2), \qquad bD(t_0) + 2\epsilon\mathbf{B}^2 \subset \mathscr{V}(t_0 - \mu_0, t_2). \tag{1}$$

We show the following:

LEMMA 1. *Suppose that $f : \bar{\Delta} \to D(t_2)$ is a continuous map that is holomorphic on Δ and satisfies $f(0) \in D(t_0)$ and $f(b\Delta) \subset \mathscr{V}(t_0, t_2)$. Given $x \in D(t_0)$, $|x - f(0)| < \epsilon$, there is a continuous map $f_1 : \bar{\Delta} \to D(t_2)$, holomorphic on Δ, satisfying $f_1(0) = x, f'_1(0) = \lambda f'(0)$ for some $\lambda > 0$, and $f_1(b\Delta) \subset \mathscr{V}(t_0, t_2)$.*

Proof. Let t'_0, $t_0 < t'_0 < t_2$, be so close to t_0 that $f(b\Delta) \subset \mathscr{V}(t'_0, t_2)$ and

$$D(t) \subset D(t_0) + \epsilon\mathbf{B}^2, \qquad bD(t) \subset bD(t_0) + \epsilon\mathbf{B}^2 \qquad (t_0 \le t \le t'_0). \tag{2}$$

By Sard's theorem there is a t, $t_0 < t < t'_0$, such that $\Omega = \{\zeta \in \Delta : \rho(f(\zeta)) < t\}$ is a relatively compact domain in Δ with $\mathscr{C}^2$ boundary. By the maximum principle, applied to the subharmonic function $\rho \circ f$, each connected component of Ω is simply connected, thus conformally equivalent to the disc. Let $\phi : \Delta \to \Omega_0$ be the conformal map onto the connected component Ω_0 of Ω containing the origin, chosen such that $\phi(0) = 0$ and $\phi'(0) > 0$. Since the boundary of Ω_0 is of class $\mathscr{C}^2$, ϕ extends to be of class $\mathscr{C}^{2-0}$ on $\bar{\Delta}$. The composition $\Phi = f \circ \phi : \bar{\Delta} \to \overline{D(t)}$ is of class $\mathscr{C}^{2-0}$, holomorphic on Δ, and satisfies $\Phi(0) = f(0)$, $\Phi'(0) = \phi'(0)f'(0)$, and $\Phi(b\Delta) \subset bD(t)$.

Let $x \in D(t_0)$, $|x - f(0)| < \epsilon$. Then by (1) and (2),

$$\zeta \in \bar{\Delta} \to g(\zeta) = \Phi(\zeta) + (x - \Phi(0))$$

is a continuous map from $\bar{\Delta}$ into $\overline{D(t_0)} + 2\epsilon\mathbf{B}^2 \subset D(t_2)$ that is holomorphic on Δ and satisfies $g(b\Delta) \subset \mathscr{V}(t_0 - \mu_0, t_2)$. Choose an $r, 0 < r < 1$, such that $g(\bar{\Delta}\backslash r\Delta) \subset \mathscr{V}(t_0 - \mu_0, t_2)$. Choose $\eta > 0$ so small that $g(r\Delta) + \eta\mathbf{B}^2 \subset D(t_2)$. By the Main Lemma applied to g there is a continuous map $h : \bar{\Delta} \to \mathbf{C}^2$, holomorphic on Δ, satisfying $h(0) = 0, h'(0) = 0$, $|h(\zeta)| < \eta$ for $|\zeta| < r$, such that the map $f_1 = g + h$ satisfies $f_1(\bar{\Delta}\backslash r\Delta) \subset D(t_2)$ and $f_1(b\Delta) \subset \mathscr{V}(t_0, t_2)$. If $\zeta \in r\Delta$, then $|h(\zeta)| < \eta$, hence $f_1(\bar{\Delta}) \subset D(t_2)$. This proves Lemma 1.

We can now complete the proof of Theorem 1. Without loss of generality we may assume that the domain $D(t_0)$ is connected. There is a point $y_0 \in bD(t_0)$ such that the complex tangent space to $bD(t_0)$ at y_0 is spanned by the given vector $X \in \mathbf{C}^2\backslash\{0\}$.

We claim that there are a point $y_1 \in D(t_0)$ close to y_0 and a map $f_0 : \bar{\Delta} \to D(t_2)$, holomorphic on Δ, satisfying $f_0(0) = y_1, f_0'(0) = \lambda X$ for some $\lambda > 0$, and $f_0(b\Delta) \subset \mathscr{V}(t_0, t_2)$. This can be seen immediately from the proof of Narasimhan's lemma [9, p. 111]: locally near y_0 we convexify the domain $D(t_0)$ by a local biholomorphic change of coordinates, we take a suitable linear disc in the convexified domain, and then pull it back to a disc in $D(t_2)$ satisfying the required properties. Of course it is essential that X is complex tangent to $bD(t_0)$ at y_0!

Using Lemma 1 a finite number of times we can slide the initial disc f_0 to a disc $f_1 : \bar{\Delta} \to D(t_2)$ satisfying the same conditions, except that the new center is $f_1(0) = y$. By a generic perturbation of f_1 we insure that f_1 intersects the boundary of $D(t_0)$ transversely. Replacing f_1 by $f_1 \circ \phi$, where ϕ is a suitable conformal map of Δ onto the connected component of $\{\zeta \in \Delta : f_1(\zeta) \in D(t_0)\}$ containing 0 (see the proof of Lemma 1), we obtain the final map F satisfying Theorem 1.

Proof of Theorem 2. Choose t_0, $T < t_0 < 0$, such that $y \in D(t_0)$. We choose sequences $t_0 < t_1 < t_2 < \cdots < 0$, $\lim_{j\to\infty} t_j = 0$, and $\epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots > 0$, $\lim_{j\to\infty} \epsilon_j = 0$, such that

$$\overline{D(t_n)} + \epsilon_{n-1}\mathbf{B}^2 \subset D(t_{n+1}) \qquad (n = 1, 2, \ldots). \tag{3}$$

We show that there are an increasing sequence of radii $r_0 < r_1 < r_2 < \cdots < 1$ with $\lim_{j\to\infty} r_j = 1$ and a sequence of continuous mappings $f_n : \bar{\Delta} \to D$ $(n = 1, 2, \ldots)$, holomorphic on Δ, such that for each $n = 1, 2, \ldots$ the following hold:

(i) $f_n(\bar{\Delta}) \subset D(t_{n+1})$,
(ii) $\rho(f_n(\zeta)) > t_{n-1}$ $(\zeta \in \bar{\Delta}\backslash r_{n-1}\Delta)$,
(iii) $t_n < \rho(f_n(\zeta)) < t_{n+1}$ $(\zeta \in \bar{\Delta}\backslash r_n\Delta)$,
(iv) $|f_{n+1}(\zeta) - f_n(\zeta)| < \epsilon_n/2^n$ $(\zeta \in r_n\Delta)$, and
(v) $f_n0) = y, f_n'(0) = \lambda X$ for some $\lambda > 0$ independent of n.

The construction is by induction on n. By Theorem 1 there is a continuous map $f_1 : \bar{\Delta} \to D(t_2)$, holomorphic on Δ, such that $f_1(0) = y$, $f_1'(0) = \lambda X$, and $t_1 < \rho(f_1(\zeta)) < t_2$ for $\zeta \in b\Delta$. Choose $r_0, r_1, 0 < r_0 < r_1 < 1$, such that (i), (ii), (iii), and (v) are satisfied for $n = 1$.

Suppose that f_j and r_j have been constructed for $1 \le j \le n$ so that (i), (ii), (iii), and (v) are satisfied. Using the Main Lemma a finite number of times we get a continuous map $f_{n+1} = f_n + g_n : \bar{\Delta} \to D$, holomorphic on Δ, and a number $r_{n+1}, r_n < r_{n+1} < 1$, such that (iv) holds and (i), (ii), (iii), and (v) hold with n replaced by $n + 1$.

Now, (iv) shows that the sequence f_n converges uniformly on compact sets in Δ to a holomorphic map F. By (v) we have $F(0) = y$ and $F'(0) = \lambda X$. For $\zeta \in r_n\Delta$ we have $|F(\zeta) - f_n(\zeta)| < \epsilon_n$ by (iv), hence (i) and (3) imply

$$F(r_n\Delta) \subset D(t_{n+1}) + \epsilon_n \mathbf{B}^2 \subset D.$$

Thus $F(\Delta) \subset D$.

It remains to show that F is a proper map into D. Let $\zeta \in r_{n+1}\Delta \backslash r_n\Delta$. By (ii) we have $\rho(f_{n+1}(\zeta)) > t_n$, and by (iv) $|F(\zeta) - f_{n+1}(\zeta)| < \epsilon_{n+1}$. Since $\epsilon_{n+1} < \epsilon_{n-2}$, (3) implies $\rho(F(\zeta)) \ge t_{n-1}$. This proves that for each n, $\rho(F(\zeta)) \ge t_{n-1}$ $(r_n < |\zeta| < 1)$, which shows that $F : \Delta \to D$ is a proper map. Theorem 2 is proved.

Proof of Corollary 3. Choose t_0, $T < t_0 < 0$, such that $y_j \in D(t_0)$ $(1 \le j \le n)$. Let $\Delta_j \subset \mathbf{C}$ be the open disc of radius one with center at $3j \in \mathbf{C}$. By Theorem 1 there exist continuous maps $F_j : \bar{\Delta}_j \to \overline{D(t_0)}$, holomorphic on Δ_j, satisfying $F_j(b\Delta) \subset bD(t_0)$, $F_j(3j) = y_j$, and $F_j'(3j) = \lambda_j X_j$ $(1 \le j \le n)$. Let K be the union of the closed discs $\bar{\Delta}_j$ $(1 \le j \le n)$ and the interval $[3, 3n] \subset \mathbf{C}$. Let $\tilde{F} : K \to \overline{D(t_0)}$ be a continuous map that equals F_j on $\bar{\Delta}_j$ and satisfies $\tilde{F}(bK) \subset bD(t_0)$. Here, bK is the topological boundary of K in $\mathbf{C}$.

Since the complement of K in $\mathbf{C}$ is connected and $\tilde{F}$ is holomorphic in the interior of K, we can apply Mergelyan's theorem to approximate $\tilde{F}$ uniformly on K by a polynomial mapping $F_0 : \mathbf{C} \to \mathbf{C}^2$ satisfying $F_0(3j) = y_j$, $F_0'(3j) = \lambda_j X_j$ $(1 \le j \le n)$. Let U be a small simply connected neighborhood of K with smooth boundary. If the approximation is close enough on K and if U is chosen sufficiently small, then $F_0(\bar{U}) \subset D$ and $F_0(bU) \subset \mathscr{V}(T, 0)$.

Since U is conformally equivalent to the disc Δ, we can now proceed as in the proof of Theorem 2 to modify the given map F_0 to a proper map $F : U \to D$, without changing the values of F_0 and its first derivative at the points $3j$, $1 \le j \le n$. (See the remark following the Main Lemma). If D is strictly pseudoconvex, we can make F smooth up to the boundary as in the proof of Theorem 1. This proves Corollary 3.

Sections three and four are devoted to the proof of the Main Lemma.

3. Technical lemmas

Recall that the disc algebra $\mathscr{A}(\Delta)$ is the set of all continuous functions on $\bar{\Delta}$ that are holomorphic on Δ.

LEMMA 2. *Let V be a compact set and let $F : \bar{\Delta} \times V \to \mathbf{C}$ be a continuous function such that for each $v \in V$ the function $\zeta \to F(\zeta, v)$ belongs to the disc algebra. Given $\epsilon > 0$ there are $n \in \mathbf{Z}_+$ and a continuous map $G : \bar{\Delta} \times V \to \mathbf{C}$ such that for each $v \in V, \zeta \to G(\zeta, v)$ is a polynomial of degree $\leq n$ satisfying $|G(\zeta, v) - F(\zeta, v)| < \epsilon$ for all $(\zeta, v) \in \bar{\Delta} \times V$.*

Proof. There is an $r, 0 < r < 1$, such that $|F(r\zeta, v) - F(\zeta, v)| < \epsilon/2$ for all $(\zeta, v) \in \bar{\Delta} \times V$. By the Cauchy formula we have

$$\begin{aligned} F(z, v) &= \frac{1}{2\pi i} \int_{b\Delta} \frac{F(\zeta, v)}{\zeta - z} \, d\zeta \\ &= \frac{1}{2\pi i} \int_{b\Delta} \left[\frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots + \frac{z^n}{\zeta^{n+1}} \right] F(\zeta, v) \, d\zeta + z^{n+1} \frac{1}{2\pi i} \int_{b\Delta} \frac{F(\zeta, v)}{\zeta^{n+1}(\zeta - z)} \, d\zeta \\ &= G_n(z) + R_n(z). \end{aligned}$$

Since F is bounded on $b\Delta \times V$, the remainder $R_n(z)$ tends to zero uniformly on $r\bar{\Delta} \times V$ as $n \to \infty$, hence $|F - G_n| < \epsilon/2$ ($|z| \leq r, v \in V$) if n is sufficiently large. Since G_n is a polynomial of degree at most n in z, Lemma 2 is proved.

Remark. If $F(0, v) = 0$ for all $v \in V$ then we may take $G(0, v) = 0$ for all $v \in V$. If $V \subset \mathbf{R}^N$ and F is smooth on $\Delta \times \operatorname{Int} V$, then G will be smooth on $\Delta \times \operatorname{Int} V$.

COROLLARY. *Let $A : b\Delta \times b\Delta \to \mathbf{C}$ be a continuous map such that for each $\zeta \in b\Delta$, $L_\zeta = \{A(\eta, \zeta) : \eta \in b\Delta\}$ is a Jordan curve with the origin contained in the bounded part of its complement. Given $\epsilon > 0$ there is a function $f \in \mathscr{A}(\Delta)$ satisfying $f(\zeta) \in L_\zeta + \epsilon\Delta$ for each $\zeta \in b\Delta, f(0) = 0$, and $f'(0) = 0$.*

Remark. The Corollary gives an approximate solution of the Riemann-Hilbert boundary value problem with the data L_ζ ($\zeta \in b\Delta$). The exact solution, i.e., the existence of functions $f \in \mathscr{A}(\Delta)$ satisfying $f(\zeta) \in L_\zeta$ ($\zeta \in b\Delta$), is a much deeper result; see the papers [3] and [12].

Proof. For each $\zeta \in b\Delta$ let D_ζ be the domain bounded by L_ζ, and let $\Phi_\zeta : \Delta \to D_\zeta$ be the conformal map that satisfies $\Phi_\zeta(0) = 0, \Phi'_\zeta(0) > 0$. Then the map

$F(\eta, \zeta) = \Phi_\zeta(\eta)$ is continuous on $\bar{\Delta} \times b\Delta$, and $\eta \to F(\eta, \zeta)$ is in the disc algebra for each $\zeta \in b\Delta$. By Lemma 2 there are $n \in \mathbf{Z}_+$ and a continuous map $G: \bar{\Delta} \times b\Delta \to \mathbf{C}$ such that for each $\zeta \in b\Delta$, $\eta \to G(\eta, \zeta)$ is a polynomial of degree at most n without constant term, satisfying

$$|G(\eta, \zeta) - F(\eta, \zeta)| < \epsilon/2, \qquad (\eta, \zeta) \in \bar{\Delta} \times b\,\Delta.$$

Write

$$G(\eta, \zeta) = \sum_{j=1}^{n} a_j(\zeta)\eta^j \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}).$$

For each j there are polynomials P_j and Q_j satisfying

$$|a_j(\zeta) - P_j(\zeta) - Q_j(1/\zeta)| < \epsilon/2n \qquad (\zeta \in b\Delta).$$

Let $m \in \mathbf{Z}_+$ be greater than the degree of each polynomial Q_j, $1 \leq j \leq n$, and set

$$f(\zeta) = \sum_{j=1}^{n} [P_j(\zeta) + Q_j(1/\zeta)](\zeta^m)^j.$$

Then f is a polynomial in ζ that vanishes at 0 to arbitrary finite order (by choosing m sufficiently large). If $\zeta \in b\Delta$ then $|f(\zeta) - G(\zeta^m, \zeta)| < \epsilon/2$ which implies that $|f(\zeta) - F(\zeta^m, \zeta)| < \epsilon$. In particular, $f_\zeta \in L_\zeta + \epsilon\Delta$. This completes the proof of the Corollary.

Remark. If $0 < R < 1$ then, by choosing m large enough, we can get f as above with the additional property $|f(\zeta)| < \epsilon$ $(|\zeta| \leq R)$.

As before we denote by $\mathbf{B}^2$ the open unit ball in $\mathbf{C}^2$.

LEMMA 3. *Let $T < t_1 < t_2 < 0$ and let $L = \{(w_1, w_2) \in \mathbf{B}^2 : w_1 = 0\}$. There is a $v_0 > 0$ and for each $z \in \mathscr{V}(t_1, t_2)$ there is a biholomorphic map $\Psi_z : \mathbf{B}^2 \to \Psi_z(\mathbf{B}^2) \subset \mathbf{C}^2$ satisfying*

(i) $\Psi_z(0) = 0$ $(z \in \mathscr{V}(t_1, t_2))$,

(ii) $z + \Psi_z(\mathbf{B}^2) \subset D$ $(z \in \mathscr{V}(t_1, t_2))$,

(iii) $(z, w) \to \Psi_z(w)$ *is smooth on* $\mathscr{V}(t_1, t_2) \times \mathbf{B}^2$,

(iv) *for each* $z \in \mathscr{V}(t_1, t_2)$ *and for each* v, $-v_0 \leq v \leq v_0$, *the set*

$$P(z, v) = \{w \in \mathbf{B}^2 : \rho(z + \Psi_z(w)) < \rho(z) + v\}$$

is a convex domain and

$$S(z, v) = \{w \in \mathbf{B}^2 : \rho(z + \Psi_z(w)) = \rho(z) + v\}$$

is a smooth surface,

(v) *for each* $z \in \mathscr{V}(t_1, t_2)$ *and for each* v, $-v_0 \le v < 0$, *we have* $\overline{P(z, v)} \cap L = \varnothing$,

(vi) *for each* $z \in \mathscr{V}(t_1, t_2)$ *and for each* v, $0 < v \le v_0$, $S(z, v) \cap L$ *is a simple closed curve.*

Remark. The convexity of $P(z, v)$ implies that if $0 < v \le v_0$ then L intersects $S(z, v)$ transversely.

Proof of Lemma 3. The proof will be split into three parts.

Part 1. Let e'_1, e'_2 be the standard basis of $\mathbf{C}^2$. Fix a point $z \in \mathscr{V}(T, 0) = \{z \in D : T < \rho(z) < 0\}$, and choose a new coordinate system in $\mathbf{C}^2$ by putting $e_1(z) = \nabla\rho(z)/|\nabla\rho(z)|$ and letting $e_2(z)$ be canonically orthogonal to $e_1(z)$, that is, if $e_1(z) = \alpha e'_1 + \beta e'_2$, then $e_2(z) = -\bar\beta e'_1 + \bar\alpha e'_2$. The Taylor formula gives

$$\rho(z + u_1 e_1(z) + u_2 e_2(z)) = \rho(z) + 2\Re\left[|\nabla\rho(z)|u_1 + 1/2 \sum_{j,k=1}^{2} (D_j D_k \rho)(z) u_j u_k\right]$$

$$+ \sum_{j,k=1}^{2} (D_j \bar D_k \rho)(z) u_j \bar u_k + o(z, |u|^2). \tag{4}$$

Since ρ is of class $\mathscr{C}^2$ we have

$$\lim_{|u| \to 0} \frac{o(z, |u|^2)}{|u|^2} = 0,$$

uniformly with respect to $z \in \mathscr{V}(t_1, t_2)$ (since this set is relatively compact in $\mathscr{V}(T, 0)$).

Part 2. For each $z \in \mathscr{V}(T, 0)$ we define the entire map $\Phi_z : \mathbf{C}^2 \to \mathbf{C}^2$ by

$$\Phi_z(u_1 e_1(z) + u_2 e_2(z)) = w_1 e'_1 + w_2 e'_2,$$

where

$$w_1 = |(\nabla\rho)(z)|u_1 + \frac{1}{2} \sum_{j,k=1}^{2} (D_j D_k \rho)(z) u_j u_k,$$

$$w_2 = u_2.$$

Note that $\Phi_z(0) = 0$. Since everything in the definition of Φ_z depends smoothly on z, it follows that $(z, w) \to \Phi_z(w)$ is a smooth map on $\mathscr{V}(T, 0) \times \mathbf{C}^2$.

For each $z \in \mathscr{V}(T, 0)$ we get, using bases $\{e_1(z), e_2(z)\}$ and $\{e'_1, e'_2\}$,

$$(D\Phi_z)(0) = \begin{pmatrix} |(\nabla\rho)(z)| & 0 \\ 0 & 1 \end{pmatrix},$$

which shows that $(D\Phi_z)(0)$ is invertible for each $z \in \mathscr{V}(T, 0)$. Let $T < s_1 < t_1 < t_2 < s_2 < 0$. By the Inverse Function Theorem there is a ball $B \subset \mathbf{C}^2$, centred at the origin, such that for each $z \in \mathscr{V}(s_1, s_2)$, Φ_z maps a neighborhood of 0 biholomorphically onto a neighborhood of 0 that contains B and such that $(z, w) \to \Phi_z^{-1}(w)$ is smooth on $\mathscr{V}(s_1, s_2) \times B$. Denote $\Psi_z = \Phi_z^{-1}|_B$. Passing to a smaller B if necessary we have

(a) $\Psi_z(0) = 0$ $(z \in \mathscr{V}(s_1, s_2))$,

(b) $z + \Psi_z(B) \subset D$ $(z \in \mathscr{V}(s_1, s_2))$,

(c) $(z, w) \to \Psi_z(w)$ is smooth on $\mathscr{V}(s_1, s_2) \times B$.

Part 3. Let $s_1 < s'_1 < t_1 < t_2 < s'_2 < s_2$. For each $z = \mathscr{V}(s_1, s_2)$ we have

$$(D\Phi_z)(0)(e_1(z)) = |(\nabla\rho)(z)|e'_1,$$

which implies that L is tangent to $S(z, 0)$ at the origin. By (c) we have

$$\Psi_z(w) = (D\Psi_z)(0)(w) + o(z, |w|),$$

where $\lim_{|w| \to 0} o(z, |w|)/|w| = 0$, uniformly with respect to $z \in \mathscr{V}(s'_1, s'_2)$. In the bases $\{e'_1; e'_2\}$ and $\{e_1(z), e_2(z)\}$ we have

$$(D\Psi_z)(0) = \begin{pmatrix} b(z) & 0 \\ 0 & 1 \end{pmatrix},$$

where $b(z) = 1/|(\nabla\rho)(z)|$. If $u = \Psi_z(w) = u_1 e_1(z) + u_2 e_2(z)$, we thus get

$$u_1 = b_1(z) w_1 + o(z, |w|),$$

$$u_2 = w_2.$$

It follows that

$$u_1\bar{u}_1 = b(z)^2 w_1\bar{w}_1 + o_1(z, |w|^2),$$

$$u_1\bar{u}_2 = b(z)w_1\bar{w}_2 + o_2(z, |w|^2),$$

$$\bar{u}_1 u_2 = b(z)\bar{w}_1 w_2 + o_3(z, |w|^2),$$

$$u_2\bar{u}_2 = w_2\bar{w}_2,$$

where $\lim_{|w|\to 0} o_j(z, |w|^2)/|w|^2 = 0$ $(1 \le j \le 3)$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. Using (4) we get

$$\rho(z + \Psi_z(w)) = \rho(z + u_1(w_1, w_2)e_1(z) + u_2(w_1, w_2)e_2(z))$$

$$= \rho(z) + 2\Re w_1 + \sum_{j,k=1}^{2} (D_j\bar{D}_k\rho)(z)u_j\bar{u}_k + o(z, |u|^2), \tag{5}$$

where $\lim_{|u|\to 0} o(z, |u|^2)/|u|^2 = 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. The ratio $|u(w)|/|w|$ is bounded from above and from below away from zero as $w \to 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. It follows that $o(z, |u|^2)$ in (5) is in fact $o(z, |w|^2)$, where $\lim_{|w|\to 0} o(z, |w|^2)/|w|^2 = 0$, uniformly with respect to $z \in \mathscr{V}(s_1', s_2')$. Thus

$$\rho(z + \Psi_z(w)) = \rho(z) + 2\Re w_1 + \sum_{j,k=1}^{2} b_{j,k}(z)w_j\bar{w}_k + o(z, |w|^2),$$

where

$$(b_{j,k}(z)) = \begin{pmatrix} b(z)^2(D_1\bar{D}_1\rho)(z) & b(z)(D_1\bar{D}_2\rho)(z) \\ b(z)(\bar{D}_1 D_2\rho)(z) & (D_2\bar{D}_2\rho)(z) \end{pmatrix}.$$

By strict plurisubharmonicity of ρ its complex Hessian is strictly positive definite. Since $b(z) = 1/|(\nabla\rho)(z) > 0$, the matrix $(b_{j,k}(z))$ is also strictly positive definite, and its eigenvalues are bounded from above and from below away from zero, uniformly for $z \in \mathscr{V}(s_1, s_2)$. The surface $S(z, v)$ in B is given by the equation

$$2\Re w_1 + \sum_{j,k=1}^{2} b_{j,k}(z)w_j\bar{w}_k + o(z, |w|^2) = v,$$

where

$$\lim_{|w|\to 0} o(z, |w|^2)/|w|^2 = 0,$$

uniformly with respect to $z \in \mathscr{V}(s'_1, s'_2)$. This implies the existence of a number $v_0 > 0$, depending only on t_1, t_2, satisfying the properties (iv), (v), and (vi) of Lemma 3, with B in place of the unit ball $\mathbf{B}^2$. To complete the proof we simply rescale B to $\mathbf{B}^2$.

LEMMA 4. *Let $T < t_1 < t_2 < 0$. There is a $v_0, 0 < v_0 < t_1 - T$, and for each $z \in \mathscr{V}(t_1, t_2)$ there is a holomorphic map $\psi_z : \Delta \to \mathbf{C}^2$ such that*

(i) *$M_z = z + \psi_z(\Delta)$ is a submanifold of an open neighborhood of z contained in D, and $\zeta \to z + \psi_z(\zeta)$ maps Δ biholomorphically to M_z,*
(ii) $\psi_z(0) = 0$,
(iii) *for each v, $-v_0 \le v < 0$, we have $\{p \in D : \rho(p) \le \rho(z) + v\} \cap M_z = \emptyset$,*
(iv) *for each v, $0 < v \le v_0$, M_z intersects $\{p \in D : \rho(p) = \rho(z) + v\}$ transversely in a simple closed curve, and*
(v) *the map $(z, \zeta) \to \psi_z(\zeta)$ is smooth on $\mathscr{V}(t_1, t_2) \times \Delta$.*

Proof. The maps $\psi_z(\zeta) = \Psi_z(0, \zeta)$ $(\zeta \in \Delta)$, where Ψ_z is given by Lemma 3, satisfy all the required properties.

4. Proof of the Main Lemma

Applying Lemma 4 to $\mathscr{V}(t'_1, t'_2)$ where $T < t'_1 < \underline{t_1 < t_2} < t'_2 < 0$ we get v_0 and the maps $\psi_z, z \in \mathscr{V}(t'_1, t'_2)$. Using the compactness of $\overline{\mathscr{V}(t'_1, t'_2)}$ we see (after replacing Δ by a slightly smaller disc) that in Lemma 4 we may assume that each ψ_z extends holomorphically across $b\Delta$ and that $(z, \psi) \to \psi_z(\zeta)$ is smooth on $\mathscr{V}(t'_1, t'_2) \times \bar{\Delta}$.

To approximate a holomorphic map on Δ smoothly on compact subsets of Δ it suffices to approximate it by holomorphic maps uniformly on Δ. Thus, given $\alpha > 0$, there is $\delta > 0$ with the following property: If $\Theta : b\Delta \times \bar{\Delta} \to \mathbf{C}^2$ is a smooth map such that $\Theta(\zeta, \cdot)$ is holomorphic on Δ for each fixed $\zeta \in b\Delta$, and if

$$|\Theta(\zeta, \eta) - \psi_{F(\zeta)}(\eta)| < \delta \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}),$$

then we have

(a) $F(\zeta) + \Theta(\zeta, \Delta) \subset D$ $(\zeta \in b\Delta)$,
(b) $\rho(F(\zeta) + \Theta(\zeta, \lambda)) > \rho(F(z)) - \alpha$ $(\zeta \in b\Delta, \lambda \in \bar{\Delta})$,
(c) for each v, $\alpha < v \le v_0$, the set

$$\Gamma_\zeta(v) = \{\lambda \in \Delta : \rho(F(\zeta) + \Theta(\zeta, \lambda)) = \rho(F(z)) + v\}$$

is a smooth simple closed curve containing 0 in its interior part, and

(d) the curves $\Gamma_\zeta(v)$ depend smoothly on $\zeta \in b\Delta$ and $v, \alpha < v \le v_0$.

Set $\mu_0 = v_0$ and choose $\alpha > 0$ so small that $\mu(\zeta) > \alpha$ $(\zeta \in b\Delta)$ and $\rho(F(\zeta)) > c + \alpha$ $(\zeta \in \bar{\Delta}\backslash r\Delta)$. Further, choose $d > 0$ so small that

$$x \in F(\bar{\Delta}\backslash r\Delta), \quad |x - y| < d \qquad \text{implies } y \in D \text{ and } \rho(y) > c. \tag{6}$$

With no loss of generality we may assume that the function μ is smooth on $b\Delta$.

By Lemma 2 there are $n \in \mathbf{Z}_+$ and a function

$$\Omega(\zeta, \eta) = a_1(\zeta)\eta + \cdots + a_n(\zeta)\eta^n$$

such that

$$|\Omega(\zeta, \eta) - \psi_{F(\zeta)}(\eta)| < \delta/2 \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}).$$

For each j, $1 \le j \le n$, we choose holomorphic polynomials P_j and Q_j such that the function

$$\Theta(\zeta, \eta) = \sum_{j=1}^{n} [P_j(\zeta) + Q_j(1/\zeta)]\eta^j$$

satisfies

$$|\Omega(\zeta, \eta) - \Theta(\zeta, \eta)| < \delta/2 \qquad (\zeta \in b\Delta, \eta \in \bar{\Delta}).$$

Consequently $|F - \Theta| < \delta$ on $b\Delta \times \bar{\Delta}$, hence the properties (a)–(d) hold.
By (b) we have the inequality

$$\rho(F(\zeta) + \Theta(\zeta, \eta)) > \rho(F(\zeta)) - \alpha \qquad (\eta \in \bar{\Delta}) \tag{7}$$

that holds initially for all $\zeta \in b\Delta$, and after passing to a larger $R < 1$ it also holds for all $\zeta \in \bar{\Delta}\backslash R\Delta$.

Choose $m \in \mathbf{Z}_+$ greater than the degrees of all Q_j, $1 \le j \le n$. Since $\alpha < \mu(\zeta) < v_0$, the properties (c) and (d) imply that

$$\Lambda_\zeta = \{\lambda \in \Delta : \rho(F(\zeta) + \Theta(\zeta, \lambda)) = \rho(F(\zeta)) + \mu(\zeta)\}$$

is a continuously varying family of smooth simple closed curves enveloping 0. There is a $\gamma > 0$ such that for all $\zeta \in b\Delta$ and $\eta \in \Lambda_\zeta + \gamma\Delta$ we have

$$|\rho(F(\zeta) + \Theta(\zeta, \eta)) - \rho(F(\zeta)) - \mu(\zeta)| < \epsilon.$$

By the Corollary (Section 2) there is a function $\omega \in \mathscr{A}(\Delta)$ such that $\zeta^m\omega(\zeta) \in \Lambda_\zeta + \gamma\Delta$ for each $\zeta \in b\Delta$. Starting with an even larger m we may assume that

$$|\Theta(\zeta, \zeta^m\omega(\zeta))| < \min\{\epsilon, d\} \qquad (|\zeta| \le R).$$

Define

$$G(\zeta) = \Theta(\zeta, \zeta^m\omega(\zeta)) \qquad (\zeta \in \bar{\Delta}).$$

Then G is continuous on $\bar{\Delta}$, holomorphic on Δ, and by construction it satisfies the properties (i), (iii), (iv), and (v) in the Main Lemma. To prove (ii), observe that by (7) we have $\rho(F(\zeta) + G(\zeta)) > c$ $(\zeta \in \bar{\Delta}\backslash R\Delta)$. If $\zeta \in \bar{\Delta}\backslash r\Delta$, $|\zeta| \le R$, then $|G(\zeta)| < d$ so by (6), $\rho(F(\zeta) + G(\zeta)) > c$. This completes the proof of the Main Lemma.

Remark. To prove the Main Lemma with the stronger condition (v′) we choose a Blaschke product $P(\zeta)$ that vanishes to second order at each point $\zeta_j \in \Delta$, $1 \le j \le n$, we choose $\omega \in \mathscr{A}(\Delta)$ satisfying $\zeta^m P(\zeta)\omega(\zeta) \in \Lambda_\zeta + \gamma\Delta$ for $\zeta \in b\Delta$, and we set

$$G(\zeta) = \Theta(\zeta, \zeta^m P(\zeta)\omega(\zeta)) \qquad (\zeta \in \bar{\Delta}).$$

5. An example

In this section we construct for each $N \ge 2$ a smoothly bounded domain $D \subset\subset \mathbf{C}^N$ with a point $x \in D$ that is not contained in the image of any proper holomorphic map $f: \Delta \to D$.

Let $\mathbf{B}^N$ be the unit ball in $\mathbf{C}^N$. For $x \in \mathbf{C}^N\backslash\{0\}$ we denote by $H(x)$ the real hyperplane through the origin that is perpendicular to x. If $\rho > 0$ write $K(x, \rho) = \{z \in b\mathbf{B}^N : |z - x| \le \rho\}$.

There are δ, $0 < \delta < 1/2$, and ζ, $0 < \alpha < 1$, such that if $1 < R < 1 + \delta$, if $x \in b\mathbf{B}^N$, and if Ω is the connected component of $\mathbf{C}^N\backslash[RK(x, 1/3) \cup (\alpha x + H(x))]$ that contains x, then

$$b\Omega = [RK(x, 1/3) \cap \bar{P}] \cap [(\alpha x + H(x)) \cap R\mathbf{B}^N], \tag{8}$$

where P is a half-space of $\mathbf{C}^N$ determined by the hyperplane $\alpha x + H(x)$.

There are $n \in \mathbf{Z}_+$ and points $x_j \in b\mathbf{B}^N$, $1 \le j \le n$, such that $\bigcup_{j=1}^n K(x_j, 1/3) = b\mathbf{B}^N$. Choose numbers R_j, $1 \le j \le n$, $1 < R_1 < R_2 < \cdots < R_n < 1 + \delta$. For each j we fatten $R_j K(x_j, 2/3)$ to get a smoothly bounded domain $U_j \subset (3/2)\mathbf{B}^N$ that contains $R_j K(x_j, 2/3)$ and has connected boundary. We can choose the domain U_j so small that their closures are pairwise disjoint and $\bar{U}_j \cap H(x_j) = \varnothing$ $(1 \le j \le n)$. Define $D = 2\mathbf{B}^N \backslash \bigcup_{j=1}^n \bar{U}_j$.

Suppose that $f: \Delta \to D$ is a proper holomorphic map such that $f(0) = 0$. Its total cluster set $C(f) = \bigcap_{0<r<1} \overline{f(\Delta \backslash r\Delta)}$ is a connected compact set contained in bD. Since $bD = b(2\mathbf{B}^N) \cup [\bigcup_{j=1}^n bU_j]$ is a disjoint union of $n+1$ compact connected sets, it follows that either $C(f) \subset bU_j$ for some j or $C(f) \subset b(2\mathbf{B}^N)$. We will show that none of these is possible.

Suppose first that $C(f) \subset bU_j$ for some j, $1 \le j \le n$. As f is bounded, the maximum principle implies that $f(\Delta)$ is contained in the closed convex hull of $C(f)$. However, $\bar{U}_j$ is a connected compact set that misses $H(x_j)$, so its convex hull does not contain the point $f(0) = 0 \in H(x_j)$, a contradiction.

Thus $C(f)$ must be contained in $b(2\mathbf{B}^N)$, hence f is a proper map from Δ to $2\mathbf{B}^N$. Since $f(\Delta)$ is connected and since $f(0) = 0$, there is a $\zeta_0 \in \Delta$ such that $f(\zeta_0) = x \in b\mathbf{B}^N$. There is a j, $1 \le j \le n$, such that $x \in K(x_j, 1/3)$, hence $K(x, 1/3) \subset K(x_j, 2/3)$, and it follows that $R_j K(x, 1/3) \subset U_j$. Recall that $1 < R_j < 1 + \delta$. Denote by Ω the connected component of $\mathbf{C}^N \backslash [R_j K(x, 1/3) \cup (\alpha x + H(x))]$ that contains x. Since $\Omega \subset (3/2)\mathbf{B}^N$, $f^{-1}(\Omega)$ is an open, relatively compact subset of Δ. Let G be the component of $f^{-1}(\Omega)$ that contains ζ_0. Since $f(\Delta) \subset D$, it follows that $f(\Delta)$ misses U_j whence it misses $R_j K(x, 1/3)$. Since $f(bG) \subset b\Omega$, (8) implies that $f(bG) \subset \alpha x + H(x)$ which, by the maximum principle, gives $f(G) \subset \alpha x + H(x)$. In particular, $f(\zeta_0) = x \in \alpha x + H(x)$ which contradicts the fact that $0 < \alpha < 1$. This shows that there is no proper holomorphic map $f: \Delta \to D$ satisfying $0 \in f(\Delta)$.

Remark. The domain D constructed above has disconnected boundary. We do not know whether there exists a domain $D \subset \mathbf{C}^N$ with smooth *connected* boundary such that all proper holomorphic maps $f: \Delta \to D$ avoid certain point $x \in D$.

REFERENCES

[1] H. ALEXANDER and J. WERMER, *Polynomial hulls with convex fibers*, Math. Ann. *271* (1985), 99–109.

[2] K. DIEDERICH and J. E. FORNÆSS, *Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. *39* (1977), 129–147.

[3] F. FORSTNERIČ, *Polynomially convex hulls of sets fibered over the circle*, Indiana Univ. Math. J. *37* (1988), 869–889.

[4] F. FORSTNERIČ, *Polynomially convex hulls with piecewise smooth boundaries*, Math. Ann. *276* (1986), 97–104.

[5] J. GLOBEVNIK, *Boundary interpolation and proper holomorphic maps from the disc to the ball*, Math. Z. *198* (1988), 143–150.
[6] J. GLOBEVNIK, *Relative embeddings of discs into convex domains*, Invent. Math. *98* (1989), 331–350.
[7] J. GLOBEVNIK, *On moduli of boundary values of holomorphic functions*, to appear in Math. Z.
[8] J. W. HELTON and D. E. MARSHALL, *Frequency domain design and analytic selections*, Indiana Univ. Math. J. *39* (1990), 157–184.
[9] S. G. KRANTZ, *Function Theory of Several Complex Variables*, Wiley-Interscience, New York, 1982.
[10] Z. SLODKOWSKI, *An analytic set-valued selection and its applications to the corona theorem, to polynomial hulls and joint spectra*, Trans. Amer. Math. Soc. *294* (1986), 367–377.
[11] Z. SLODKOWSKI, *Polynomial hulls with convex fibers and complex geodesics*, J. Funct. Anal. *94* (1990), 156–176.
[12] Z. SLODKOWSKI, *Polynomial hulls in* $\mathbf{C}^2$ *and quasicircles*, Ann. Sc. Norm. Sup. Pisa, cl. s.c., Ser. IV, *26* (1989), 367–391.

Institute of Mathematics
University of Ljubljana
Jadranska 19
61000 LJUBLJANA
Yugoslavia

Received March 23, 1991